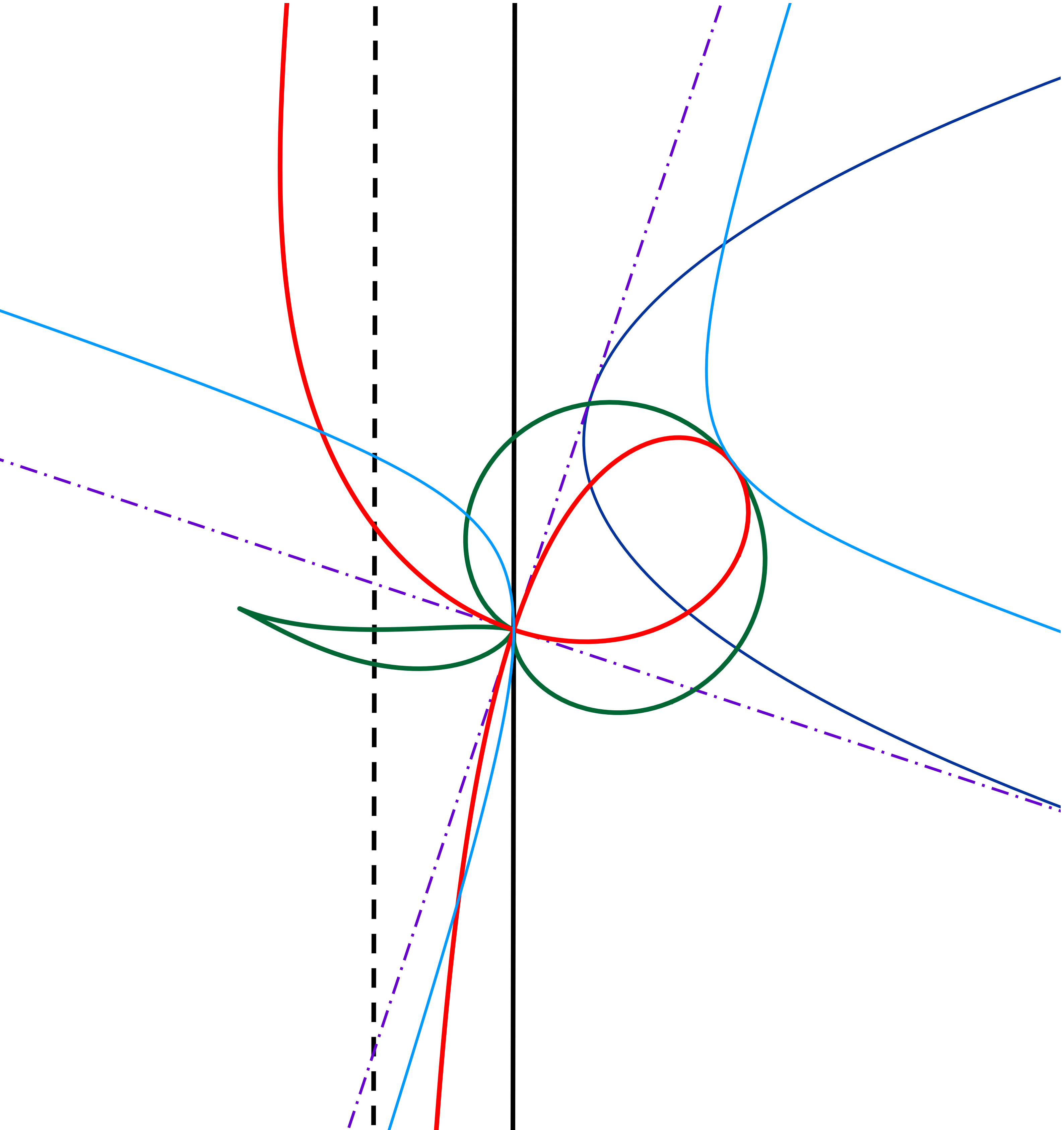
Exploring properties of strophoids through classical geometry

Bohdan Zhelyabovskyy



Abstract

Strophoids are a family of cubic plane curves, which are usually investigated using algebraic methods. This article is unique in that it takes a purely geometric approach to the analysis of strophoids. In this article, strophoids are shown to arise from a range of geometric constructions, each providing a distinct way of exploring the strophoids' properties.

The article takes place solely in the Euclidean plane.

Key words: Strophoid, inversion, bisector hyperbola, cyclinear conjugate, pedal curve, inscribed conics.

1. Primary geometric definition of a strophoid

Let there be points F, D; with a line $\mathscr{L}$ passing through D. There is a circle ω passing through D, with center P lying on $\mathscr{L}$. Line FP intersects ω in points N and M. The strophoid is the trace of points: N, M; as P slides on $\mathscr{L}$.

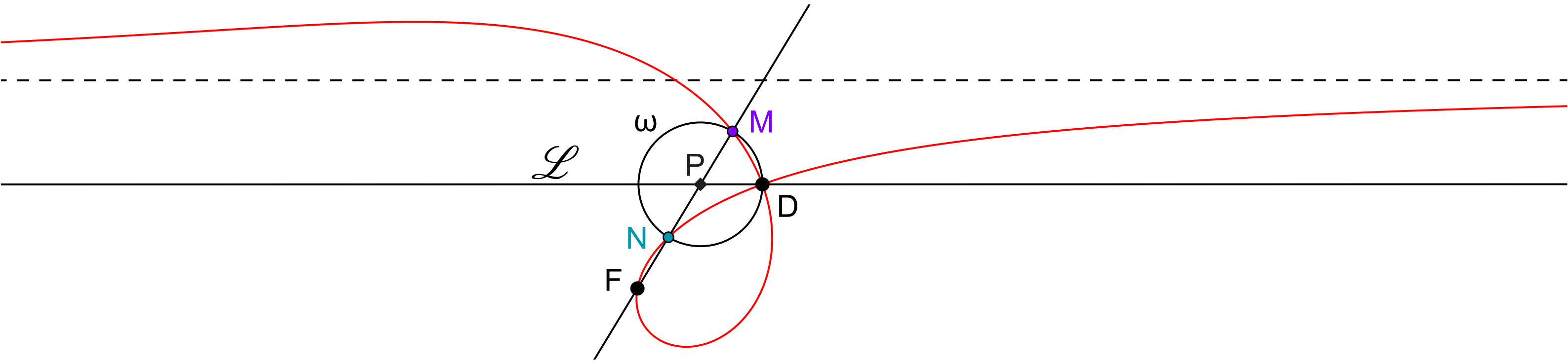


Diagram 1

Defining the features of the strophoid:

1) The **node** or the **double-point** of the strophoid is point **D**.
2) The **focus** is point **F**.
3) The axis of the strophoid is a line ($\mathscr{L}$) through the node that is parallel to the asymptote.

