ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{3}{2}(a+b+c)=\frac{15}{2}$, then prove that :

$$a^2+b^2+c^2 \le 3$$

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$$\text{Firstly, } \left(\sum_{cyc} ab\right)^3 \overset{A-G}{\ge} 27a^2b^2c^2 \Rightarrow abc \le \sqrt{\frac{\left(\sum_{cyc} ab\right)^3}{27}} \to (1) \text{ and also,}$$

$$\frac{15}{2} = \sum_{cyc}\left(\frac{1}{a}+a\right)+\frac{1}{2}\left(\sum_{cyc} a\right) \overset{A-G}{\ge} 6+\frac{1}{2}\left(\sum_{cyc} a\right) \Rightarrow \sum_{cyc} a \le 3 \to (2)$$

$$\text{Now, we assume : } \sum_{cyc} a^2 > 3 \text{ and we have : } \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{3}{2}(a+b+c)=\frac{15}{2}$$

$$\Rightarrow \sum_{cyc} ab + \frac{3}{2}abc\left(\sum_{cyc} a\right) = \frac{15}{2}abc \Rightarrow \sum_{cyc} ab = \frac{3}{2}abc\left(5-\sum_{cyc} a\right)$$

$$\overset{via\ (1)}{\le} \frac{3}{2}\cdot\sqrt{\frac{\left(\sum_{cyc} ab\right)^3}{27}}\cdot\left(5-\sum_{cyc} a\right) \left(\because \text{via (2)}, \sum_{cyc} a \le 3 < 5 \Rightarrow 5-\sum_{cyc} a > 0\right)$$

$$= \frac{1}{2}\cdot\left(\sum_{cyc} ab\right)\cdot\sqrt{\frac{\sum_{cyc} ab}{3}}\cdot\left(5-\sqrt{\sum_{cyc} a^2+2\sum_{cyc} ab}\right)$$

$$\overset{via\ assumption}{<} \frac{1}{2}\cdot\left(\sum_{cyc} ab\right)\cdot\sqrt{\frac{\sum_{cyc} ab}{3}}\cdot\left(5-\sqrt{3+2\sum_{cyc} ab}\right)$$

$$\Rightarrow \boxed{\sqrt{\frac{t}{3}}\cdot\left(5-\sqrt{3+2t}\right)-2>0} \to (*) \left(t=\sum_{cyc} ab\right)$$

$$\text{Now, } t=\sum_{cyc} ab \le \frac{1}{3}\left(\sum_{cyc} a\right)^2 \overset{via\ (2)}{\le} \frac{1}{3}.9 \Rightarrow \boxed{\boxed{0<t\le 3}} \to (3) \text{ and we denote}$$

$$f(t)=\sqrt{\frac{t}{3}}\cdot\left(5-\sqrt{3+2t}\right)-2 \ \forall\ t\in(0,3] \text{ and then : } f'(t)=\frac{5.\sqrt{2t+3}-(4t+3)}{2.\sqrt{3t(2t+3)}}$$

$$=\frac{25(2t+3)-(4t+3)^2}{2.\sqrt{3t(2t+3)}.\left(5.\sqrt{2t+3}+(4t+3)\right)}=\frac{-2(8t^2-13t-33)}{2.\sqrt{3t(2t+3)}.\left(5.\sqrt{2t+3}+(4t+3)\right)}$$

$$=\frac{(3-t)(8t+11)}{\sqrt{3t(2t+3)}.\left(5.\sqrt{2t+3}+(4t+3)\right)}\overset{\text{via }(3)}{\geq}0\Rightarrow f'(t)\geq 0\ \forall\ t\in(0,3]$$

$$\Rightarrow f(t)\text{ is }\uparrow\text{ on }(0,3]\Rightarrow f(t)\leq f(3)=0\Rightarrow\boxed{\sqrt{\frac{t}{3}}.\left(5-\sqrt{3+2t}\right)-2\leq 0}$$

which contradicts $(*)$ and hence, we conclude that our assumption is incorrect

$$\Rightarrow a^2+b^2+c^2\leq 3,''=''\text{ iff } a=b=c=1\ (\text{QED})$$