ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $ab+bc+ca+abc=4$, then prove that :

$$\frac{2-\sqrt{ab}}{\sqrt{c}}+\frac{2-\sqrt{bc}}{\sqrt{a}}+\frac{2-\sqrt{ca}}{\sqrt{b}}\geq 3\sqrt{abc}$$

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Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\left((2+b)(2+c)\right)-(2+a)(2+b)(2+c)=4-(ab+bc+ca+abc)$$

$$=0\ \therefore\sum_{cyc}\frac{1}{2+a}=1\to(m)$$

$$\text{Now,}\ \frac{1}{2+a}\overset{a>0}{<}\frac{1}{2}\ \therefore\ \text{we can set}:\frac{1}{2+a}=\frac{1}{2}-x\ \left(x>0\ \text{and}\ x<\frac{1}{2}\right)$$

$$\therefore a+2=\frac{2}{1-2x}\Rightarrow a=\frac{2}{1-2x}-2=\frac{2x}{\frac{1}{2}-x}\to(1)$$

$$\text{Similarly, we set}:\frac{1}{2+b}=\frac{1}{2}-y\ \text{and}\ \frac{1}{2+c}=\frac{1}{2}-z$$

$$\therefore 1\overset{via\,(m)}{=}\frac{1}{2+a}+\frac{1}{2+b}+\frac{1}{2+c}=\frac{1}{2}-x+\frac{1}{2}-y+\frac{1}{2}-z\Rightarrow x+y+z=\frac{1}{2}\to(i)$$

$$\therefore(1)\ \text{and}\ (i)\Rightarrow a=\frac{2x}{y+z}\ \text{and analogously,}\ b=\frac{2y}{z+x}\ \text{and}\ c=\frac{2z}{x+y}\ \text{and hence}:$$

$$\frac{2-\sqrt{ab}}{\sqrt{c}}+\frac{2-\sqrt{bc}}{\sqrt{a}}+\frac{2-\sqrt{ca}}{\sqrt{b}}\geq 3.\sqrt{abc}\ \text{transforms into}:$$

$$2\sum_{cyc}\sqrt{\frac{x+y}{2z}}-\sum_{cyc}\left(\sqrt{\frac{x+y}{2z}}.\sqrt{\frac{2x}{y+z}.\frac{2y}{z+x}}\right)\geq 3\sqrt{\frac{8xyz}{(y+z)(z+x)(x+y)}}$$

$$\Leftrightarrow\sum_{cyc}\sqrt{\frac{x+y}{z}}-\sqrt{\frac{xyz}{(y+z)(z+x)(x+y)}}.\sum_{cyc}\frac{x+y}{z}\geq 6\sqrt{\frac{xyz}{(y+z)(z+x)(x+y)}}$$

$$\Leftrightarrow\sum_{cyc}\sqrt{\frac{x+y}{z}}\geq\left(6+\sum_{cyc}\frac{x+y}{z}\right).\sqrt{\frac{xyz}{(y+z)(z+x)(x+y)}}$$

$$\Leftrightarrow\sum_{cyc}\frac{x+y}{z}+2\sum_{cyc}\sqrt{\frac{(y+z)(z+x)}{xy}}\geq\frac{xyz}{(y+z)(z+x)(x+y)}.\left(6+\sum_{cyc}\frac{x+y}{z}\right)^2$$

$$\Leftrightarrow\frac{\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)}{xyz}-3+2\sum_{cyc}\sqrt{\frac{(y+z)(z+x)}{xy}}$$

$$\boxed{\overset{(*)}{\geq}}\ \frac{xyz}{(y+z)(z+x)(x+y)}.\left(\frac{\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)}{xyz}+3\right)^2$$

Assigning $y+z=A,z+x=B,x+y=C\Rightarrow A+B-C=2z>0,B+C-A=2x$

> 0 and $C + A - B = 2y > 0 \Rightarrow A + B > C, B + C > A, C + A > B \Rightarrow A, B, C$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc} x = \sum_{cyc} A = 2s \Rightarrow \sum_{cyc} x = s \to (1) \Rightarrow x = s - A, y = s - B, z = s - C$

$$\Rightarrow xyz = r^2 s \to (2) \text{and} \sum_{cyc} xy = \sum_{cyc}(s-A)(s-B) = 4Rr + r^2 \to (3)$$

$$\therefore \text{via } (1),(2),(3),(*) \Leftrightarrow \frac{s(4Rr + r^2)}{r^2 s} - 3 + 2\sum_{cyc}\sqrt{\frac{AB}{(s-A)(s-B)}}$$

$$\geq \frac{r^2 s}{4Rrs}\cdot\left(\frac{s(4Rr+r^2)}{r^2 s} + 3\right)^2 \Leftrightarrow \frac{4R - 2r}{r} + 2\sum_{cyc}\operatorname{cosec}\frac{\alpha}{2} \geq \frac{r^2(4R+4r)^2}{4Rr.r^2}$$

$$(\alpha, \beta, \gamma \to \text{angles of } \Delta \text{ with sides A, B and C}) \Leftrightarrow \frac{2R - r}{r} + \sum_{cyc}\operatorname{cosec}\frac{\alpha}{2} \boxed{\overset{(**)}{\geq}} \frac{2(R+r)^2}{Rr}$$

$$\text{Now, LHS of } (**) \overset{\text{Jensen}}{\geq} \frac{2R - r}{r} + 3\operatorname{cosec}\frac{\pi}{6} = \frac{2R - r}{r} + 6 = \frac{2R + 5r}{r} \overset{?}{\geq} \frac{2(R+r)^2}{Rr}$$

$$\Leftrightarrow 2R^2 + 5Rr \overset{?}{\geq} 2R^2 + 4Rr + 2r^2 \Leftrightarrow Rr \overset{?}{\geq} 2r^2 \to \text{true via Euler} \Rightarrow (**) \Rightarrow (*)$$

$$\text{is true} \therefore \frac{2 - \sqrt{ab}}{\sqrt{c}} + \frac{2 - \sqrt{bc}}{\sqrt{a}} + \frac{2 - \sqrt{ca}}{\sqrt{b}} \geq 3.\sqrt{abc}\ \forall\, a, b, c > 0$$

$$\Big|\, ab + bc + ca + abc = 4, ''='' \text{ iff } a = b = c = 1 \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morroco

By AM – GM inequality, we have

$$(2 + \sqrt{ab})(2 - \sqrt{ab}) = 4 - ab = bc + ca + abc \geq 2c\sqrt{ab} + abc = c\sqrt{ab}(2 + \sqrt{ab})$$

$$\Rightarrow\ 2 - \sqrt{ab} \geq c\sqrt{ab}\ \Rightarrow\ \frac{2 - \sqrt{ab}}{\sqrt{c}} \geq \sqrt{abc}\ \text{(and analogs)}$$

Therefore

$$\frac{2 - \sqrt{ab}}{\sqrt{c}} + \frac{2 - \sqrt{bc}}{\sqrt{a}} + \frac{2 - \sqrt{ca}}{\sqrt{b}} \geq 3\sqrt{abc}.$$

Equality holds iff $a = b = c = 1$.