A strophoid is called a right strophoid if it has a line of symmetry $\Longrightarrow FD \perp \mathscr{L}$

2. Isogonal conjugate

The isogonal conjugate of P, P* or I(P), about some triangle ABC is constructed by reflecting lines AP, BP, CP, about the corresponding angle bisector, and finding the intersection point.

It is known that the isogonal conjugate of every point of a line is some conic that is passing through every vertex of the triangle (a circumconic).

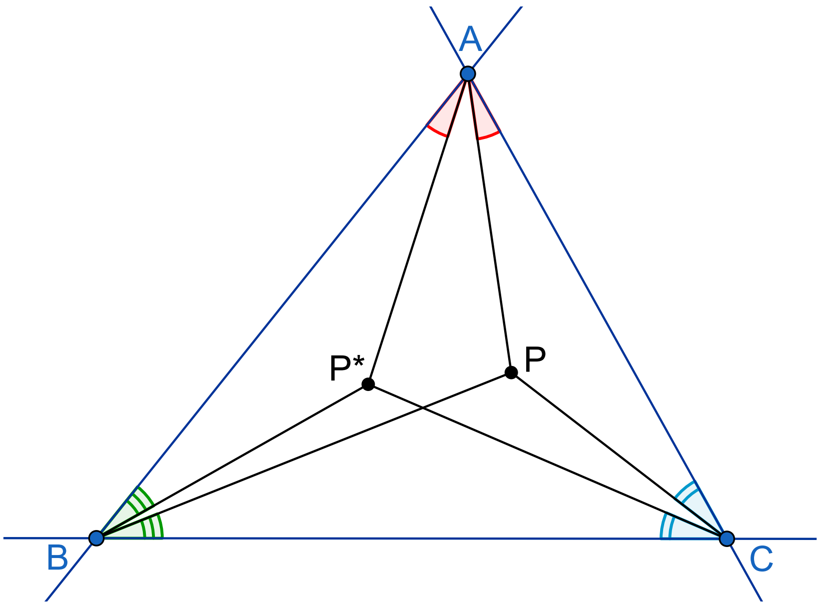


Diagram 2

3. Strophoids as inverses

<u>Statement</u>: A strophoid inverted with the center of inversion at the node is a rectangular hyperbola.

Continuing with the construction in the primary definition, let h be the perpendicular bisector of **DF**, Ω be a circle with center **D** passing through **F**, and (ψ)' denote the inversion of an object ψ in Ω.

ω' becomes a line perpendicular to ℒ, (PF)' passes through M', N', **F, D**. Since points D and F are fixed the center of (PF)', O, will move along h, as P moves (diagram 3.1).

Now the inverse of the strophoid can be redefined as loci of N', M'; as L moves on h (diagram 3.2).

(PF)' is a circle through **F,** L, **D**; ω' is a line through O at a constant angle 2φ with h ⟹ LN' makes a constant angle φ with h.

Let A be a point such that the angle bisector of **FAD** is parallel to LN' ⟹ ∠N'**F**A=∠N'**D**A=δ.

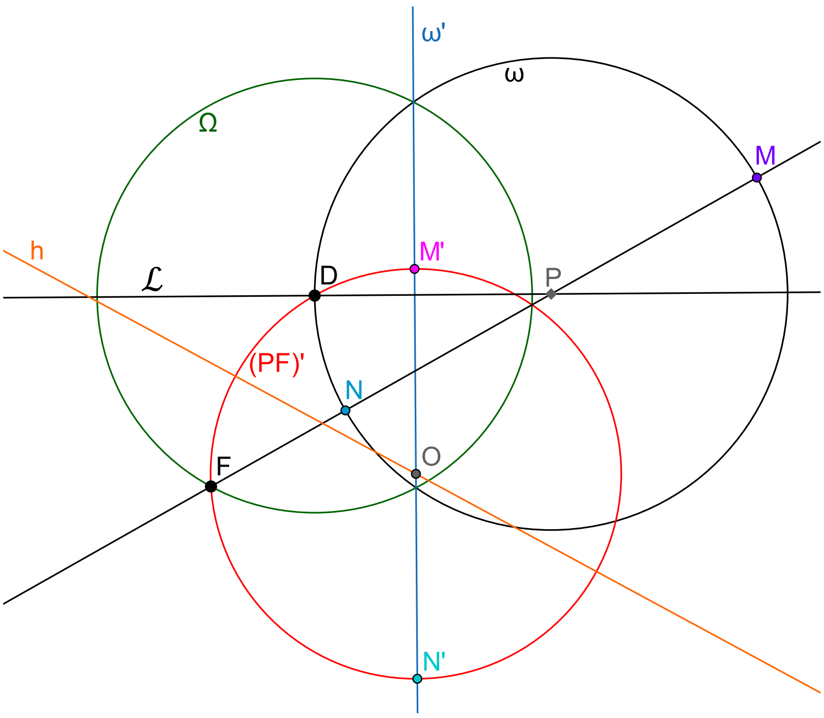


Diagram 3.1

Isogonal conjugate N' with respect to A**D**F, N'*, therefore, forms angles ∠N'***FD**=∠N'***DF**=δ ⟹ N'* lies on h. Since the locus of N'* is a line, the locus of N' is some circumconic[1] (in pink). This conic is known to be a rectangular hyperbola because it passes through the orthocenter of A**D**F (I(O_{ADF})).

Thus, the inversion of the strophoid at its node is a rectangular hyperbola.

[1] Such conic, that is the locus of points T such that the angle bisector of DTF is parallel to the one at DAF, will be called the bisector hyperbola of △DAF (notice the symmetry of the notation).

