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FUNDAMENTAL SYMMETRIC POLYNOMIALS IN ALGEBRAIC INEQUALITIES

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Definition: Polinomyal $p \in R[x, y, z]$ **it is symmetric in the variables** x, y, z **if for any permutation** σ **of the variables** x, y, z **we have:** $\sigma(p) = p$ **, i.e. the polynomial** p **satisfy** $p(x, y, z) = p(x, z, y) = p(y, x, z) = p(y, z, x) = p(z, x, y) = p(z, y, x)$ **.**

Polynomials: $\sigma_1 \in R[x, y, z]$ **,** $\sigma_1 = x + y + z$ **;** $\sigma_2 \in R[x, y, z]$ **,** $\sigma_2 = xy + yz + zx$ **;** $\sigma_3 \in R[x, y, z]$ **,** $\sigma_3 = xyz$ **they are called fundamental symmetric polynomials.**

Lemma. Polynomials $S_k = x^k + y^k + z^k$ **can be expressed using polynomials** $\sigma_1, \sigma_2, \sigma_3$ **.**

Proof. We have that:

$$S_k = \sigma_1 S_{k-1} - \sigma_2 S_{k-2} + \sigma_3 S_{k-3},\ \forall k \geq 3\ (S_0 = 3, S_1 = \sigma_1, S_2 = \sigma_2),\ (1).$$

Indeed, substituting $S_{k-3}, S_{k-2}, S_{k-1}, \sigma_1, \sigma_2, \sigma_3$ through their expressions we obtain

$$(x + y + z)(x^{k-1} + y^{k-1} + z^{k-1}) - (xy + yz + zx)(x^{k-2} + y^{k-2} + z^{k-2}) +$$

$$+ xyz(x^{k-3} + y^{k-3} + z^{k-3}) = x^k + y^k + z^k.$$

Theorem 1. Any symmetric polynomial in three variables x, y, z **can be expressed uniquely with the help of fundamental symmetric polynomials.Waring's formula is deduced from formula (1):**

$$\frac{S_k}{k} = \sum \frac{(-1)^{k-\lambda_1-\lambda_2-\lambda_3}(\lambda_1 + \lambda_2 + \lambda_3 - 1)!\sigma_1^{\lambda_1}\sigma_2^{\lambda_2}\sigma_3^{\lambda_3}}{\lambda_1!\lambda_2!\lambda_3!}$$ **, unde** $\lambda_1 + 2\lambda_2 + 3\lambda_3 = k$ **, (2).**

By (2), we obtain $S_k\ (k = \overline{3,6})$**:**

1. $S_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3$ **;**

2. $S_4 = \sigma_1^4 - 4\sigma_1^2\sigma_2 + 2\sigma_2^2 + 4\sigma_1\sigma_3$ **;**

3. $S_5 = \sigma_1^5 - 5\sigma_1^3\sigma_2 + 5\sigma_1\sigma_2^2 + 5\sigma_1^2\sigma_3 - 5\sigma_2\sigma_3$ **;**

4. $S_6 = \sigma_1^6 - 6\sigma_1^4\sigma_2 + 9\sigma_1^2\sigma_2^2 - 2\sigma_2^3 + 6\sigma_1^3\sigma_3 - 12\sigma_1\sigma_2\sigma_3 + 3\sigma_3^2$ **.**

Example. If $p \in R[x,y,z]$, $p = x^n(y^m + z^m) + y^n(x^m + z^m) + z^n(x^m + y^m)$, $m,n \in N$, then

$$p = x^n(y^m + z^m + x^m) - x^{m+n} + y^n(x^m + y^m + z^m) - y^{m+n} + z^n(x^m + y^m + z^m) - z^{m+n} =$$

$$= S_m S_n - S_{m+n}.$$

Theorem 2. If $x > 0, y > 0, z > 0$, then $\sigma_1 > 0, \sigma_2 > 0, \sigma_3 > 0$.

If $x, y, z \in R$ and $\sigma_1 > 0, \sigma_2 > 0, \sigma_3 > 0$, then $x > 0, y > 0, z > 0$.

