ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b}, \mathrm{c} \geq 0$, then prove that :

$$(a+\mathrm{bc})^2+(\mathrm{b}+\mathrm{c}a)^2+(\mathrm{c}+a\mathrm{b})^2 \geq \sqrt{2}(a+\mathrm{b})(\mathrm{b}+\mathrm{c})(\mathrm{c}+a)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$\boxed{\text{Case 1}}$ Exactly 2 variables equal to zero and WLOG we may assume :
$\mathrm{b}=\mathrm{c}=0$ and then : $\mathrm{LHS}-\mathrm{RHS}=a^2>0$

$\boxed{\text{Case 2}}$ Exactly 1 variable equals to 0 and WLOG we may assume
$a=0$ and then : $\mathrm{LHS}=\mathrm{b}^2\mathrm{c}^2+\mathrm{b}^2+\mathrm{c}^2=\left(\frac{\mathrm{b}^2\mathrm{c}^2}{2}+\mathrm{b}^2\right)+\left(\frac{\mathrm{b}^2\mathrm{c}^2}{2}+\mathrm{c}^2\right)$

$$\overset{A-G}{\geq} 2\mathrm{b}^2.\sqrt{\frac{\mathrm{c}^2}{2}}+2\mathrm{c}^2.\sqrt{\frac{\mathrm{b}^2}{2}}=\sqrt{2}\mathrm{bc}(\mathrm{b}+\mathrm{c})=\mathrm{RHS} \Rightarrow \mathrm{LHS} \geq \mathrm{RHS},$$

$''=''$ iff $\left(a=0, \mathrm{b}=\mathrm{c}=\sqrt{2}\right)$ and permutations

$\boxed{\text{Case 3}}$ $a, \mathrm{b}, \mathrm{c}>0$ and assigning $\mathrm{b}+\mathrm{c}=x, \mathrm{c}+a=\mathrm{y}, a+\mathrm{b}=\mathrm{z} \Rightarrow x+\mathrm{y}-\mathrm{z}=2\mathrm{c}>0, y+z-x=2a>0$ and $\mathrm{z}+x-\mathrm{y}=2\mathrm{b}>0 \Rightarrow x+\mathrm{y}>z, \mathrm{y}+\mathrm{z}>x, \mathrm{z}+x>y$ $\Rightarrow x, \mathrm{y}, \mathrm{z}$ form sides of a triangle with semiperimeter, circumradius and inradius $=\mathrm{s}, \mathrm{R}, \mathrm{r}$ (say); so $2\sum_{\text{cyc}} a=\sum_{\text{cyc}} x=2\mathrm{s} \Rightarrow \sum_{\text{cyc}} a=\mathrm{s} \rightarrow (1) \Rightarrow a=\mathrm{s}-x, \mathrm{b}=\mathrm{s}-\mathrm{y}$,

$\mathrm{c}=\mathrm{s}-\mathrm{z} \therefore a\mathrm{bc}=\mathrm{r}^2\mathrm{s} \rightarrow (2)$ and such substitutions $\Rightarrow \sum_{\text{cyc}} a\mathrm{b}=\sum_{\text{cyc}}(\mathrm{s}-x)(\mathrm{s}-\mathrm{y})$

$$\Rightarrow \sum_{\text{cyc}} a\mathrm{b}=4\mathrm{Rr}+\mathrm{r}^2 \rightarrow (3) \text{ and } \sum_{\text{cyc}} a^2=\left(\sum_{\text{cyc}} a\right)^2-2\sum_{\text{cyc}} a\mathrm{b} \overset{\text{via (1) and (3)}}{=}$$

$$\mathrm{s}^2-2(4\mathrm{Rr}+\mathrm{r}^2) \Rightarrow \sum_{\text{cyc}} a^2=\mathrm{s}^2-8\mathrm{Rr}-2\mathrm{r}^2 \rightarrow (4) \text{ and also,}$$

$$\sum_{\text{cyc}} a^2\mathrm{b}^2=\left(\sum_{\text{cyc}} a\mathrm{b}\right)^2-2a\mathrm{bc}\left(\sum_{\text{cyc}} a\right) \overset{\text{via (1),(2) and (3)}}{=} (4\mathrm{Rr}+\mathrm{r}^2)^2-2\mathrm{r}^2\mathrm{s}.\mathrm{s}$$

$$\Rightarrow \sum_{\text{cyc}} a^2\mathrm{b}^2=\mathrm{r}^2((4\mathrm{R}+\mathrm{r})^2-2\mathrm{s}^2) \rightarrow (5)$$

$$\text{Now, } (a+\mathrm{bc})^2+(\mathrm{b}+\mathrm{c}a)^2+(\mathrm{c}+a\mathrm{b})^2=\sum_{\text{cyc}} a^2+\sum_{\text{cyc}} a^2\mathrm{b}^2+6a\mathrm{bc}$$