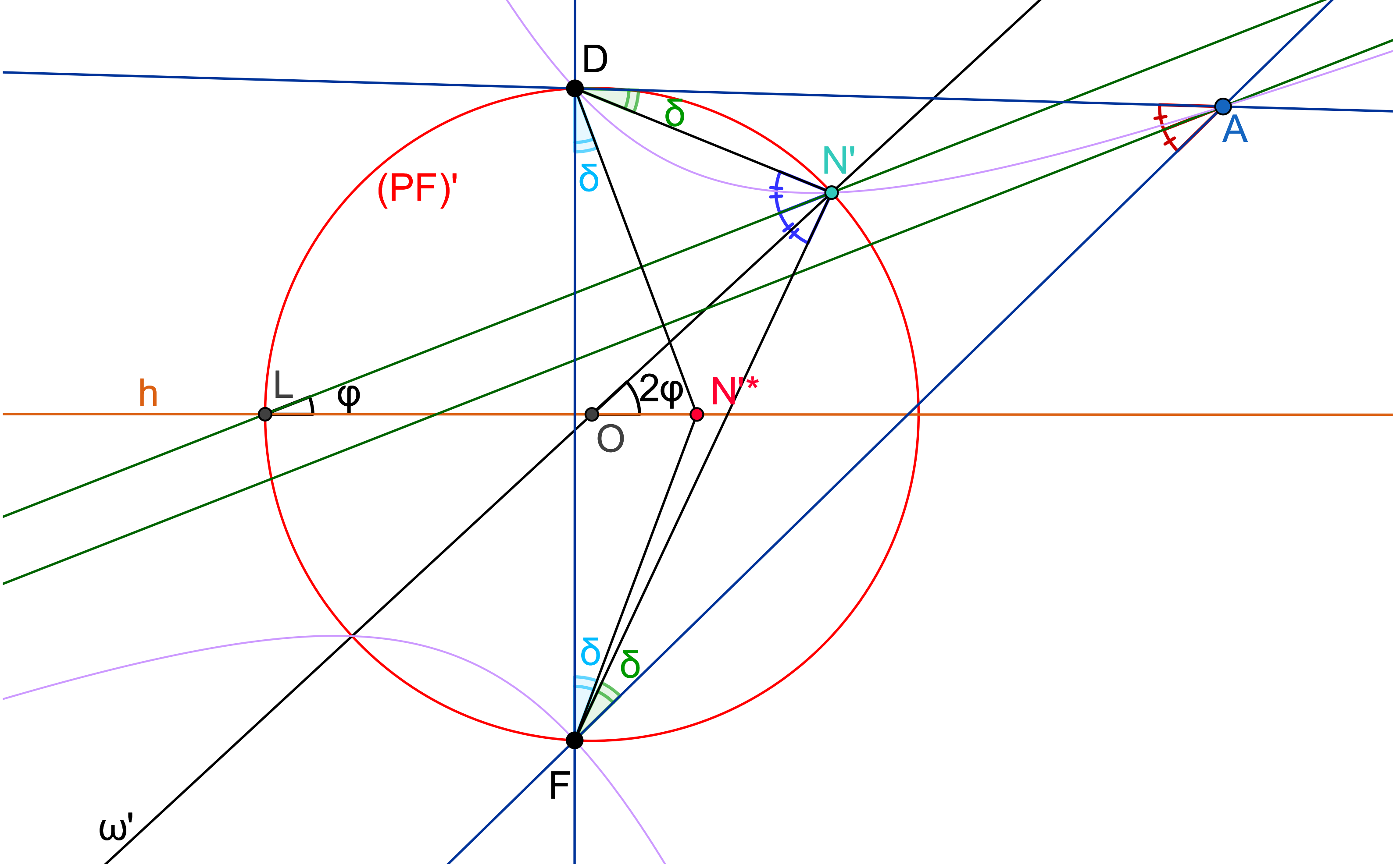


Diagram 3.2

The fact that strophoids are inversions of a second degree curve implies that a circle can have at most 4 real intersections with a strophoid.

4. Cyclinear conjugate

The cyclinear conjugate of point P with respect to △BAC, P° or $\Theta_A(P)$, is defined as the point cyclic with P, B, C; and collinear with A, P.

It is easy to see that this transformation is also involutory, like the isogonal transformation.

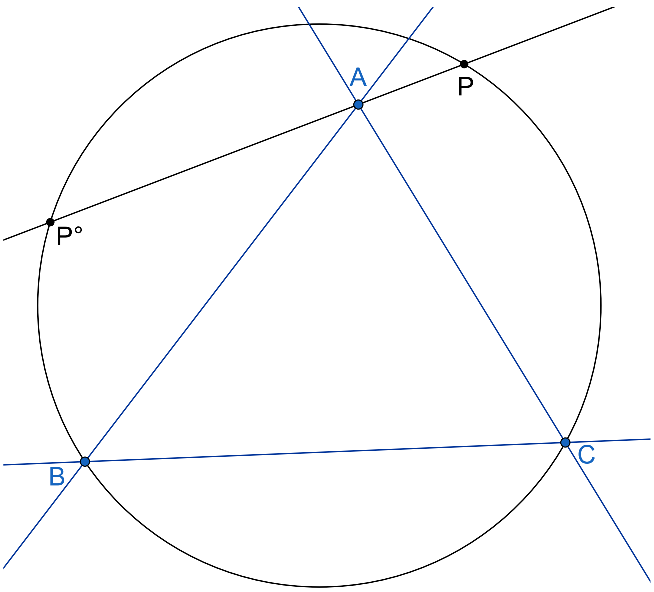


Diagram 4.1

We now will go on to show that isogonal and cyclinear transformations are commutative:

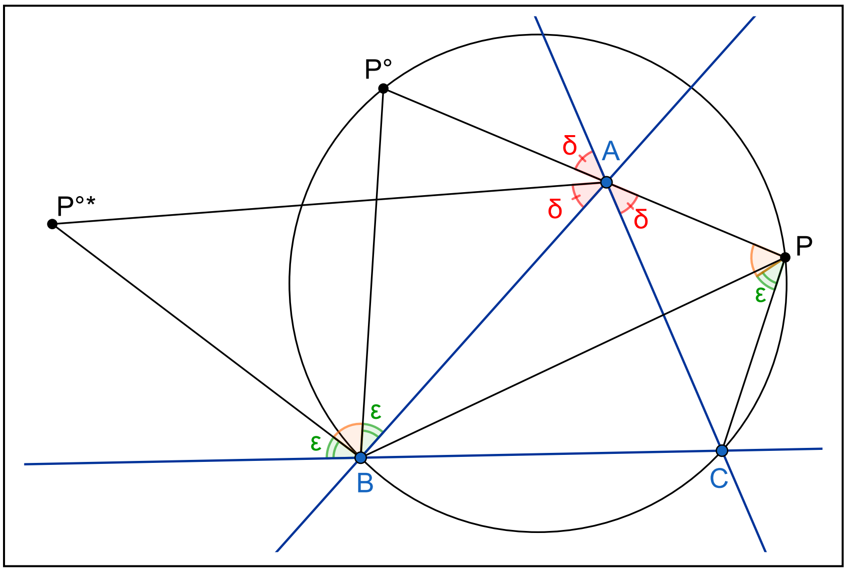


Diagram 4.2

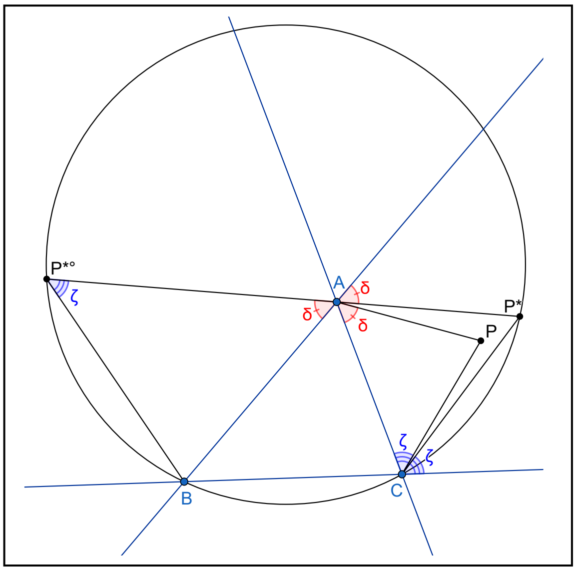


Diagram 4.3

Let $\angle BAC, \angle PAC = \alpha, \delta$ respectively
Then:

$$\left.\begin{array}{c}\angle PAC = \angle BAP^{\circ *} = BAP^{* \circ} \\ \angle P^{* \circ}BA = \angle CPA \\ \angle P^{\circ *}BA = \angle CPA\end{array}\right\} \Longrightarrow \left.\begin{array}{r}\begin{cases}\triangle \mathrm{P}^{* \circ}\mathrm{BA} \sim \triangle \mathrm{CAP} \\ \triangle \mathrm{P}^{\circ *}\mathrm{BA} \sim \triangle \mathrm{CAP}\end{cases} \\ \angle \mathrm{PAP}^{\circ *} = \alpha + 2\delta = \angle \mathrm{PAP}^{* \circ}\end{array}\right\} \Longrightarrow \mathrm{P}^{* \circ} \equiv \mathrm{P}^{\circ *}$$

Therefore, we define a new transformation:

$$R_A(X) = I\big(\Theta_A(X)\big) = \Theta_A\big(I(X)\big)$$

Using the aforementioned triangle similarity, we can find that:

$$\frac{|AP^{\circ *}|}{|AB|} = \frac{|AC|}{|AP|}$$

$$|AP^{\circ *}| \cdot |AP| = |AB| \cdot |AC|$$

Therefore $R_A(P)$ is equivalent to reflecting the point P about the angle bisector of BAC, and inverting it about the circle with radius $\sqrt{|AB| \cdot |AC|}$, center A.

One can analogously prove that: $\Theta_B(\Theta_C(X)) = \Theta_C(\Theta_B(X)) = R_A(X)$

Since all the transformations are commutative and associative (as well as involutory) notation similar to the one in group theory will be used.

From the two observations made earlier the following list of properties can be deduced:

$I\Theta_A(X) = R_A(X)$	$R_A(B) = C$
$R_A(X) = \Theta_B\Theta_C(X)$	$R_A(\boldsymbol{a}) = \circ\,(ABC)$
$R_AR_BR_C(X) = X$	$R_A(\mathcal{J}) = \mathcal{J}_A$
$\Theta_A\Theta_B\Theta_C(X) = I(X)$	$R_A(\mathcal{J}_B) = \mathcal{J}_C$

Where $\boldsymbol{a}$ *is the side BC,* $\mathcal{J}$ *is the incenter,* $\mathcal{J}_A$ *is the excentre of A.*

The set of points which are invariant under cyclinear transformation is a curve known as a focal cubic, that we will call $\mathcal{f}$. A point on $\mathcal{f}$ is constructed by drawing a tangent from A to some circle through B,C; and finding the point of tangency. It is easy to see that if some point P is on $\mathcal{f}$, then $R_A(P)$ remains on $\mathcal{f}$. This means that $\mathcal{f}$ is also self-isogonal.

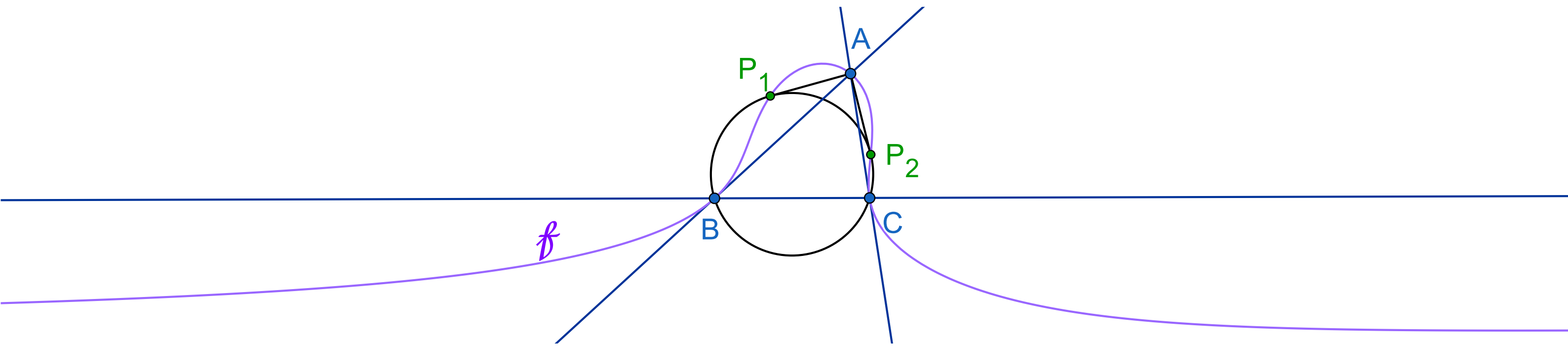


Diagram 4.4

A particularly interesting property of curve $\mathcal{f}$, is that when points B and C are «glued» together (B becomes equal to C, preserving the direction of BC), $\mathcal{f}$ becomes a strophoid.

5. Strophoids in triangles

Let ℓ be a line through the circumcenter O of triangle ABC. Then $I(\ell)$ is a circumconic $\mathscr{h}$, which is a rectangular hyperbola since it passes through the orthocenter of ABC.

$$\Theta_A(\ell) = I\Theta_A I(\ell) = R_A\big(I(\ell)\big) = R_A(\mathscr{h})$$

$\Longrightarrow$ the cyclinear conjugate of ℓ is an inversion of a rectangular hyperbola, a strophoid.

$$\Theta_A(\ell) = I\Theta_A I(\ell) = I\big(R_A(\ell)\big)$$

$\Longrightarrow$ strophoids are also isogonal conjugates of some circle $c = R_A(\ell)$.
It can be easily seen that c passes through A and has its center on BC.

The node of the strophoid is at A, and it is trivial to show that it will pass through the other two vertices of the triangle. As established, strophoid is a third degree curve, therefore it intersects the base of the triangle 3 times.