Proof. By equation $u : u^3 - \sigma_1 u^2 + \sigma_2 u - \sigma_3 = 0$, if $u < 0$, then $u^3 - \sigma_1 u^2 + \sigma_2 u - \sigma_3 < 0$. So, $u > 0$ and $x > 0, y > 0, z > 0$.

Theorem 3. If $x > 0, y > 0, z > 0$, then we have the following inequalities:

(1) $\sigma_1^2 - 3\sigma_2 \geq 0$;

(2) $\sigma_1^3 - 3\sigma_1\sigma_2 \geq 0$;

(3) $\sigma_2^2 \geq 3\sigma_1\sigma_3$;

(4) $\sigma_1\sigma_2 \geq 9\sigma_3$;

(5) $\sigma_1^3 - 4\sigma_1\sigma_2 + 9\sigma_3 \geq 0$ (***Schur***);

(6) $\sigma_2^3 \geq 27\sigma_3^2$;

(7) $\sigma_1^3 \geq 27\sigma_3$;

(8) $2\sigma_1^3 - 7\sigma_1\sigma_2 + 9\sigma_3 \geq 0$;

(9) $8\sigma_1^3 - 27\sigma_1\sigma_2 + 27\sigma_3 \geq 0$.

Proof.

(1) By $(x-y)^2 + (y-z)^2 + (z-x)^2 \geq 0$, we have $(x+y+z)^2 \geq 3(xy+yz+zx)$, i.e. $\sigma_1^2 - 3\sigma_2 \geq 0$.

(2) Yields by (1) by multiplying with σ_1.

(3) Yields by $(a+b+c)^2 \geq 3(ab+bc+ca)$ with $a = xy, b = yz, c = zx$.

(4) $(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\geq 9$ it is $\sigma_1\sigma_2 \geq 9\sigma_3$.

(5) In $(a+b)(b+c)(c+a)\geq 8abc$ we take $a+b=x, b+c=y, c+a=z$

$xyz \geq (-x+y+z)(x-y+z)(x+y-z)$, then $\sigma_1^3-4\sigma_1\sigma_2+9\sigma_3\geq 0$;

(6) $\sigma_2^3=\sigma_2\sigma_2^2\overset{(3)}{\geq}\sigma_2\cdot 3\sigma_1\sigma_3=3\sigma_3(\sigma_1\sigma_2)\overset{(4)}{\geq}27\sigma_3^2$;

(7) $\sigma_1^4=\sigma_1^2\sigma_1^2\overset{(1)}{\geq}\sigma_1^2\cdot 3\sigma_2=3\sigma_1(\sigma_1\sigma_2)\overset{(4)}{\geq}3\sigma_1 9\sigma_3$, so $\sigma_1^3\geq 27\sigma_3$.

(8) $2\sigma_1^3-7\sigma_1\sigma_2+9\sigma_3=(\sigma_1^3-4\sigma_1\sigma_2+9\sigma_3)+(\sigma_1^3-3\sigma_1\sigma_2)\underset{(5)}{\overset{(2)}{\geq}}0$.

(9) $8\sigma_1^3-27\sigma_1\sigma_2+27\sigma_3=4(2\sigma_1^3-7\sigma_1\sigma_2+9\sigma_3)+(\sigma_1\sigma_2-9\sigma_3)\underset{(8)}{\overset{(4)}{\geq}}0$.

Applications

1) $(x+y+z)(x^2+y^2+z^2)\geq 9xyz$, $\forall x,y,z>0$.

Solution. The inequality becomes $\sigma_1(\sigma_1^2-2\sigma_2)\geq 9\sigma_3 \Leftrightarrow \sigma_1^3\geq 2\sigma_1\sigma_2+9\sigma_3$.

By (1) we have $\sigma_1^2\geq 3\sigma_2 \Leftrightarrow \sigma_1^3\geq 3\sigma_1\sigma_2 \Leftrightarrow \sigma_1^3\geq 2\sigma_1\sigma_2+\sigma_1\sigma_2$, than by (4) we deduce

$$\sigma_1^3\geq 2\sigma_1\sigma_2+\sigma_1\sigma_2 \Rightarrow \sigma_1^3\geq 2\sigma_1\sigma_2+9\sigma_3.$$

2) $2(x^3+y^3+z^3)\geq x^2(y+z)+y^2(z+x)+z^2(x+y)$, $\forall x,y,z>0$.

