ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $ab+bc+ca+abc=4$, then prove that :

$$\sqrt{abc}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\le 3$$

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$$\sum_{cyc}\left((2+b)(2+c)\right)-(2+a)(2+b)(2+c)=4-(ab+bc+ca+abc)$$

$$=0\ \therefore \sum_{cyc}\frac{1}{2+a}=1\to(m)$$

Now, $\frac{1}{2+a}\overset{a>0}{<}\frac{1}{2}\ \therefore$ we can set : $\frac{1}{2+a}=\frac{1}{2}-x\ \left(x>0 \text{ and } x<\frac{1}{2}\right)\ \therefore a+2=$

$$\frac{2}{1-2x}\Rightarrow a=\frac{2}{1-2x}-2=\frac{2x}{\frac{1}{2}-x}\to(1)$$

Similarly, we set : $\frac{1}{2+b}=\frac{1}{2}-y$ and $\frac{1}{2+c}=\frac{1}{2}-z\ \therefore 1\overset{\text{via }(m)}{=}$

$$\frac{1}{2+a}+\frac{1}{2+b}+\frac{1}{2+c}=\frac{1}{2}-x+\frac{1}{2}-y+\frac{1}{2}-z\Rightarrow x+y+z=\frac{1}{2}\to(i)$$

$\therefore (1)$ and $(i)\Rightarrow a=\frac{2x}{y+z}$ and analogously, $b=\frac{2y}{z+x}$ and $c=\frac{2z}{x+y}$

and hence : $\sqrt{abc}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)=\sqrt{\frac{2x}{y+z}\cdot\frac{2y}{z+x}\cdot\frac{2z}{x+y}}\cdot\sqrt{\frac{2x}{y+z}+\frac{2y}{z+x}+\frac{2z}{x+y}}$

$$\overset{CBS}{\le}4\cdot\sqrt{\frac{xyz}{\prod_{cyc}(x+y)}}\cdot\sqrt{\sum_{cyc}x}\cdot\sqrt{\sum_{cyc}\frac{1}{y+z}}=4\cdot\sqrt{\frac{xyz\sum_{cyc}x}{\prod_{cyc}(x+y)^2}}\cdot\sqrt{\left(\sum_{cyc}x\right)^2+\sum_{cyc}xy}$$

$$\le 4\cdot\sqrt{\frac{xyz\sum_{cyc}x}{\left(\frac{8}{9}\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)\right)^2}}\cdot\sqrt{\frac{4}{3}\left(\sum_{cyc}x\right)^2}\overset{\text{via }(i)}{=}4\cdot\sqrt{\frac{81xyz\sum_{cyc}x}{16\left(\sum_{cyc}xy\right)^2}}\cdot\sqrt{\frac{4}{3}\cdot\frac{1}{4}}$$

$$\le\sqrt{\frac{27xyz\sum_{cyc}x}{3xyz\sum_{cyc}x}}=3\ \therefore \sqrt{abc}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\le 3$$

$$\forall\, a,b,c>0\ \middle|\ ab+bc+ca+abc=4, \text{"="} \text{ iff } a=b=c=1\ (QED)$$