If $x, y, z > 0$ and $xyz = 1$, then prove that:

$$\frac{1}{\sqrt{4x^2+x+4}}+\frac{1}{\sqrt{4y^2+y+4}}+\frac{1}{\sqrt{4z^2+z+4}}\le 1$$

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Since $xyz = 1$, we can assign $x=\frac{a}{b}, y=\frac{b}{c}, z=\frac{c}{a}$ and then:

$$\frac{1}{\sqrt{4x^2+x+4}}+\frac{1}{\sqrt{4y^2+y+4}}+\frac{1}{\sqrt{4z^2+z+4}}$$
$$=\frac{b}{\sqrt{4a^2+ab+4b^2}}+\frac{c}{\sqrt{4b^2+bc+4c^2}}+\frac{a}{\sqrt{4c^2+ca+4a^2}}$$
$$=\frac{1}{\sqrt{\prod_{cyc}(4a^2+ab+4b^2)}}\cdot\sum_{cyc}\left(b.\sqrt{(4b^2+bc+4c^2)(4c^2+ca+4a^2)}\right)$$
$$\overset{CBS}{\le}\frac{1}{\sqrt{\prod_{cyc}(4a^2+ab+4b^2)}}\cdot\sqrt{\sum_{cyc}(4b^2c^2+cab^2+4a^2b^2)}\cdot\sqrt{\sum_{cyc}(4b^2+bc+4c^2)}\overset{?}{\le}1$$

$$\Leftrightarrow (4a^2+ab+4b^2)(4b^2+bc+4c^2)(4c^2+ca+4a^2)\overset{?}{\ge}$$
$$\left(8\sum_{cyc}a^2+\sum_{cyc}ab\right).\left(8\sum_{cyc}a^2b^2+abc\sum_{cyc}a\right)$$

$$\Leftrightarrow 64\sum_{cyc}(a^4b^2+a^2b^4)+16abc\sum_{cyc}a^3+16\sum_{cyc}a^3b^3+129a^2b^2c^2$$
$$+20abc\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)\overset{?}{\ge}$$
$$64\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^2b^2\right)+8\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2b^2\right)+8abc\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2\right)$$
$$+abc\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)$$

$$\Leftrightarrow 64\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^2b^2\right)-192a^2b^2c^2+16abc\sum_{cyc}a^3+16\sum_{cyc}a^3b^3$$
$$+129a^2b^2c^2+20abc\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-60a^2b^2c^2\overset{?}{\ge}$$

$$64\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right)+8\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2b^2\right)+8abc\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2\right)$$

$$+abc\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right)$$

$$\Leftrightarrow abc\left(16\sum_{cyc} a^3+19\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right)-8\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2\right)\right)-123a^2b^2c^2$$

$$+16\sum_{cyc} a^3b^3-8\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2b^2\right)\boxed{\overset{?}{\underset{(*)}{\geq}}}0$$

Assigning $b+c=X, c+a=Y, a+b=Z \Rightarrow X+Y-Z=2c>0, Y+Z-X=2a>0$ and $Z+X-Y=2b>0 \Rightarrow X+Y>Z, Y+Z>X, Z+x>Y \Rightarrow X,Y,Z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say);

$$\text{so } 2\sum_{cyc} a=\sum_{cyc} X=2s \Rightarrow \sum_{cyc} a=s \to (1) \Rightarrow a=s-X, b=s-Y, c=s-Z$$

$$\therefore abc=r^2s \to (2)\text{and such substitutions} \Rightarrow \sum_{cyc} ab=\sum_{cyc}(s-x)(s-y)$$

$$\Rightarrow \sum_{cyc} ab=4Rr+r^2 \to (3), \text{and} \sum_{cyc} a^2=\left(\sum_{cyc} a\right)^2-2\sum_{cyc} ab \overset{\text{via }(1)\text{ and }(3)}{=}$$

$$s^2-2(4Rr+r^2) \Rightarrow \sum_{cyc} a^2=s^2-8Rr-2r^2 \to (4) \text{ and also,}$$

$$\sum_{cyc} a^2b^2=\left(\sum_{cyc} ab\right)^2-2abc\left(\sum_{cyc} a\right) \overset{\text{via }(1),(2)\text{ and }(3)}{=} (4Rr+r^2)^2-2r^2s.s$$

$$\Rightarrow \sum_{cyc} a^2b^2=r^2((4R+r)^2-2s^2) \to (5) \text{ and moreover,} \sum_{cyc} a^3=$$

$$\left(\sum_{cyc} a\right)^3-3(a+b)(b+c)(c+a) \overset{\text{via }(1)}{=} s^3-3.4Rrs \Rightarrow \sum_{cyc} a^3=s^3-12Rrs$$

$$\to (6) \text{ and finally,} \sum_{cyc} a^3b^3=\left(\sum_{cyc} ab\right)^3-3(ab+bc)(bc+ca)(ca+ab)$$

$$\overset{\text{via }(2)\text{ and }(3)}{=} (4Rr+r^2)^3-3r^2s.4Rrs \Rightarrow \sum_{cyc} a^3b^3=(4Rr+r^2)^3-12Rr^3s^2 \to (7)$$

and via $(1),(2),(3),(4),(5),(6)$ and (7), $(*)$ transforms into :

$$r^2s\left(16(s^3-12Rrs)+19s(4Rr+r^2)-8s(s^2-8Rr-2r^2)\right)-123r^4s^2$$

$$+16((4Rr+r^2)^3-12Rr^3s^2)-8(4Rr+r^2).r^2((4R+r)^2-2s^2)\geq 0$$

$$\Leftrightarrow 2s^4-(45Rr+18r^2)s^2+2r(4R+r)^3 \boxed{\geq} 0 \to \text{true} \because 2s^4-(45Rr+18r^2)s^2$$

$$+2r(4R+r)^3 \text{ is } a \text{ qu}a\text{dr}a\text{tic polynomi}al \text{ in } s^2 \text{ with discrimin}a\text{nt} =$$

$$(45Rr+18r^2)^2-16r(4R+r)^3=-r(1024R^3-1257R^2r-1428Rr^2-308r^3)$$

$$=-r(R-2r)(1024R^2+791Rr+154r^2)\overset{\text{Euler}}{\leq} 0 \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{1}{\sqrt{4x^2+x+4}}+\frac{1}{\sqrt{4y^2+y+4}}+\frac{1}{\sqrt{4z^2+z+4}}\leq 1$$

$$\forall x,y,z>0 \,\Big|\, xyz=1, ''='' \text{ iff } x=y=z=1 \text{ (QED)}$$