Solution. The inequality becomes

$2(\sigma_1^3-3\sigma_1\sigma_2+3\sigma_3)\geq \sigma_1\sigma_2-3\sigma_3 \Leftrightarrow 2\sigma_1^3-7\sigma_1\sigma_2+9\sigma_3\geq 0$, true by (8).

3) $\frac{2}{x+y}+\frac{2}{y+z}+\frac{2}{z+x}\geq \frac{9}{x+y+z}$, $\forall x,y,z>0$.

Solution. The inequality becomes

$2(x+y+z)\big((x+y)(x+z)+(y+x)(y+z)+(z+x)(z+y)\big)\geq 9(x+y)(y+z)(z+x)$ or

$$2\sigma_1(S_2+3\sigma_2)\geq 9(\sigma_1\sigma_2-\sigma_3) \Leftrightarrow 2\sigma_1(\sigma_1^2-2\sigma_2+3\sigma_2)\geq 9(\sigma_1\sigma_2-\sigma_3) \Leftrightarrow$$

$$\Leftrightarrow 2\sigma_1^3 - 7\sigma_1\sigma_2 + 9\sigma_3 \ge 0,$$ true by (8).

4) $\dfrac{x^3+y^3+z^3}{x^2+y^2+z^2} \ge \dfrac{x+y+z}{3}$, $\forall x,y,z>0$.

Solution. The inequality becomes successively

$$3S_3 \ge S_1S_2 \Leftrightarrow 3(\sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3) \ge \sigma_1(\sigma_1^2 - 2\sigma_2) \Leftrightarrow 2\sigma_1^3 - 7\sigma_1\sigma_2 + 9\sigma_3 \ge 0,$$

true by (8).

5) $(x+y+z)^3 \ge 9(x^3+y^3+z^3)$, $\forall x,y,z>0$.

Solution. The inequality becomes successively $\sigma_1^3 \ge 9S_3 \Leftrightarrow 8\sigma_1^3 - 27\sigma_1\sigma_2 + 27\sigma_3 \ge 0$,

true by (9).

6) If $x,y,z>0$ **such that** $x+y+z=1$**, then**

$$\frac{xy}{1+z}+\frac{yz}{1+x}+\frac{xz}{1+y} \le \frac{1}{4} - \frac{m}{4}(1-3(xy+yz+zx)) \le \frac{1}{4},$$ where $m=\min(x,y,z)$.

Solution. $\dfrac{1}{1+z} = 1-\dfrac{z}{1+z} \Rightarrow \dfrac{xy}{1+z} = xy\left(1-\dfrac{z}{1+z}\right) = xy - \dfrac{xyz}{1+z}$. Analogusly:

$$\frac{yz}{1+x} = yz - \frac{xyz}{1+x}$$ şi $$\frac{xz}{1+x} = xz - \frac{xyz}{1+y}$$; then $$\sum\frac{xy}{1+z} = \sum xy - xyz\sum\frac{1}{1+z}.$$

$$\left(\sum\frac{1}{1+z}\right)\left(\sum(1+z)\right) \ge 9 \Rightarrow \sum\frac{1}{1+z} \ge \frac{9}{4};$$ so, $$\sum\frac{xy}{1+z} = \sum xy - xyz\sum\frac{1}{1+z} \le \sigma_2 - \frac{9}{4}\sigma_3.$$

From $\sigma_1 = 1 \Rightarrow m(1-3\sigma_2) \le 1 - 4\sigma_2 + 9\sigma_3 \Rightarrow \sigma_2 - \dfrac{9}{4}\sigma_3 \le \dfrac{1}{4} - \dfrac{m}{4}(1-3\sigma_2)$.

