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If $x,y,z \geq 0$ and $x^2+y^2+z^2=3$, then prove that :

$$\frac{16}{\sqrt{x^2y^2+y^2z^2+z^2x^2+1}}+\frac{xy+yz+zx+1}{x+y+z}\geq\frac{28}{3}$$

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$\boxed{\text{Case 1}}$ Exactly two variables equal to zero and WLOG we may assume

$$y=z=0\ (x=\sqrt{3})\text{ and then : LHS}=16+\frac{1}{\sqrt{3}}>\frac{28}{3}$$

$\boxed{\text{Case 2}}$ Exactly one variable equals to zero and WLOG we may assume

$x=0$ with $y,z>0$ *such that* : $y^2+z^2=3$ *and then* :

$$\text{LHS}=\frac{16}{\sqrt{y^2z^2+1}}+\frac{yz+1}{y+z}\rightarrow(1)$$

$$\text{Now, } yz\overset{A-G}{\leq}\frac{y^2+z^2}{2}=\frac{3}{2}\therefore y^2z^2+1\leq\frac{13}{4}\Rightarrow\frac{16}{\sqrt{y^2z^2+1}}\geq\frac{32}{\sqrt{13}}\rightarrow(\text{i})$$

$$\textit{and also},\frac{yz+1}{y+z}\overset{?}{>}\frac{1}{2}\Leftrightarrow 2yz+2\overset{?}{>}y+z\Leftrightarrow y(2z-1)+2-z\overset{?}{>}0$$

$$\Leftrightarrow\sqrt{3-z^2}.(2z-1)+2-z\underbrace{\overset{?}{>}}_{(\blacksquare)}0$$

$(\blacksquare)$ is true when $z\geq\frac{1}{2}$ $(\because 2>\sqrt{3}>z\text{ as }y^2+z^2=3)$ *and so*, we now

consider the case when : $z<\frac{1}{2}$ *and then* : $(\blacksquare)\Leftrightarrow 2-z\overset{?}{>}\sqrt{3-z^2}.(1-2z)$

$$\Leftrightarrow(2-z)^2\overset{?}{>}(3-z^2)(1-2z)^2\Leftrightarrow 4z^4-4z^3-10z^2+8z+1\overset{?}{>}0$$

$$\Leftrightarrow z^2(1-2z)^2+8z(1-2z)+5z^2+1\overset{?}{>}0\rightarrow\text{true}\because 0<z<\frac{1}{2}$$

$$\Rightarrow(\blacksquare)\text{ is true and so},\frac{yz+1}{y+z}>\frac{1}{2}\rightarrow(\text{ii})\forall\, y,z>0\,|\,y^2+z^2=3\therefore(\text{i})+(\text{ii})\Rightarrow$$

$$\frac{16}{\sqrt{y^2z^2+1}}+\frac{yz+1}{y+z}>\frac{32}{\sqrt{13}}+\frac{1}{2}\approx 9.3752>\frac{28}{3}\overset{\text{via }(1)}{\Rightarrow}\text{LHS}>\frac{28}{3}$$

$\boxed{\text{Case 3}}$ $x,y,z>0$ *and assigning* $y+z=a,z+x=b,x+y=c\Rightarrow a+b-c=2z>0,b+c-a=2x>0$ *and* $c+a-b=2y>0\Rightarrow a+b>c,b+c>a,c+a>b$ $\Rightarrow a,b,c$ form sides of *a* triangle with semiperimeter, circumradius *and* inradius

$$=s,R,r\ (\textit{say})\text{ yielding }2\sum_{\text{cyc}}x=\sum_{\text{cyc}}a=2s\Rightarrow\sum_{\text{cyc}}x\overset{(\bullet)}{=}s\Rightarrow x=s-a,y=s-b,$$

$$z=s-c\therefore xyz\overset{(\bullet\bullet)}{=}r^2s\text{ and},\sum_{\text{cyc}}xy=\sum_{\text{cyc}}(s-a)(s-b)=4Rr+r^2$$

$$\Rightarrow\sum_{\text{cyc}}xy\overset{(\bullet\bullet\bullet)}{=}4Rr+r^2\text{ and also},\sum_{\text{cyc}}x^2=\left(\sum_{\text{cyc}}x\right)^2-2\sum_{\text{cyc}}xy\overset{\text{via }(\bullet)\text{ and }(\bullet\bullet\bullet)}{=}$$

$$s^2-2(4Rr+r^2)\Rightarrow\sum_{\text{cyc}}x^2\overset{(\bullet\bullet\bullet\bullet)}{=}s^2-8Rr-2r^2\text{ and also},\sum_{\text{cyc}}x^2y^2=$$

$$\left(\sum_{cyc} xy\right)^2 - 2xyz\left(\sum_{cyc} x\right) \overset{\text{via } (\bullet),(\bullet\bullet) \text{ and } (\bullet\bullet\bullet)}{=} \left(4Rr+r^2\right)^2 - 2r^2s^2$$

$$\Rightarrow \sum_{cyc} x^2y^2 \overset{(\bullet\bullet\bullet\bullet\bullet)}{=} r^2\left((4R+r)^2 - 2s^2\right)$$

$$\text{Now, LHS} \overset{x^2+y^2+z^2=3}{=} \frac{16}{\sqrt{\sum_{cyc} x^2y^2 + \frac{\left(\sum_{cyc} x^2\right)^2}{9}}} + \frac{\sum_{cyc} xy + \frac{\sum_{cyc} x^2}{3}}{\left(\sum_{cyc} x\right).\sqrt{\frac{\sum_{cyc} x^2}{3}}} \overset{x^2+y^2+z^2=3}{=}$$

