ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b}, \mathrm{c} > 0$ and $a + \mathrm{b} + \mathrm{c} = a\mathrm{bc}$, then prove that :

$$\frac{3\sqrt{3}}{4} \le \frac{\mathrm{bc}}{a(1+\mathrm{bc})} + \frac{\mathrm{c}a}{\mathrm{b}(1+\mathrm{c}a)} + \frac{a\mathrm{b}}{\mathrm{c}(1+a\mathrm{b})} \le \frac{a+\mathrm{b}+\mathrm{c}}{4}$$

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$$\frac{\mathrm{bc}}{a(1+\mathrm{bc})} + \frac{\mathrm{c}a}{\mathrm{b}(1+\mathrm{c}a)} + \frac{a\mathrm{b}}{\mathrm{c}(1+a\mathrm{b})} \le \frac{a+\mathrm{b}+\mathrm{c}}{4} \overset{a+\mathrm{b}+\mathrm{c}=a\mathrm{bc}}{\Leftrightarrow}$$

$$\sum_{\mathrm{cyc}} \frac{\mathrm{bc}}{a\left(\frac{a\mathrm{bc}}{a+\mathrm{b}+\mathrm{c}} + \mathrm{bc}\right)} \le \frac{a+\mathrm{b}+\mathrm{c}}{4} \Leftrightarrow (a+\mathrm{b}+\mathrm{c}).\sum_{\mathrm{cyc}} \frac{1}{a(2a+\mathrm{b}+\mathrm{c})} \le \frac{a+\mathrm{b}+\mathrm{c}}{4}$$

$$\overset{a+\mathrm{b}+\mathrm{c}=a\mathrm{bc}}{\Leftrightarrow} \frac{a\mathrm{bc}}{a+\mathrm{b}+\mathrm{c}}.\sum_{\mathrm{cyc}} \frac{1}{a(2a+\mathrm{b}+\mathrm{c})} \le \frac{1}{4} \Leftrightarrow \sum_{\mathrm{cyc}} \frac{\mathrm{bc}}{(a+\mathrm{b})+(\mathrm{c}+a)} \boxed{\overset{(*)}{\le}} \frac{a+\mathrm{b}+\mathrm{c}}{4}$$

Assigning $\mathrm{b}+\mathrm{c} = x, \mathrm{c}+a = \mathrm{y}, a+\mathrm{b} = \mathrm{z} \Rightarrow x+\mathrm{y}-\mathrm{z} = 2\mathrm{c} > 0, y+z-x = 2a > 0$ and $\mathrm{z}+x-\mathrm{y} = 2\mathrm{b} > 0 \Rightarrow x+\mathrm{y} > z, \mathrm{y}+\mathrm{z} > x, \mathrm{z}+x > y \Rightarrow x, \mathrm{y}, \mathrm{z}$ form sides of a triangle with semiperimeter, circumradius and inradius $= \mathrm{s}, \mathrm{R}, \mathrm{r}$ (say);

$$\text{so } 2\sum_{\mathrm{cyc}} a = \sum_{\mathrm{cyc}} x = 2\mathrm{s} \Rightarrow \sum_{\mathrm{cyc}} a = \mathrm{s} \to (1) \Rightarrow a = \mathrm{s}-x, \mathrm{b} = \mathrm{s}-\mathrm{y}, \mathrm{c} = \mathrm{s}-\mathrm{z}$$

$$\therefore a\mathrm{bc} = \mathrm{r}^2\mathrm{s} \to (2) \text{ and such substitutions} \Rightarrow \sum_{\mathrm{cyc}} a\mathrm{b} = \sum_{\mathrm{cyc}} (\mathrm{s}-x)(\mathrm{s}-\mathrm{y})$$

$$\Rightarrow \sum_{\mathrm{cyc}} a\mathrm{b} = 4\mathrm{Rr}+\mathrm{r}^2 \to (3) \therefore \sum_{\mathrm{cyc}} \frac{\mathrm{bc}}{(a+\mathrm{b})+(\mathrm{c}+a)} = \sum_{\mathrm{cyc}} \frac{(\mathrm{s}-\mathrm{y})(\mathrm{s}-\mathrm{z})}{\mathrm{y}+\mathrm{z}}$$

$$= \frac{1}{2\mathrm{s}(\mathrm{s}^2+2\mathrm{Rr}+\mathrm{r}^2)}.\sum_{\mathrm{cyc}} \left((\mathrm{s}-\mathrm{y})(\mathrm{s}-\mathrm{z})(\mathrm{z}+x)(x+\mathrm{y})\right)$$

$$= \frac{1}{2\mathrm{s}(\mathrm{s}^2+2\mathrm{Rr}+\mathrm{r}^2)}.\sum_{\mathrm{cyc}} \left((\mathrm{s}-\mathrm{y})(\mathrm{s}-\mathrm{z})\left(x^2+\sum_{\mathrm{cyc}} xy\right)\right)$$

