ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $a^2 + b^2 + c^2 + abc = 4$ then prove that:

$$2(a + b + c) + ab + bc + ca - abc \le 8$$

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By AM − GM inequality, we have

$$4 - a^2 = b^2 + c^2 + abc \ge 2bc + abc = bc(2 + a) \Rightarrow 2 - a \ge bc \overset{a<2}{\Rightarrow} (2 - a - bc)(2 - a) \ge 0$$

$$\Rightarrow 4 \ge 4a + 2bc - a^2 - abc \ \ (\textit{and analogs})$$

Adding this inequality with similar ones, we get

$$12 \ge 4(a + b + c) + 2(ab + bc + ca) - \left(a^2 + b^2 + c^2 + abc\right) - 2abc$$

$$= 4(a + b + c) + 2(ab + bc + ca) - 4 - 2abc$$

$$\Leftrightarrow 8 \ge 2(a + b + c) + ab + bc + ca - abc.$$

Equality holds iff $a = b = c = 1$.