Therefore, $\dfrac{xy}{1+z}+\dfrac{yz}{1+x}+\dfrac{xz}{1+y} \le \dfrac{1}{4} - \dfrac{m}{4}(1-3\sigma_2)$.

$$xy+yz+zx \le \frac{(x+y+z)^2}{3} \Rightarrow \sigma_2 \le \frac{1}{3} \Rightarrow 1-3\sigma_2 \ge 0 \Rightarrow -\frac{m}{4}(1-3\sigma_2) \le 0$$, so

$$\frac{xy}{1+z}+\frac{yz}{1+x}+\frac{xz}{1+y}\le\frac{1}{4}-\frac{m}{4}(1-3(xy+yz+zx))\le\frac{1}{4}.$$

Equality occurs iff $x=y=z=\frac{1}{3}$.

7) Prove that in any triangle ABC holds:

a) $2(ab+bc+ca)>a^2+b^2+c^2$**;**

b) $(a^2+b^2+c^2)(a+b+c)>2(a^3+b^3+c^3)$**.**

Solution. a) We denote $x=a+b-c, y=a-b+c, z=-a+b+c$, where $x,y,z>0$.

We have $a=\frac{x+y}{2}, b=\frac{x+z}{2}, c=\frac{y+z}{2}$ and the inequality becomes

$$\frac{(x+y)(x+z)+(y+x)(y+z)+(z+x)(z+y)}{2}>\frac{(x+y)^2+(y+z)^2+(z+x)^2}{4}$$

$$\Leftrightarrow 2(S_2+3\sigma_2)>2S_2+2\sigma_2\Leftrightarrow\sigma_2>0, \text{ true.}$$

b) The inequality becomes successively

$$\left[\left(\frac{x+y}{2}\right)^2+\left(\frac{y+z}{2}\right)^2+\left(\frac{z+x}{2}\right)^2\right](x+y+z)>2\left[\left(\frac{x+y}{2}\right)^3+\left(\frac{y+z}{2}\right)^3+\left(\frac{z+x}{2}\right)^3\right]$$

$$\Leftrightarrow\sigma_1\sigma_2+3\sigma_3>0, \text{ true.}$$

8) Prove that among all triangles with the same perimeter, the equilateral triangle has the maximum area.

Solution. Let a,b,c be the sides of triangle and p the semiperimeter, so $2p=\sigma_1$.

The area of triangle is $S=\sqrt{\frac{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}{16}}$.

We denote $x=a+b-c, y=a-b+c, z=-a+b+c$, then:

$$S=\sqrt{\frac{(x+y+z)xyz}{16}}=\frac{1}{4}\sqrt{\sigma_1\sigma_3}\overset{(7)}{\le}\frac{1}{4}\sqrt{\sigma_1\cdot\frac{\sigma_1^3}{27}}=\frac{\sigma_1^2\sqrt{3}}{36}.$$

Hence, $\frac{\sigma_1^2\sqrt{3}}{36}$ represents the area of an equilateral triangle with the side $l=\frac{\sigma_1}{3}$.

9) (IMO, 1977) $x^3+y^3+z^3+3xyz\ge x^2(y+z)+y^2(x+z)+z^2(x+y), \forall x,y,z>0$**.**

Solution. The inequality becomes successively

$\sigma_1^3-3\sigma_1\sigma_2+3\sigma_3+3\sigma_3\ge\sigma_1\sigma_2-3\sigma_3\Leftrightarrow\sigma_1^3-4\sigma_1\sigma_2+9\sigma_3\ge 0$, i.e. the inequality of *Schur.*

10) (IMO, 1964) Prove that in any triangle ABC is true the following inequality:

$$a^2(b+c-a)+b^2(a+c-b)+c^2(a+b-c)\le 3abc.$$

***Solution*.** Denoting $x=b+c-a, y=a-b+c, z=a+b-c$ ($x,y,z>0$)the inequality becomes:

$$\sum(y+z)^2\frac{x}{4}\le\frac{3}{8}(y+z)(z+x)(x+y)\Leftrightarrow 2\sum(y^2+z^2+2yz)x\le 3\left(\sum x^2y+2xyz\right)\Leftrightarrow$$

$$\Leftrightarrow\sum(y+z)^2\frac{x}{4}\le\frac{3}{8}(y+z)(z+x)(x+y)\Leftrightarrow\sigma_1\sigma_2-3\sigma_3\ge 6\sigma_3\Leftrightarrow\sigma_1\sigma_2\ge 9\sigma_3,$$

true by (4).

