ROMANIAN MATHEMATICAL MAGAZINE

**If $a,b,c>0$ and $a^2+b^2+c^2=1$, then prove that :**

$$(1+9abc-a-b-c)\left(\frac{1}{1-ab}+\frac{1}{1-bc}+\frac{1}{1-ca}\right)\le\frac{9}{2}$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution 1 by Soumava Chakraborty-Kolkata-India*

$$\text{If } (1+9abc-a-b-c)\le 0,\text{ then : LHS}\le 0$$

$$\left(\because\sum_{cyc}a^2=1\wedge a,b,c>0\Rightarrow a,b,c<1\Rightarrow(1-ab),(1-bc),(1-ca)>0 \atop \Rightarrow\frac{1}{1-ab}+\frac{1}{1-bc}+\frac{1}{1-ca}>0\right)$$

$$<\frac{9}{2}\text{ and so, we now concentrate on the scenario when : }(1+9abc-a-b-c)$$

$$>0\text{ and then : }(1+9abc-a-b-c)\left(\frac{1}{1-ab}+\frac{1}{1-bc}+\frac{1}{1-ca}\right)$$

$$=(1+9abc-a-b-c)\left(\sum_{cyc}\frac{1}{a^2+b^2+c^2-bc}\right)\le$$

$$(1+9abc-a-b-c)\left(\sum_{cyc}\frac{1}{a^2+\frac{(b+c)^2}{4}}\right)\overset{A-G}{\le}$$

$$(1+9abc-a-b-c)\left(\sum_{cyc}\frac{1}{\frac{1}{2}\left(a+\frac{b+c}{2}\right)^2}\right)$$

$$=8(1+9abc-a-b-c).\sum_{cyc}\frac{1}{\left((c+a)+(a+b)\right)^2}\overset{?}{\le}\frac{9}{2}\overset{a^2+b^2+c^2=1}{\Leftrightarrow}$$

$$8\sum_{cyc}\frac{1}{\left((c+a)+(a+b)\right)^2}\boxed{\overset{?}{\underset{(*)}{\le}}}$$

$$8\left(\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2\right)-9abc\right).\sum_{cyc}\frac{1}{\left((c+a)+(a+b)\right)^2}+\frac{9}{2}$$

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a$
$>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y\Rightarrow x,y,z$ form
sides of $a$ triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say);

$$\text{so }2\sum_{cyc}a=\sum_{cyc}x=2s\Rightarrow\sum_{cyc}a=s\to(1)\Rightarrow a=s-x, b=s-y, c=s-z$$

$$\Rightarrow abc=r^2s\to(2)\text{ and such substitutions}\Rightarrow\sum_{cyc}ab=\sum_{cyc}(s-x)(s-y)$$

$$\Rightarrow\sum_{cyc}ab=4Rr+r^2\to(2)\text{ and }\sum_{cyc}a^2=\left(\sum_{cyc}a\right)^2-2\sum_{cyc}ab\overset{\text{via (1) and (2)}}{=}$$

$$s^2-2(4Rr+r^2)\Rightarrow\sum_{cyc}a^2=s^2-8Rr-2r^2\to(3)$$

Now, via (1), (2) and (3), $8\left(\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2\right)-9abc\right).\sum_{cyc}\frac{1}{\left((c+a)+(a+b)\right)^2}$

$$=8\left(s(s^2-8Rr-2r^2)-9r^2s\right).\sum_{cyc}\frac{1}{(y+z)^2}\geq(s^2-8Rr-11r^2).\sum_{cyc}\frac{1}{(y+z)^2}$$

$$\left(\begin{array}{c}\because s^2-8Rr-11r^2=s^2-16Rr+5r^2+8r(R-2r)\overset{\substack{\text{Gerretsen}\\ and\\ \text{Euler}}}{\geq}0\ and\\ s=\sum_{cyc}a=\sqrt{\sum_{cyc}a^2+2\sum_{cyc}ab}>\sqrt{\sum_{cyc}a^2}\overset{a^2+b^2+c^2=1}{=}1\end{array}\right)$$

$$=8(s^2-8Rr-11r^2).\left(\left(\sum_{cyc}\frac{1}{y+z}\right)^2-2\sum_{cyc}\frac{1}{(y+z)(z+x)}\right)$$

$$=8(s^2-8Rr-11r^2).\left(\frac{(5s^2+4Rr+r^2)^2}{4s^2(s^2+2Rr+r^2)^2}-\frac{2.4s}{2s(s^2+2Rr+r^2)}\right)$$

$$=8(s^2-8Rr-11r^2).\left(\frac{(5s^2+4Rr+r^2)^2-16s^2(s^2+2Rr+r^2)}{4s^2(s^2+2Rr+r^2)^2}\right)$$

