ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c\geq 0$ and $a^2+b^2+c^2+abc=4$ then prove that:

$$a+b+c\leq\sqrt{2-a}+\sqrt{2-b}+\sqrt{2-c}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We will prove that :

$$\sqrt{2-a}\geq\frac{b+c}{2}\quad(1)$$

We have

$$(1)\ \Leftrightarrow 4(2-a)\geq b^2+c^2+2bc=4-a^2-abc+2bc=(2-a)(2+a+bc)$$

$$\Leftrightarrow(2-a)(2-a-bc)\geq 0$$

By AM − GM inequality, we have

$$(2-a)(2+a)=4-a^2=b^2+c^2+abc\geq 2bc+abc=bc(2+a)\Rightarrow 2-a\geq bc,$$

then (1) is true. Adding this inequality with similar ones yields the desired inequality.

Equality holds iff $a=b=c=1$.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\text{Firstly, }(4-b^2),(4-c^2)=(a^2+c^2+abc),(a^2+b^2+abc)\overset{a,b,c\geq 0}{\geq}0$$

$$\text{and we have : }a^2+b^2+c^2+abc=4\Rightarrow a^2+a.bc+(b^2+c^2-4)=0$$

$$\Rightarrow a=\frac{-bc\pm\sqrt{b^2c^2-4(b^2+c^2-4)}}{2}=\frac{-bc\pm\sqrt{b^2(c^2-4)-4(c^2-4)}}{2}$$

$$=\frac{-bc\pm\sqrt{(b^2-4)(c^2-4)}}{2}$$

$$\Rightarrow a=\frac{-bc-\sqrt{(4-b^2)(4-c^2)}}{2}\text{ or }a=\frac{-bc+\sqrt{(4-b^2)(4-c^2)}}{2}$$

$$\boxed{\text{When }a=\frac{-bc-\sqrt{(4-b^2)(4-c^2)}}{2}}\text{, then : }b,c\geq 0$$

$$\Rightarrow a=\frac{-bc-\sqrt{(4-b^2)(4-c^2)}}{2}\leq 0\Rightarrow a=0\ (\because a\geq 0)$$

$$\therefore\overbrace{bc}^{\geq 0}+\overbrace{\sqrt{(4-b^2)(4-c^2)}}^{\geq 0}=0\Rightarrow(b=0,c=2)\text{ or }(c=0,b=2)$$

$$\text{and in either case : }\boxed{a+b+c=2}\text{ and }\boxed{\sqrt{2-a}+\sqrt{2-b}+\sqrt{2-c}=\sqrt{2}+\sqrt{2}}$$

$$=2\sqrt{2}\Rightarrow a+b+c<\sqrt{2-a}+\sqrt{2-b}+\sqrt{2-c}$$

$$\boxed{\text{When } a=\frac{-bc+\sqrt{(4-b^2)(4-c^2)}}{2}}, \text{then}: 2-a=$$

$$2-\frac{-bc+\sqrt{(4-b^2)(4-c^2)}}{2}=\frac{8+2bc-2\sqrt{(4-b^2)(4-c^2)}}{4}$$

$$\overset{A-G}{\geq}\frac{8+2bc-\left(8-b^2-c^2\right)}{4}=\frac{b^2+c^2+2bc}{4}=\frac{(b+c)^2}{4}$$

$$\Rightarrow \sqrt{2-a}\geq\frac{b+c}{2} \text{ and analogously}, \sqrt{2-b}\geq\frac{c+a}{2} \text{ and } \sqrt{2-c}\geq\frac{a+b}{2}$$

and summing up, we get : $\sqrt{2-a}+\sqrt{2-b}+\sqrt{2-c}\geq a+b+c \ \forall\, a,b,c\geq 0$,

$'' = ''$ iff $a=b=c=1$ (QED)