ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $ab+bc+ca+abc=4$, then prove that :

$$\sqrt{a}+\sqrt{b}+\sqrt{c}+2\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right)\le\sqrt{abc}+\frac{8}{\sqrt{abc}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{\text{cyc}}\big((2+b)(2+c)\big)-(2+a)(2+b)(2+c)=$$

$$=4-(ab+bc+ca+abc)=0\ \therefore\ \sum_{\text{cyc}}\frac{1}{2+a}=1\to(\text{m})$$

$$\text{Now, }\frac{1}{2+a}\overset{a>0}{<}\frac{1}{2}\ \therefore\ \text{we can set : }\frac{1}{2+a}=\frac{1}{2}-x\ \left(x>0\text{ and }x<\frac{1}{2}\right)$$

$$\therefore\ a+2=\frac{2}{1-2x}\Rightarrow a=\frac{2}{1-2x}-2=\frac{2x}{\frac{1}{2}-x}\to(1)$$

$$\text{Similarly, we set : }\frac{1}{2+b}=\frac{1}{2}-y\text{ and }\frac{1}{2+c}=\frac{1}{2}-z\ \therefore\ 1\overset{\text{via (m)}}{=}$$

$$\frac{1}{2+a}+\frac{1}{2+b}+\frac{1}{2+c}=\frac{1}{2}-x+\frac{1}{2}-y+\frac{1}{2}-z\Rightarrow x+y+z=\frac{1}{2}\to(\text{i})$$

$$\therefore\ (1)\text{ and (i)}\Rightarrow a=\frac{2x}{y+z}\text{ and analogously, }b=\frac{2y}{z+x}\text{ and }c=\frac{2z}{x+y}\text{ and hence :}$$

$$\sqrt{a}+\sqrt{b}+\sqrt{c}+2\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right)\le\sqrt{abc}+\frac{8}{\sqrt{abc}}\Leftrightarrow\sqrt{abc}.\sum_{\text{cyc}}\sqrt{a}+2\sum_{\text{cyc}}\sqrt{ab}$$

$$\le abc+8\overset{ab+bc+ca+abc=4}{\Leftrightarrow}\Leftrightarrow\sqrt{abc}.\sum_{\text{cyc}}\sqrt{a}+2\sum_{\text{cyc}}\sqrt{ab}\le 4-\sum_{\text{cyc}}ab+8$$

$$\Leftrightarrow\sqrt{abc}.\sum_{\text{cyc}}\sqrt{a}+2\sum_{\text{cyc}}\sqrt{ab}+\sum_{\text{cyc}}ab\le 12$$

$$\Leftrightarrow\sqrt{\frac{xyz}{(y+z)(z+x)(x+y)}}.\sum_{\text{cyc}}\sqrt{\frac{x}{y+z}}+\sum_{\text{cyc}}\sqrt{\frac{xy}{(y+z)(z+x)}}$$

$$+\sum_{\text{cyc}}\frac{xy}{(y+z)(z+x)}\overset{(*)}{\le}3$$

Assigning $y+z=A,\ z+x=B,\ x+y=C\Rightarrow A+B-C=2z>0,\ B+C-A=2x>0$ **and** $C+A-B=2y>0\Rightarrow A+B>C,\ B+C>A,\ C+A>B\Rightarrow A,B,C$ *form* **sides of** a **triangle with semiperimeter, circumradius and inradius** $=s,R,r$ (*say*)

$$\text{yielding }2\sum_{\text{cyc}}x=\sum_{\text{cyc}}A=2s\Rightarrow\sum_{\text{cyc}}x=s\Rightarrow x=s-A,\ y=s-B,\ z=s-C$$

$$\Rightarrow xyz=r^2s\text{ and via such substitutions, LHS of }(*)\text{ becomes :}$$

$$\sqrt{\frac{r^2 s}{\underbrace{16R^2r^2s^2}_{(y+z)^2(z+x)^2(x+y)^2}}}\cdot\sum_{cyc}\sqrt{BC(s-A)}+\sum_{cyc}\sqrt{\frac{(s-A)(s-B)}{AB}}+\sum_{cyc}\frac{(s-A)(s-B)}{AB}$$

$$\overset{CBS}{\leq}\sqrt{\frac{1}{16R^2s}}\cdot\sqrt{\sum_{cyc}(s-A)}\cdot\sqrt{\frac{\sum_{cyc}BC}{\underbrace{}_{s^2+4Rr+r^2}}}+\sum_{cyc}\sin\frac{\alpha}{2}+\sum_{cyc}\sin^2\frac{\alpha}{2}$$

$$(\alpha,\beta,\gamma\to \text{angles of triangle with sides } A, B \text{ and } C)\ \overset{\substack{CBS\\ and\\ Jensen}}{\leq}$$

$$\frac{\sqrt{4R^2+8Rr+4r^2}}{4R}+\frac{3}{2}+\frac{2R-r}{2R}=\frac{R+r+3R+2R-r}{2R}=3\Rightarrow(*)\text{ is true}$$

$$\therefore\sqrt{a}+\sqrt{b}+\sqrt{c}+2\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right)\leq\sqrt{abc}+\frac{8}{\sqrt{abc}}$$

$$\forall\, a,b,c>0 \,\big|\, ab+bc+ca+abc=4,''='' \text{ iff } a=b=c=1 \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality, we have

$$(2+\sqrt{bc})(2-\sqrt{bc})=4-bc=ab+ca+abc\geq 2a\sqrt{bc}+abc=a\sqrt{bc}(2+\sqrt{bc})$$

$$\Rightarrow\ 2-\sqrt{bc}\geq a\sqrt{bc}\ \overset{bc\leq 4}{\Leftrightarrow}\ (2-\sqrt{bc}-a\sqrt{bc})(2-\sqrt{bc})\geq 0$$

$$\Rightarrow 4+bc+abc\geq 2\sqrt{bc}(a+2)\ \Rightarrow\ \frac{4+bc}{2\sqrt{abc}}+\frac{\sqrt{abc}}{2}\geq\sqrt{a}+\frac{2}{\sqrt{a}}\ \ \text{(and analogs)}$$

Adding this inequality with similar ones, we get

$$\sqrt{a}+\sqrt{b}+\sqrt{c}+2\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right)\leq\frac{12+ab+bc+ca}{2\sqrt{abc}}+\frac{3\sqrt{abc}}{2}=$$

$$=\frac{16-abc}{2\sqrt{abc}}+\frac{3\sqrt{abc}}{2}=\sqrt{abc}+\frac{8}{\sqrt{abc}},$$

as desired. Equality holds iff $a=b=c=1$.