$$\frac{16\sum_{cyc} x^2}{\sqrt{\left(\sum_{cyc} x^2\right)^2 + 9\sum_{cyc} x^2y^2}} + \frac{\left(\sum_{cyc} x\right)^2 + \sum_{cyc} xy}{\left(\sum_{cyc} x\right).\sqrt{3\sum_{cyc} x^2}}$$

$$\Rightarrow \text{LHS}^2 = \frac{256\left(\sum_{cyc} x^2\right)^2}{\left(\sum_{cyc} x^2\right)^2 + 9\sum_{cyc} x^2y^2} + \frac{\left(\left(\sum_{cyc} x\right)^2 + \sum_{cyc} xy\right)^2}{\left(\sum_{cyc} x\right)^2 .3\sum_{cyc} x^2}$$

$$+ \frac{32\sum_{cyc} x^2}{\sqrt{\left(\sum_{cyc} x^2\right)^2 + 9\sum_{cyc} x^2y^2}} \cdot \frac{\left(\sum_{cyc} x\right)^2 + \sum_{cyc} xy}{\left(\sum_{cyc} x\right).\sqrt{3\sum_{cyc} x^2}}$$

$$\overset{\text{CBS}}{\geq} \frac{256\left(\sum_{cyc} x^2\right)^2}{\left(\sum_{cyc} x^2\right)^2 + 9\sum_{cyc} x^2y^2} + \frac{\left(\left(\sum_{cyc} x\right)^2 + \sum_{cyc} xy\right)^2}{\left(\sum_{cyc} x\right)^2 .3\sum_{cyc} x^2}$$

$$+ \frac{32\sum_{cyc} x^2}{\sqrt{\left(\sum_{cyc} x^2\right)^2 + 3\left(\sum_{cyc} x^2\right)^2}} \cdot \frac{\left(\sum_{cyc} x\right)^2 + \sum_{cyc} xy}{\sqrt{3\sum_{cyc} x^2}.\sqrt{3\sum_{cyc} x^2}} \overset{?}{\geq} \frac{784}{9}$$

$$\overset{\text{via } (\bullet),(\bullet\bullet\bullet),(\bullet\bullet\bullet\bullet) \text{ and } (\bullet\bullet\bullet\bullet\bullet)}{\Leftrightarrow}$$

$$\frac{768s^2\left(s^2-8Rr-2r^2\right)^3 + \left(s^2+4Rr+r^2\right)\left(17s^2+4Rr+r^2\right)\begin{pmatrix} s^2\left(s^2-8Rr-2r^2\right)^2 + \\ 9r^2\left((4R+r)^2-2s^2\right)\end{pmatrix}}{3s^2\left(s^2-8Rr-2r^2\right)\left(s^2\left(s^2-8Rr-2r^2\right)^2 + 9r^2\left((4R+r)^2-2s^2\right)\right)} \overset{?}{\geq} \frac{784}{9}$$

$$\Leftrightarrow 1571s^8 - \left(37080Rr - 3924r^2\right)s^6 + r^2\left(186144R^2 - 23712Rr - 17562\right)s^4 +$$

$$r^3\left(169088R^3 + 125952R^2r + 31272Rr^2 + 2588r^3\right)s^2 + 39r^4(4R+r)^4 \boxed{\overset{?}{\underset{(*)}{\geq}}} 0 \text{ and}$$

$$\because \text{via Gerretsen}, P = 1571\left(s^2-16Rr+5r^2\right)^4 +$$

$$4r(15866R - 6874r)\left(s^2-16Rr+5r^2\right)^3 +$$

$$4r^2\left(204840R^2 - 196830Rr + 39807\right)\left(s^2-16Rr+5r^2\right)^2 \geq 0 \therefore \text{in order}$$

to prove $(*)$, it suffices to prove : LHS of $(*) \geq P$

$$\Leftrightarrow \left(105860R^3 - 181650R^2r + 80709Rr^2 - 9781r^3\right)s^2 \boxed{\overset{(**)}{\geq}}$$

$$r\left(1648584R^4 - 3282264R^3r + 2056212R^2r^2 - 524997Rr^3 + 47673r^4\right)$$

$$\text{Again, LHS of } (**) \overset{\text{Gerretsen}}{\geq} \begin{pmatrix} 105860R^3 - 181650R^2r + \\ 80709Rr^2 - 9781r^3 \end{pmatrix}\left(16Rr - 5r^2\right)$$

$$\overset{?}{\geq} r\left(1648584R^4 - 3282264R^3r + 2056212R^2r^2 - 524997Rr^3 + 47673r^4\right)$$

$$\Leftrightarrow 22588t^4 - 76718t^3 + 71691t^2 - 17522t + 616 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(22588t^2+13634t+35875)+71442\right) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \frac{16}{\sqrt{x^2y^2+y^2z^2+z^2x^2+1}} + \frac{xy+yz+zx+1}{x+y+z} \geq \frac{28}{3}$$

under *case* (3) *and* combining *all* cases,

$$\frac{16}{\sqrt{x^2y^2+y^2z^2+z^2x^2+1}} + \frac{xy+yz+zx+1}{x+y+z} \geq \frac{28}{3}$$

$$\forall\, x,y,z \geq 0 \,\middle|\, x^2+y^2+z^2 = 3, \text{''}=\text{''} \text{ iff } x=y=z=1 \text{ (QED)}$$