Through some quick inspection, one obtains the following list of properties of features of the strophoid $\mathscr{s}$:

- The focus is the inverse of the center of circle c about ○(ABC).
- The focus is the cyclinear conjugate of the second intersection of ℓ with ○(BOC).
- The axis is the line through A and intersection of ℓ and BC.
- The line through A and the third intersection of BC with $\mathscr{s}$ is parallel to ℓ.

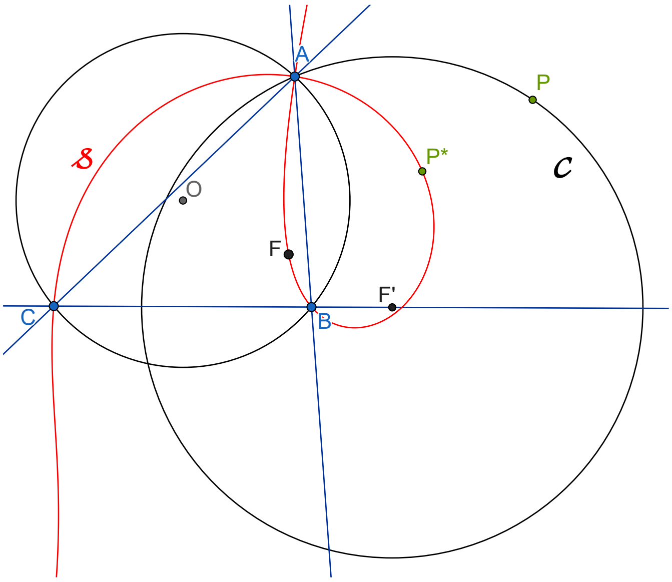


Diagram 5.1

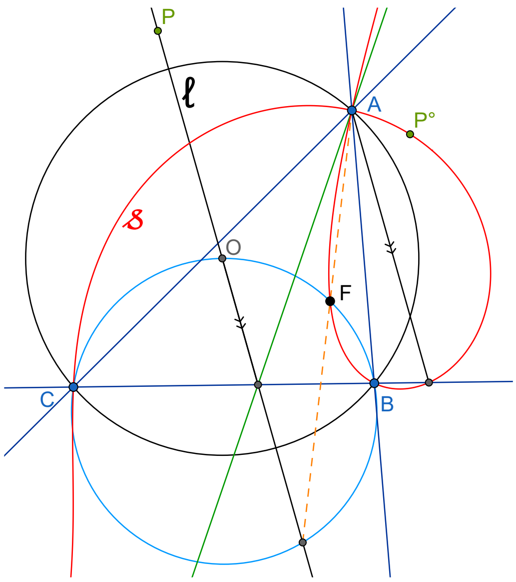


Diagram 5.2

If $\mathscr{s}$ is a right strophoid, it can be shown that the line through the center of c and the intersection of c and ○(ABC) is perpendicular to BC. As an exercise, one could try to show that finding such centers of c is equivalent to intersecting some strophoid with BC.

Using the property that $R_B R_A(X) = R_C(X)$ on the hyperbola $\mathscr{h}$, we can see that strophoids are self-inversive at any point other than the node, as:

$$R_B R_A(\mathscr{h}) = R_C(\mathscr{h}) \Longrightarrow R_B(\mathscr{s}) = R_C(\mathscr{h})$$

6. Strophoids and inscribed conics

Let h be the perpendicular bisector of BC. Then $\Theta_A(\text{h})$ is the locus of foci of conic sections that are tangent to AB, AC; at points B, C; respectively.

The reason for thit is the fact that for some point P on h, line PP° is the angle bisector of BP°C. Thus, A lies on the angle bisector of BP°C, which is exactly the property of the foci of the aforementioned conic.

Therefore, this strophoid is called the "bisector strophoid of △BAC".

The isogonal conjugate of this strophoid is the Apollonius circle of vertex A, and $R_A(\mathscr{s}) = I(\text{h})$ is the bisector hyperbola of △BAC.

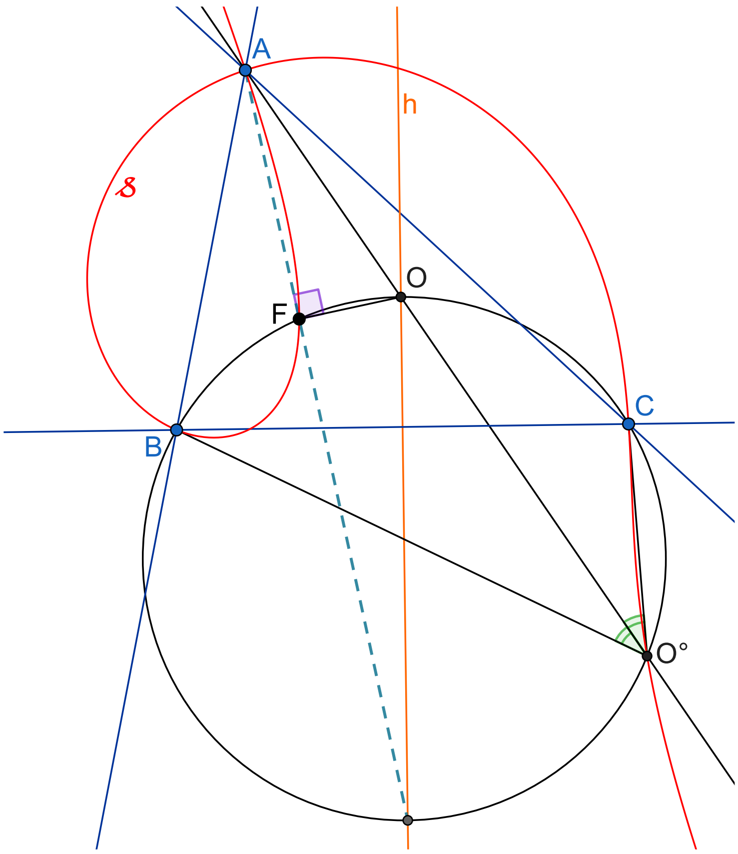


Diagram 6

Using the line definition, we can see that the axis of the bisector strophoid is the *median,* the third intersection with BC is at the projection of A on BC, and the focus lies on the *symmedian* from A.

The three bisector strophoids, with the nodes being each of the vertices of the triangle, all intersect at both Fermat–Torricelli points. This is another proof of Apollonius and Torricelli points being isogonal conjugates.