11) $x^4+y^4+z^4+xyz(x+y+z)\ge 2(x^2y^2+y^2z^2+z^2x^2)$, $\forall x,y,z>0$.

Solution. The inequality becomes successively

$$S_4+\sigma_1\sigma_3\ge 2(\sigma_2^2-3\sigma_3\sigma_1)\Leftrightarrow\sigma_1^4-4\sigma_1^2\sigma_2+2\sigma_2^2+4\sigma_1\sigma_3+\sigma_1\sigma_3\ge 2\sigma_2^2-4\sigma_3\sigma_1\Leftrightarrow$$

$\Leftrightarrow\sigma_1^3-4\sigma_1\sigma_2+9\sigma_3\ge 0$, i.e. *Schur's* inequality.

12) If $x,y,z>0$ **such that** $x+y+z=1$ **, then** $\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right)\ge 64$**.**

Solution. The inequality becomes successively

$1+x+y+z+xy+yz+zx\ge 63xyz\Leftrightarrow 2\sigma_1+\sigma_2\ge 63\sigma_3$, true since $6\sigma_2\le 2=2\sigma_1$, $2\sigma_1^3\le 54\sigma_3$ and $\sigma_2\le 9\sigma_3$.

13) (Moldavia Republic , National Math Olympiad 1993)

$$x(x-z)^2 + y(y-z)^2 \geq (x-z)(y-z)(x+y+z),\ \forall x,y,z>0.$$

Solution. The inequality becomes successively

$$x^3+y^3+z^3+3xyz \geq xy^2+yx^2+yz^2+zy^2+zx^2+xz^2 \Leftrightarrow S_3+3\sigma_3 \geq \sigma_1\sigma_2-3\sigma_3 \Leftrightarrow$$

$$\Leftrightarrow \sigma_1^3-4\sigma_1\sigma_2+9\sigma_3 \geq 0,$$ true (*Schur*).

14) If $x,y,z>0$ such that $xyz=1$, then

$$2(x+y+z)^3+(xy+yz+zx)^3+27 \geq 18(x+y+z)(xy+yz+zx).$$

Solution. The inequality becomes successively $2\sigma_1^3+3\sigma_2^3+27 \geq 18\sigma_1\sigma_2$.

$\sigma_1^3-4\sigma_1\sigma_2+9\sigma_3 \geq 0 \Leftrightarrow 2\sigma_1^3+18 \geq 8\sigma_1\sigma_2$; $\sigma_2^3+9\sigma_3^2 \geq 4\sigma_1\sigma_2 \Leftrightarrow 3\sigma_2^3+27 \geq 12\sigma_1\sigma_2$ and $\sigma_1\sigma_2 \geq 9\sigma_3 \Leftrightarrow 2\sigma_1\sigma_2 \geq 18$, which by adding up yields to the desired inequality la inegalitatea.

15) $a^3+b^3+c^3-3abc \geq (a^2+b^2+c^2)\sqrt{a^2+b^2+c^2}$ **,** $\forall a,b,c \in R$

Solution. Denoting $x=\frac{a}{\sqrt{a^2+b^2+c^2}}$, $y=\frac{b}{\sqrt{a^2+b^2+c^2}}$, $z=\frac{c}{\sqrt{a^2+b^2+c^2}}$,

($a \neq 0, b \neq 0, c \neq 0$) and we have $x^2+y^2+z^2=1$. The case $a=b=c=0$ it is obvious.

The inequality becomes successively :

$$x^3+y^3+z^3-3xyz \leq 1 \Leftrightarrow (x+y+z)(x^2+y^2+z^2-xy-yz-zx) \leq 1 \Leftrightarrow \sigma_1(3-\sigma_1^2) \leq 2 \Leftrightarrow$$

$$\Leftrightarrow (\sigma_1-1)^2(\sigma_1+2) \geq 0,$$ true.

Equality occurs if $\sigma_1=1 \Leftrightarrow ab+bc+ca=0$ or if

$$\sigma_1=-2 \Leftrightarrow 2\sum ab = 3\sum a^2 \Leftrightarrow 2\sum(a-b)^2+\sum a^2=0 \Leftrightarrow a=b=c=0.$$

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