ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z \in \mathbb{R}$ such that : $x^2 + y^2 + z^2 = 8$ and $xy + yz + zx = -4$,

then prove that : $-\frac{\sqrt{3}}{6} \leq \frac{x^3 + y^3 + z^3}{x^4 + y^4 + z^4} \leq \frac{\sqrt{3}}{6}$

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$$x^2 + y^2 + z^2 + 2(xy + yz + zx) = 8 - 8 = 0 \Rightarrow (x + y + z)^2 = 0$$
$$\Rightarrow x + y + z = 0 \rightarrow (1)$$

$$xy + yz + zx = -4 \Rightarrow xy + z(x + y) = -4 \overset{\text{via (1)}}{\Rightarrow} xy + z(-z) = -4 \Rightarrow z^2 - 4 \overset{(2)}{=} xy$$
$$\leq \frac{(x + y)^2}{4} \; (\because (x - y)^2 \geq 0 \; \forall \, x, y \in \mathbb{R}) \overset{\text{via (1)}}{=} \frac{z^2}{4} \Rightarrow \frac{3z^2}{4} \leq 4 \Rightarrow 3z^2 - 16 \leq 0 \rightarrow (3)$$

$$\text{Now,} \frac{x^3 + y^3 + z^3}{x^4 + y^4 + z^4} = \frac{\left(\sum_{\text{cyc}} x\right)^3 - 3\left(\left(\sum_{\text{cyc}} x\right)\left(\sum_{\text{cyc}} xy\right) - xyz\right)}{\left(\sum_{\text{cyc}} x^2\right)^2 - 2\left(\left(\sum_{\text{cyc}} xy\right)^2 - 2xyz\left(\sum_{\text{cyc}} x\right)\right)} \overset{\text{via (1)}}{=} \frac{3xyz}{64 - 32}$$

$$\left(\because \sum_{\text{cyc}} x^2 = 8 \text{ and } \sum_{\text{cyc}} xy - 4\right) \Rightarrow \left(\frac{x^3 + y^3 + z^3}{x^4 + y^4 + z^4}\right)^2 = \frac{9x^2y^2z^2}{1024} \overset{\text{via (2)}}{=} \frac{9z^2(z^2 - 4)^2}{1024}$$

$$\overset{?}{\leq} \frac{1}{12} \Leftrightarrow 27z^6 - 216z^4 + 432z^2 - 256 \overset{?}{\leq} 0 \Leftrightarrow (3z^2 - 16)(3z^2 - 4)^2 \overset{?}{\leq} 0$$

$$\rightarrow \text{true} \because 3z^2 - 16 \overset{\text{via (3)}}{\leq} 0 \therefore \left(\frac{x^3 + y^3 + z^3}{x^4 + y^4 + z^4}\right)^2 \leq \frac{1}{12}$$

$$\Rightarrow -\frac{1}{\sqrt{12}} = -\frac{\sqrt{3}}{6} \leq \frac{x^3 + y^3 + z^3}{x^4 + y^4 + z^4} \leq \frac{1}{\sqrt{12}} = \frac{\sqrt{3}}{6},$$

$$'' ='' \text{ iff } \left(x = \frac{2}{\sqrt{3}}, y = \frac{2}{\sqrt{3}}, z = -\frac{4}{\sqrt{3}}\right) \text{ or } \left(x = -\frac{2}{\sqrt{3}}, y = -\frac{2}{\sqrt{3}}, z = \frac{4}{\sqrt{3}}\right)$$
$$\text{or } \left(x = -\frac{4}{\sqrt{3}}, y = \frac{2}{\sqrt{3}}, z = \frac{2}{\sqrt{3}}\right) \text{ or } \left(x = \frac{2}{\sqrt{3}}, y = -\frac{4}{\sqrt{3}}, z = \frac{2}{\sqrt{3}}\right)$$
$$\text{or } \left(x = \frac{4}{\sqrt{3}}, y = -\frac{2}{\sqrt{3}}, z = -\frac{2}{\sqrt{3}}\right) \text{ or } \left(x = -\frac{2}{\sqrt{3}}, y = \frac{4}{\sqrt{3}}, z = -\frac{2}{\sqrt{3}}\right) \text{ (QED)}$$