7. Strophoids as pedal curves

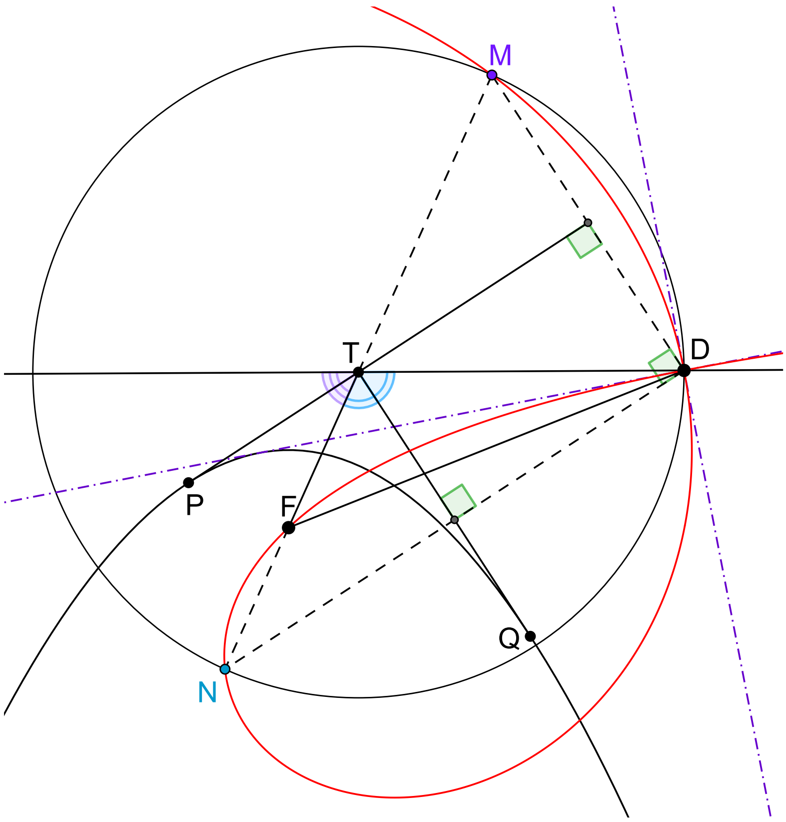


Diagram 7

<u>Statement</u>:
A pedal curve of a parabola, with the axel on the directrix, is a strophoid.

Let there be a parabola with focus F and directrix $\mathscr{L}$. Some points **D** and T are placed on $\mathscr{L}$. PT and QT are tangents to the parabola. M, N; are the reflections of **D** in PT, QT; respectively.

It is known that QT bisects the angle between **FT** and $\mathscr{L}$, hence N, the reflection of D, lies on FT. Analogously, point M lies on FT $\Longrightarrow$ N, M, T, **F**; are collinear.

Another consequence of the bisection property is that that $\angle PTQ = 90° \Longrightarrow \angle M\mathbf{D}N = 90° \Longrightarrow \circ(MND)$ is centered at T.

Thus, as T moves on $\mathscr{L}$, points M and N can be defined in the same way as they are in the primary definition. Therefore, the pedal curve of a parabola, which is a homothety $\left(D, \frac{1}{2}\right)$ of this construction, is a strophoid.

By inspection, tangents of the strophoid at the double point are orthogonal to each other. Furthermore, they are angle bisectors between **FD** and $\mathscr{L}$.

Due to properties of pedal curves, it is known[2] that tangent to the strophoid at M is also tangent to $\circ$(MP**F**) $\Longrightarrow$ tangent at M is perpendicular to MP.

Let U, V; be distinct points on the strophoid such that the tangents at those points are parallel to the axis. It can be proven that the line UV passes through **F** and is perpendicular to **FD**. Hence, U and V are equidistant from $\mathscr{L}$. In fact, the distance is exactly **FD**.

[2] See "Note on Curvature of Pedal and Reciprocal Curves" by Benjamin H. Steede

8. Circles inscribed into strophoids

Let O, O_1, be the centers of ○(ABC), ○(BOC); and ℓ intersect:

BC, AC, AB; at points X, Y, Z.
○(ABC) at points K, R.
○(BOC) at point E ≠ O.

respectively.

Notice that $\Theta_A(Y) = B$, therefore ○(BYC) is tangent to the strophoid at B. Same applies for points Z and C. If there is some circle through B, C, which is tangent to the strophoid both at B and C, then points B, C, Z, Y; are cyclic ⟹ $\ell \perp OA$.

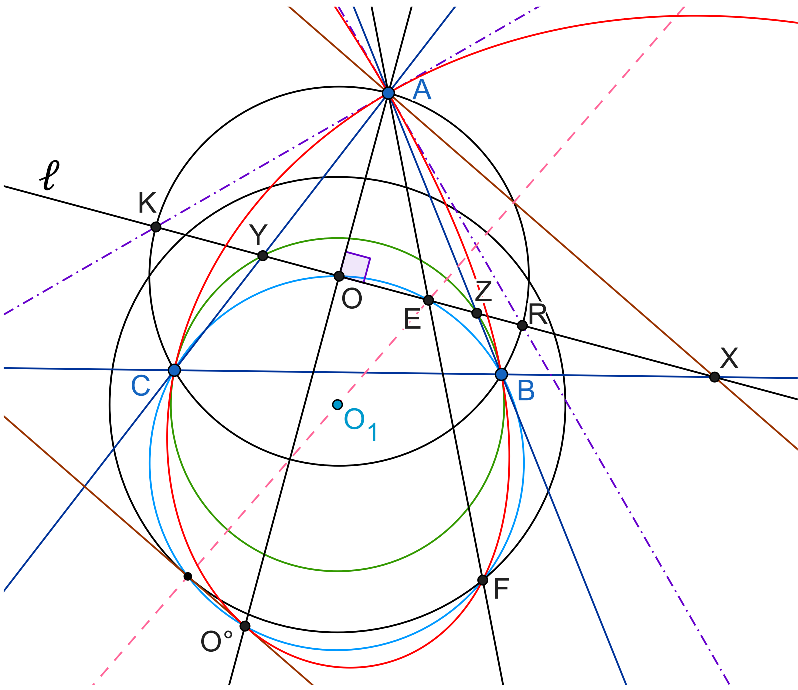


Diagram 8.1

Now consider $\Theta_A(K)$ and $\Theta_A(R)$. Both are equal to A ⟹ lines AK and AR are tangent to the strophoid.

$$\left.\begin{aligned} OK &= OR \\ KR &\perp OA \end{aligned}\right\} \Longrightarrow \angle OAR = \angle OAK$$

⟹ O lies on the angle bisector of the tangents at the node.

Another consequence of $\ell \perp$ OA is that that $O°\boldsymbol{F} \perp A\boldsymbol{F}$. Notice that we previously established (section 5) that the tangent to the strophoid at such points ($O°$) is parallel to the axis (AX). Then we can notice that the circle concentric with ○(BCZY) passing through **F** touches the tangent to $\mathcal{s}$ at $O°$. To show thit we should consider the line O_1E and prove that it is perpendicular to AX. To do this one should assume that O_1E is perpendicular to AX, and consider circles ○(BCZY), ○(ABC), ○(AOE) and the power of point X to show that E lies on ○(BOC).