$$= \frac{1}{2\mathrm{s}(\mathrm{s}^2+2\mathrm{Rr}+\mathrm{r}^2)}.\begin{pmatrix} \sum_{\mathrm{cyc}} \left(x^2(-\mathrm{s}^2+\mathrm{s}x+\mathrm{yz})\right) \\ +(\mathrm{s}^2+4\mathrm{Rr}+\mathrm{r}^2).\sum_{\mathrm{cyc}} (\mathrm{s}-\mathrm{y})(\mathrm{s}-\mathrm{z}) \end{pmatrix}$$

$$= \frac{-2\mathrm{s}^2(\mathrm{s}^2-4\mathrm{Rr}-\mathrm{r}^2)+2\mathrm{s}^2(\mathrm{s}^2-6\mathrm{Rr}-3\mathrm{r}^2)+8\mathrm{Rrs}^2+(\mathrm{s}^2+4\mathrm{Rr}+\mathrm{r}^2)(4\mathrm{Rr}+\mathrm{r}^2)}{2\mathrm{s}(\mathrm{s}^2+2\mathrm{Rr}+\mathrm{r}^2)}$$

$$\overset{?}{\le} \frac{a+\mathrm{b}+\mathrm{c}}{4} \overset{\text{via }(1)}{=} \frac{\mathrm{s}}{4} \Leftrightarrow 2\mathrm{r}\left((8\mathrm{R}-3\mathrm{r})\mathrm{s}^2+\mathrm{r}(4\mathrm{R}+\mathrm{r})^2\right) \overset{?}{\le} \mathrm{s}^2(\mathrm{s}^2+2\mathrm{Rr}+\mathrm{r}^2)$$

$$\Leftrightarrow \mathrm{s}^4-(14\mathrm{Rr}-7\mathrm{r}^2)\mathrm{s}^2-2\mathrm{r}^2(4\mathrm{R}+\mathrm{r})^2 \boxed{\underset{(**)}{\overset{?}{\ge}}} 0$$

$$\text{Now, LHS of } (**) \overset{\text{Gerretsen}}{\ge} (2\mathrm{Rr}+2\mathrm{r}^2)\mathrm{s}^2-2\mathrm{r}^2(4\mathrm{R}+\mathrm{r})^2 \overset{\text{Gerretsen}}{\ge}$$

$$\left(2Rr+2r^2\right)\left(16Rr-5r^2\right)-2r^2(4R+r)^2=3r^3(R-2r)\overset{\text{Euler}}{\geq}0\Rightarrow(**)\Rightarrow(*)$$

$$\text{is true}\therefore\frac{bc}{a(1+bc)}+\frac{ca}{b(1+ca)}+\frac{ab}{c(1+ab)}\leq\frac{a+b+c}{4}$$

$$\text{Again},\frac{bc}{a(1+bc)}+\frac{ca}{b(1+ca)}+\frac{ab}{c(1+ab)}\overset{a+b+c=abc}{=}\sum_{\text{cyc}}\frac{bc}{a+\sum_{\text{cyc}}a}$$

$$=\sum_{\text{cyc}}\frac{b^2c^2}{2abc+b^2c+bc^2}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{\text{cyc}}ab\right)^2}{6abc+\left(\sum_{\text{cyc}}a\right)\left(\sum_{\text{cyc}}ab\right)-3abc}\overset{?}{\geq}\frac{3\sqrt{3}}{4}\overset{a+b+c=abc}{=}$$

$$\frac{3\sqrt{3}}{4}\cdot\sqrt{\frac{abc}{a+b+c}}\overset{\text{via (1),(2) and (3)}}{\Leftrightarrow}\frac{r^4(4R+r)^4}{\left(3r^2s+s(4Rr+r^2)\right)^2}\overset{?}{\geq}\frac{27}{16}\cdot\frac{r^2s}{s}$$

$$\Leftrightarrow(4R+r)^4\boxed{\underset{(***)}{\overset{?}{\geq}}}27s^2(R+r)^2$$

$$\text{Now},(4R+r)^4\overset{\text{Trucht or Doucet}}{\geq}3s^2(4R+r)^2\overset{?}{\geq}27s^2(R+r)^2\Leftrightarrow4R+r\overset{?}{\geq}3R+3r$$

$$\Leftrightarrow R\overset{?}{\geq}2r\rightarrow\text{true via Euler}\Rightarrow(***)\text{ is true}\therefore\frac{bc}{a(1+bc)}+\frac{ca}{b(1+ca)}+\frac{ab}{c(1+ab)}$$

$$\geq\frac{3\sqrt{3}}{4}\text{ and so},\frac{3\sqrt{3}}{4}\leq\frac{bc}{a(1+bc)}+\frac{ca}{b(1+ca)}+\frac{ab}{c(1+ab)}\leq\frac{a+b+c}{4}$$

$$\forall a,b,c>0\,\middle|\,a+b+c=abc,''=''\text{ iff }a=b=c=\sqrt{3}\text{ (QED)}$$