$$\therefore \text{RHS of }(*)-\text{LHS of }(*)\overset{a^2+b^2+c^2=1}{\geq}$$

$$\frac{9}{2}+8(s^2-8Rr-11r^2).\left(\frac{(5s^2+4Rr+r^2)^2-16s^2(s^2+2Rr+r^2)}{4s^2(s^2+2Rr+r^2)^2}\right)$$

$$-8\left(\frac{(5s^2+4Rr+r^2)^2-16s^2(s^2+2Rr+r^2)}{4s^2(s^2+2Rr+r^2)^2}\right)\left(\sum_{cyc}a^2\right)\overset{\text{via }(3)}{=}$$

$$\frac{9}{2}+8\left(s^2-8Rr-11r^2-(s^2-8Rr-2r^2)\right).\left(\frac{(5s^2+4Rr+r^2)^2-16s^2(s^2+2Rr+r^2)}{4s^2(s^2+2Rr+r^2)^2}\right)$$

$$\overset{?}{\geq}0$$

$$\Leftrightarrow\frac{1}{2}\overset{?}{\geq}\frac{2r^2\left((5s^2+4Rr+r^2)^2-16s^2(s^2+2Rr+r^2)\right)}{s^2(s^2+2Rr+r^2)^2}$$

$$\Leftrightarrow s^6+(4Rr-34r^2)s^4+r^2s^2(4R^2-28Rr+25r^2)-4r^4(4R+r)^2\ \boxed{\underset{(**)}{\overset{?}{\geq}}}\ 0$$

$$\text{Now, LHS of }(**)\overset{\text{Gerretsen}}{\geq}(20Rr-39r^2)s^4+r^2s^2(4R^2-28Rr+25r^2)$$

$$-4r^4(4R+r)^2\overset{\text{Gerretsen}}{\geq}\left((20Rr-39r^2)(16Rr-5r^2)+r^2(4R^2-28Rr+25r^2)\right)s^2$$

$$-4r^4(4R+r)^2=4r^2\left((81R^2-188Rr+55r^2)s^2-r^2(4R+r)^2\right)\overset{\text{Gerretsen}}{\geq}$$

$$4r^2\left((81R^2-188Rr+55r^2)(16Rr-5r^2)-r^2(4R+r)^2\right)$$

$$\left(\because 81R^2-188Rr+55r^2=(R-2r)(81R-26r)+3r^2\overset{\text{Euler}}{\geq}3r^2>0\right)\overset{?}{\geq}0$$

$$\Leftrightarrow 432t^3 - 1143t^2 + 604t - 92 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)\left(432t^2 - 279t + 46\right) \overset{?}{\geq} 0$$

$$\to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore (1 + 9abc - a - b - c)\left(\frac{1}{1-ab} + \frac{1}{1-bc} + \frac{1}{1-ca}\right) \leq \frac{9}{2}$$

$$\forall\, a, b, c > 0 \,\middle|\, a^2 + b^2 + c^2 = 1, '' ='' \text{ iff } a = b = c = \frac{1}{\sqrt{3}} \text{ (QED)}$$

## Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality, we have

$$a + b + c = (a^2 + b^2 + c^2)(a + b + c) \geq 3\sqrt[3]{a^2b^2c^2}.3\sqrt[3]{abc} = 9abc$$

$$\Rightarrow\ 1 + 9abc - a - b - c \leq 1.$$

Now, let $p := a + b + c, q := ab + bc + ca \leq 1, r := abc.$ We have $p^2 = 1 + 2q.$

By Schur's inequality, we have

$$r \geq \frac{p(4q - p^2)}{9} = \frac{(1+2q)(2q-1)}{p.9} = \frac{4q^2 - 1}{9p} \quad (1)$$

$$\sum_{cyc} \frac{1}{1 - bc} = \sum_{cyc}\left(2 - \frac{1 - 2bc}{1 - bc}\right)$$

$$= 6 - \sum_{cyc} \frac{1 - 2bc}{1 - bc} \overset{CBS\ \&\ 1 \geq 2bc}{\leq} 6 - \frac{\left(\sum_{cyc}(1 - 2bc)\right)^2}{\sum_{cyc}(1 - 2bc)(1 - bc)}$$

$$= 6 - \frac{(3 - 2q)^2}{3 - 3q + 2(q^2 - 2pr)} \overset{(1)}{\leq} 6 - \frac{(3 - 2q)^2}{3 - 3q + 2q^2 - \frac{4(4q^2 - 1)}{9}} = \frac{3(35 - 18q - 8q^2)}{31 - 27q + 2q^2}$$

$$= \frac{9}{2} - \frac{3(1 - q)(23 - 22q)}{31 - 27q + 2q^2} \overset{q \leq 1}{\leq} \frac{9}{2}.$$

Therefore

$$(1 + 9abc - a - b - c)\left(\frac{1}{1 - ab} + \frac{1}{1 - bc} + \frac{1}{1 - ca}\right) \leq 1.\frac{9}{2} = \frac{9}{2}.$$

Equality holds iff $a = b = c = \dfrac{\sqrt{3}}{3}.$