Centers of all circles inscribed into the strophoid are equidistant from **F** and one of the parallel tangents ⟹ The locus of all centers of such circles is a pair of parabolas (diagram 8.2, in green). Those parabolas have their foci at **F**, and directrices at the aforementioned tangents parallel to the axis.

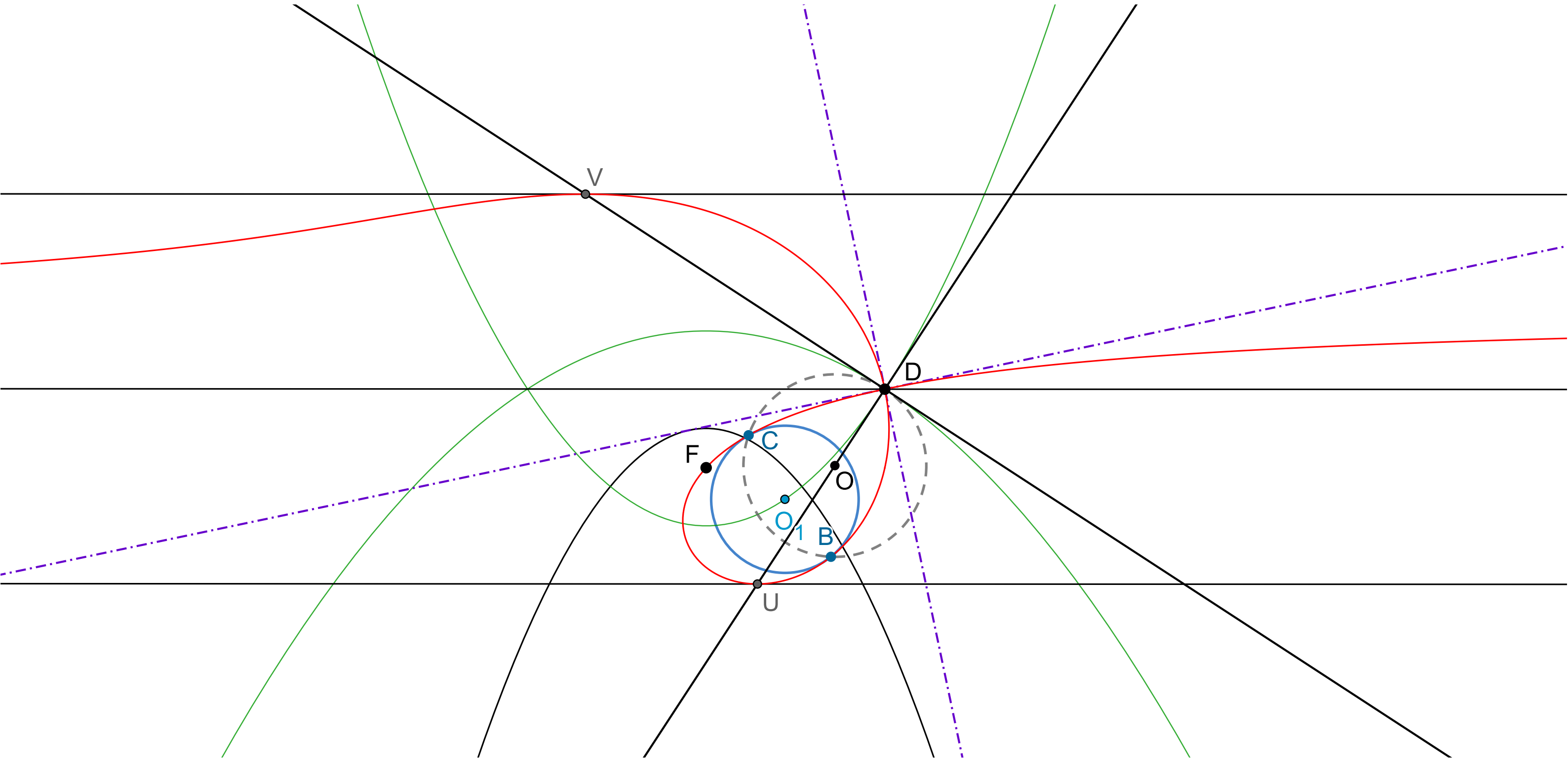


Diagram 8.2

9. Locus of an orthocenter

Statement: Given a circle ω with center O, points A, B on ω and some point C such that ∠ACO=90°, the locus of orthocenter of ABC as B moves along ω is a strophoid.

On some conic section $\mathcal{C}$ place points E, **D** such that line E**D** is a normal to $\mathcal{C}$ at **D**. Then choose points P, Q, on $\mathcal{C}$ so that ∠P**D**Q=90°.

According to Frégier's theorem[3], PQ intersects E**D** at some fixed point I. L is the point of intersection of lines PE and **D**Q. The polar line of L passes through I ⟹ L lies on the polar line ($\mathscr{L}$) of I with respect to $\mathcal{C}$.

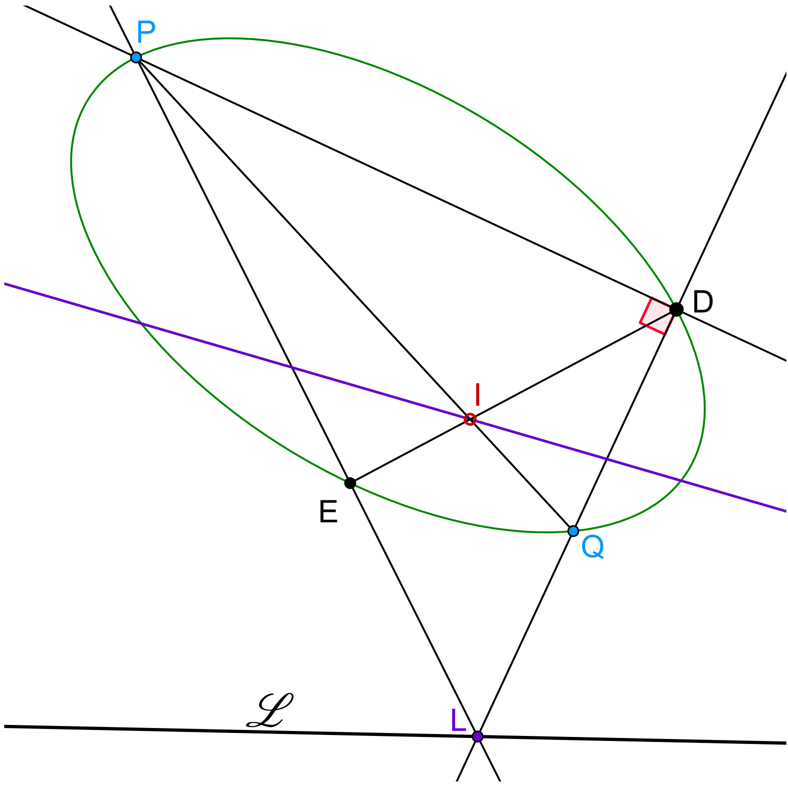


Diagram 9.1

[3] See Frégier's Theorem on Wolfram MathWorld

Now, “reversing” the order of construction, we can start by setting point L on some line $\mathscr{L}$, and drawing arbitrary points **D** and E. Point P is constructed by intersecting LE and the perpendicular to L**D** from **D**.