$$\overset{A-G}{\geq} 2\sqrt{\left(\sum_{\text{cyc}} a^2\right)\left(\sum_{\text{cyc}} a^2\mathrm{b}^2\right)}+6a\mathrm{bc} \overset{?}{>} \sqrt{2}(a+\mathrm{b})(\mathrm{b}+\mathrm{c})(\mathrm{c}+a)$$

$$\Leftrightarrow 4\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right) + 36a^2b^2c^2 + 24abc.\sqrt{\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right)}$$

$$\overset{?}{>} 2(a+b)^2(b+c)^2(c+a)^2 \overset{\text{via (2),(4) and (5)}}{\Leftrightarrow}$$

$$12r^2s.\sqrt{r^2(s^2-8Rr-2r^2)((4R+r)^2-2s^2)} \overset{?}{>}$$

$$16R^2r^2s^2 - 18r^4s^2 - 2r^2(s^2-8Rr-2r^2)((4R+r)^2-2s^2) \Leftrightarrow$$

$$3s.\sqrt{r^2(s^2-8Rr-2r^2)((4R+r)^2-2s^2)} \boxed{\underset{(*)}{\underbrace{\overset{?}{>}}}}$$

$s^4 - s^2(4R^2+12Rr+7r^2) + r(4R+r)^3$ *and* $\because$ LHS of $(*) > 0 \therefore$ when : RHS of $(*) \leq 0$, then : $(*)$ is true *and* so, we now focus on the scen*a*rio when : $s^4 - s^2(4R^2+12Rr+7r^2) + r(4R+r)^3 > 0$ *and* then : $(*) \Leftrightarrow$

$$9r^2s^2(s^2-8Rr-2r^2)((4R+r)^2-2s^2) \overset{?}{>}$$

$$(s^4 - s^2(4R^2+12Rr+7r^2) + r(4R+r)^3)^2$$

$$\Leftrightarrow (s^4 - s^2(4R^2+12Rr+7r^2) + r(4R+r)^3)^2 -$$

$$9r^2s^2(s^2-8Rr-2r^2)((4R+r)^2-2s^2) \boxed{\underset{(**)}{\underbrace{\overset{?}{<}}}} 0$$

Now, Rouche $\Rightarrow s^2 - (m-n) \geq 0$ and $s^2 - (m+n) \leq 0$, where $m = 2R^2 + 10Rr - r^2$ and $n = 2(R-2r).\sqrt{R^2-2Rr}$

$$\therefore \left(s^2-(m+n)\right)\left(s^2-(m-n)\right) \leq 0 \Rightarrow s^4 - s^2(2m) + m^2 - n^2 \leq 0$$

$$\Rightarrow s^4 - s^2(4R^2+20Rr-2r^2) + r(4R+r)^3 \overset{(i)}{\leq} 0 \Rightarrow$$

$$(s^4 - s^2(4R^2+20Rr-2r^2) + r(4R+r)^3)\begin{pmatrix} s^4 - s^2(4R^2+12Rr+7r^2) \\ +r(4R+r)^3 \end{pmatrix} \leq 0$$

$\therefore$ in order to prove $(**)$, it suffices to prove : LHS of $(**) <$

$$(s^4 - s^2(4R^2+20Rr-2r^2) + r(4R+r)^3)\begin{pmatrix} s^4 - s^2(4R^2+12Rr+7r^2) \\ +r(4R+r)^3 \end{pmatrix}$$

$$\Leftrightarrow (8R+9r)s^4 - (32R^3+204R^2r+164Rr^2-18r^3)s^2$$

$$+r(512R^4+960R^3r+528R^2r^2+116Rr^3+9r^4) \boxed{\overset{(***)}{<}} 0 \text{ and via (i)},$$

$(8R+9r)(s^4 - s^2(4R^2+20Rr-2r^2) + r(4R+r)^3) \leq 0 \therefore$ in order to prove $(***)$, it suffices to prove : LHS of $(***) <$ $(8R+9r)(s^4 - s^2(4R^2+20Rr-2r^2) + r(4R+r)^3) \Leftrightarrow -8R^2rs^2 < 0 \rightarrow$ true

$\therefore (***) \Rightarrow (**) \Rightarrow (*)$ is true $\therefore (a+bc)^2 + (b+ca)^2 + (c+ab)^2 > \sqrt{2}(a+b)(b+c)(c+a)$ *and* so, combining ***all*** cases,

$(a+bc)^2 + (b+ca)^2 + (c+ab)^2 \geq \sqrt{2}(a+b)(b+c)(c+a)\ \forall\, a,b,c \geq 0$, $''=''$ iff $\left(a=0, b=c=\sqrt{2}\right)$ or $\left(b=0, c=a=\sqrt{2}\right)$ or $\left(c=0, a=b=\sqrt{2}\right)$ (QED)