ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > -1$, then prove that:

$$\frac{1+x^2}{1+y+z^2}+\frac{1+y^2}{1+z+x^2}+\frac{1+z^2}{1+x+y^2}\geq 2$$

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$$\text{We have } y\leq\frac{1+y^2}{2} \text{ and analogs } \forall\, y\in\mathbb{R}$$

$$\Rightarrow \frac{1+x^2}{1+y+z^2}+\frac{1+y^2}{1+z+x^2}+\frac{1+z^2}{1+x+y^2}\geq$$

$$\geq \frac{1+x^2}{1+\frac{1+y^2}{2}+z^2}+\frac{1+y^2}{1+\frac{1+z^2}{2}+x^2}+\frac{1+z^2}{1+\frac{1+x^2}{2}+y^2}=2\left(\frac{a}{b+2c}+\frac{b}{c+2a}+\frac{c}{a+2b}\right)$$

$$(a=1+x^2, b=1+y^2, c=1+z^2; a,b,c\geq 1>0)$$

$$=2\left(\frac{a^2}{ab+2ca}+\frac{b^2}{bc+2ab}+\frac{c^2}{ca+2bc}\right)\overset{\text{Bergstrom}}{\geq}\frac{2\left(\sum_{cyc}a\right)^2}{3\sum_{cyc}ab}\geq 2$$

$$\therefore \frac{1+x^2}{1+y+z^2}+\frac{1+y^2}{1+z+x^2}+\frac{1+z^2}{1+x+y^2}\geq 2$$

$$\forall\, x,y,z>-1, ''='' \text{ iff } x=y=z=1 \text{ (QED)}$$