If $a,b,c>0$ and $a+b+c=3$, then prove that :

$$\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+5\ge(a+b)(b+c)(c+a)$$

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Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sqrt[3]{a.1.1}\overset{\text{GM-HM}}{\ge}\frac{3a}{2a+1}\text{ and analogs}\Rightarrow\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+5\ge$$

$$\sum_{\text{cyc}}\frac{3a^2}{2a^2+a}+5\overset{\text{Bergstrom}}{\ge}\frac{3\left(\sum_{\text{cyc}}a\right)^2}{2\sum_{\text{cyc}}a^2+\sum_{\text{cyc}}a}+5\overset{a+b+c=3}{=}$$

$$\frac{\frac{3}{27}\left(\sum_{\text{cyc}}a\right)^5}{2\sum_{\text{cyc}}a^2+\frac{\left(\sum_{\text{cyc}}a\right)^2}{3}}+\frac{5}{27}.\left(\sum_{\text{cyc}}a\right)^3=\frac{\left(\sum_{\text{cyc}}a\right)^5}{18\sum_{\text{cyc}}a^2+3\left(\sum_{\text{cyc}}a\right)^2}+\frac{5}{27}.\left(\sum_{\text{cyc}}a\right)^3$$

$$\overset{?}{\ge}(a+b)(b+c)(c+a)$$

$$\Leftrightarrow\frac{27\left(\sum_{\text{cyc}}a\right)^5}{18\sum_{\text{cyc}}a^2+3\left(\sum_{\text{cyc}}a\right)^2}+5\left(\sum_{\text{cyc}}a\right)^3\boxed{\underset{(*)}{\overset{?}{\ge}}}27(a+b)(b+c)(c+a)$$

Assigning $b+c=x,c+a=y,a+b=z\Rightarrow x+y-z=2c>0,y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z,y+z>x,z+x>y\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say);

$$\text{so }2\sum_{\text{cyc}}a=\sum_{\text{cyc}}x=2s\Rightarrow\sum_{\text{cyc}}a=s\to(1)\Rightarrow a=s-x,b=s-y,c=s-z\text{ and}$$

$$\text{such substitutions}\Rightarrow\sum_{\text{cyc}}ab=\sum_{\text{cyc}}(s-x)(s-y)\Rightarrow\sum_{\text{cyc}}ab=4Rr+r^2\to(2)\text{ and}$$

$$\sum_{\text{cyc}}a^2=\left(\sum_{\text{cyc}}a\right)^2-2\sum_{\text{cyc}}ab\overset{\text{via (1) and (2)}}{=}s^2-2\left(4Rr+r^2\right)$$

$$\Rightarrow\sum_{\text{cyc}}a^2=s^2-8Rr-2r^2\to(3)$$

$$\text{Now, via (1) and (3), }(*)\Leftrightarrow\frac{27s^5}{18(s^2-8Rr-2r^2)+3s^2}+5s^3\ge27.4Rrs$$

$$\Leftrightarrow11s^4-\left(249Rr+15r^2\right)s^2+324Rr^2(4R+r)\boxed{\overset{(**)}{\ge}}0\text{ and }\because$$

$$11\left(s^2-16Rr+5r^2\right)^2\overset{\text{Gerretsen}}{\ge}0,\therefore\text{ in order to prove }(**)\text{, it suffices to prove :}$$

$$\text{LHS of }(**)\ge11\left(s^2-16Rr+5r^2\right)^2$$

$$\Leftrightarrow(103R-125r)s^2\boxed{\overset{(***)}{\ge}}r\left(1520R^2-2084Rr+275r^2\right)$$

$$\text{Again, }(103R-125r)s^2\overset{\text{Gerretsen}}{\ge}(103R-125r)\left(16Rr-5r^2\right)\overset{?}{\ge}$$

$$r\left(1520R^2-2084Rr+275r^2\right)\Leftrightarrow128R^2-431Rr+350r^2\overset{?}{\ge}0$$

$$\Leftrightarrow (\mathrm{R}-2\mathrm{r})(128\mathrm{R}-175\mathrm{r}) \overset{?}{\geq} 0 \rightarrow \text{true} \because \mathrm{R} \overset{\text{Euler}}{\geq} 2\mathrm{r} \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore \sqrt[3]{a}+\sqrt[3]{\mathrm{b}}+\sqrt[3]{\mathrm{c}}+5 \geq (a+\mathrm{b})(\mathrm{b}+\mathrm{c})(\mathrm{c}+a)$$

$$\forall\, a,\mathrm{b},\mathrm{c}>0 \,\big|\, a+\mathrm{b}+\mathrm{c}=3, ''='' \text{ iff } a=\mathrm{b}=\mathrm{c}=1 \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality, we have

$$a^3+3\sqrt[3]{a} \geq 4\sqrt[4]{a^3.\left(\sqrt[3]{a}\right)^3} = 4a \quad \Rightarrow \quad \sqrt[3]{a} \geq \frac{4a-a^3}{3} \quad \text{(and analogs)}$$

Then

$$\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+5 \geq \frac{4(a+b+c)-(a^3+b^3+c^3)}{3}+5$$

$$=\frac{4.3-(a+b+c)^3+3(a+b)(b+c)(c+a)}{3}+5$$

$$=(a+b)(b+c)(c+a),$$

Equality holds iff $a=b=c=1$.