We can see that the intersections of $\mathscr{L}$ with the circle on diameter E**D** counts the number and the angle of asymptotes of the locus of P with respect to L. Therefore, we can infer that the locus of P is a rectangular hyperbola, if the midpoint of E**D** lies on $\mathscr{L}$.

Let Ω be a circle with center **D** passing through E. Applying inversion about Ω, P’ is the intersection of the perpendicular to **D**L’ at **D** and ○(L’E**D**), and the reflection of **D** about E lies on $\mathscr{L}$’ ⟹ ∠OE**D**=90°, where O is the center of $\mathscr{L}$’.

Let H be the orthocenter of L’**D**E. Then we can notice that H**D**P’E is a parallelogram ⟹ H is P’ reflected about midpoint of **D**E, M.

Since the locus of P’ is an inversion of a rectangular hyperbola, the locus of P’ is a strophoid ⟹ the locus of H, orthocenter of L’E**D**, as L’ moves on $\mathscr{L}$’ is a strophoid.

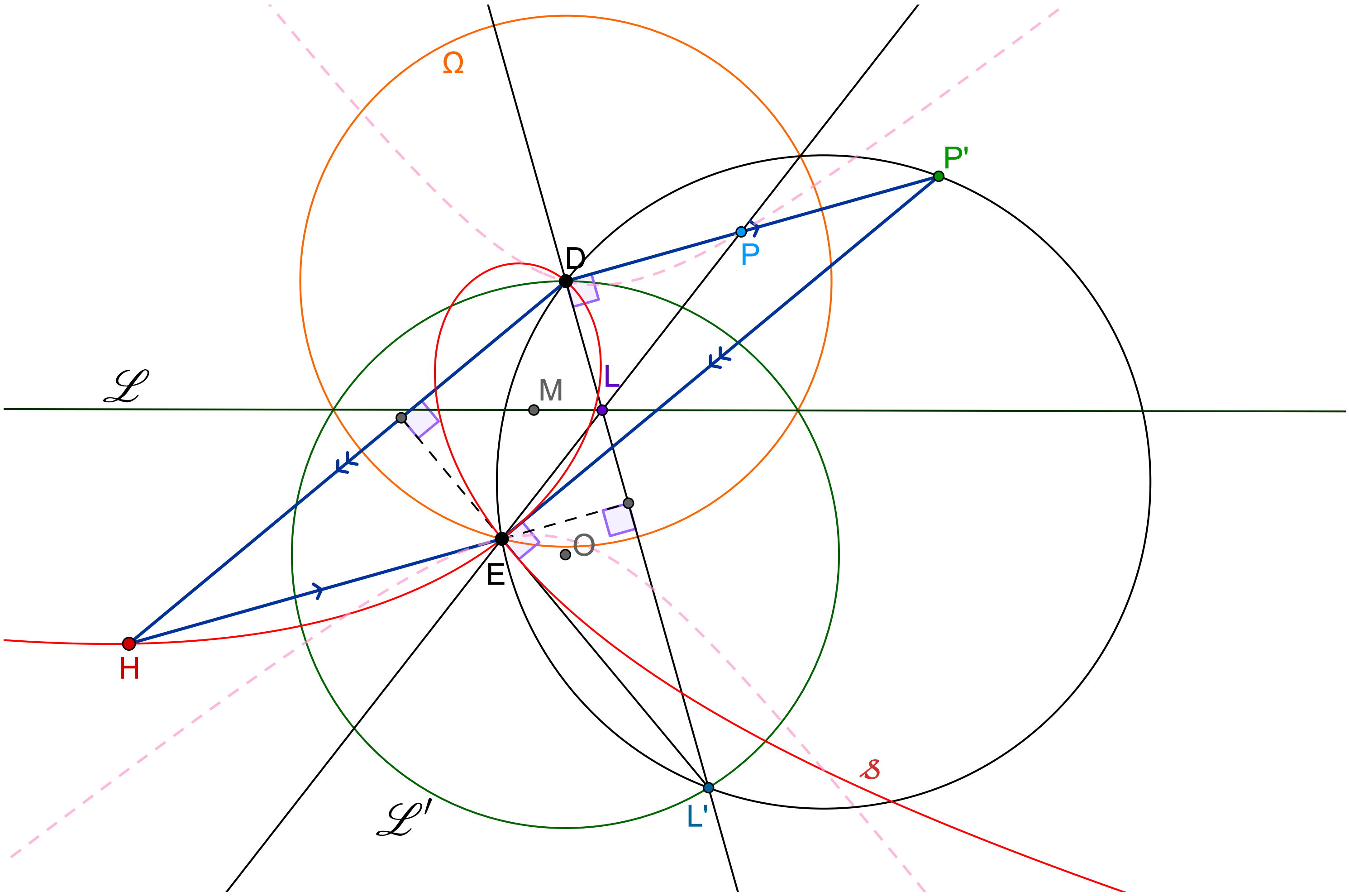


Diagram 9.2

10. Conclusion

An attentive reader might notice that in almost every definition of the strophoid presented in this article there is some “unnatural” restriction on the relative positions of the construction elements. If those restrictions are removed the loci generalize to circular rational cubics, or CRCs for short. CRCs still share some properties with strophoids, in particular being the inverses of conics and pedals of parabolas. However, there is no simple generalization to the primary definition of the strophoid, which makes working with CRCs geometrically not an easy task.

PS: The green “apple like” curve on the title page is the pedal curve of the strophoid. No remarkable properties other than the shape are known.

11. Further reading

1. STROPHOID – mathcurve.com by Robert FERRÉOL

2. Hellmuth STACHEL (2015). *Strophoids, a family of cubic curves with remarkable properties.*

3. Hellmuth STACHEL (2015). *Strophoide are auto-isogonal cubics.*

Author:
Bohdan Zhelyabovskyy
Email: bohdan.zhelyabovsky@gmail.com