ROMANIAN MATHEMATICAL MAGAZINE

$$\textit{If } x,y,z>0 \textit{ and } xyz=27, \textit{then prove that}$$

$$\frac{1}{\sqrt{x^2+21x+9}}+\frac{1}{\sqrt{y^2+21y+9}}+\frac{1}{\sqrt{z^2+21z+9}}\geq\frac{1}{3}$$

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Firstly, for all $x>0$, *we have*

$$\sqrt{x^2+21x+9}\leq x+\sqrt{3x}+3 \overset{Squaring}{\Leftrightarrow} 0\leq 2\sqrt{3x^3}-12x+6\sqrt{3x}=2\sqrt{3x}.\left(\sqrt{x}-\sqrt{3}\right)^2,$$

which is true.

Now, since $xyz=27$, then there exist $a,b,c>0$ such that $x=\frac{3bc}{a^2}, y=\frac{3ca}{b^2}, z=\frac{3ab}{c^2}$, and

$$\sum_{cyc}\frac{1}{\sqrt{x^2+21x+9}}\geq\sum_{cyc}\frac{1}{x+\sqrt{3x}+3}$$

$$=\sum_{cyc}\frac{a^2}{3\left(bc+a\sqrt{bc}+a^2\right)}\overset{AM-GM}{\geq}\frac{1}{3}\sum_{cyc}\frac{a^2}{bc+\frac{ab+ac}{2}+a^2}$$

$$\overset{CBS}{\geq}\frac{(a+b+c)^2}{3\sum_{cyc}\left(bc+\frac{ab+ac}{2}+a^2\right)}=\frac{1}{3}.$$

Equality holds iff $x=y=z=3$.