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Inequalities in the integral

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Abstract

In algebra and analysis, integrals hold particular significance. One fascinating branch of integration involves problems directly posed in terms of integrals and inequalities, or those inequalities that require the use of integrals in their solution. In this article, we aim to elucidate several famous integral inequalities. Furthermore, by solving numerous examples, we will learn how to employ these inequalities and transform certain problems into the framework of these integrals.

Why integral?

In the world of algebra and inequalities, some problems require important concepts such as integration, which greatly assist us in proving matters. For instance, a problem that might be resolved with several pages of solution could ultimately be proven by writing a simple integral. Let's examine a few simple integrals before we begin :

1 - For all integrable functions $f, g : [a, b] \to R$ such as $f \leq g$ we have :

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

In simple terms, the inequality states that if we have two functions that are integrable and one of them is less than or equal to the other, then the integral of the smaller function is less than or equal to the integral of the larger function.

Let's say we have two functions f and g that are both integrable on the interval $[a, b]$. If f is always less than or equal to g, that is, for every number x in the interval $[a, b]$ we have that $f(x) \leq g(x)$. Then, the integral of function f on the interval $[a, b]$ will also be less than or equal to the integral of function g on the interval $[a, b]$.

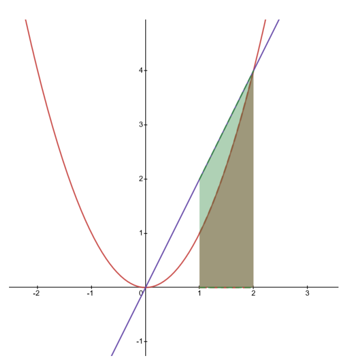

Figure 1: Here, two functions, x^2 and $2x$, were examined over the interval from 1 to 2. It is intuitively evident that the area under the curve of the function $2x$ compared to x^2 is significantly greater.

2 - For all integrable functions $f : [a, b] \to R$ we have :

$$\int_a^b f^2(x)\,dx \geq 0$$

In simple terms, the integral states that if a function f is integrable on the interval $[a, b]$, then the value of the integral of f on this interval is always non-negative.

Since the graph of f on the interval $[a, b]$ always lies above the x-axis, the area under the graph of f on the interval $[a, b]$ is always positive or zero.

Therefore, the value of the integral of f on the interval $[a, b]$ is always non-negative.

In some issues, creativity is required. For example, the following problem requires a double counting approach to calculate the area under the curve :

Example : If $x_i > 0$ then proof :

$$(\sum_{i=1}^{n} x_i)(1 - (\sum_{k=1}^{n} x_k)^2) < \frac{2}{3}$$

Solution : We simply need to state that:

$$\sum x_i \leq 1$$

If you notice, every term exiting the summation is approximately equal to the area of the i-th rectangle. Hence, the sum of these algebraic terms is less than the area under the curve in the interval from 0 to 1. Therefore:

$$\int_0^1 (1 - x^2)dx = \frac{2}{3}$$

Now, let's discuss several well-known integrals:

Example : Show that :

$$0 \leq \int_0^1 sin(x^2) \leq \frac{1}{3}$$

Hint : Proof $0 \leq sin(x^2) \leq 1$

Cauchy–Schwarz inequality

Augustin-Louis Cauchy was a French mathematician and engineer who lived in the 18th century. He was among the first to express and rigorously prove the theorems of differential and integral calculus. One of Cauchy's notable theorems is Cauchy's Schwartz Inequality, which is described as follows:

$$\left| \int_a^b f(x)g(x)dx \right| \leq \int_a^b f^2(x)dx . \int_a^b g^2(x)dx$$

For a better understanding of this inequality, let's assume that $f(x)$ and $g(x)$ are two functions that take positive values on the interval $[a, b]$. In this case, it can be said that the square root of the product of the squares of these two functions indicates the relative magnitude of the two functions. In other words, if the square root of the product of the squares of two functions is large, it implies that both functions take relatively large values on the interval $[a, b]$.

We can also present another form of the Schwartz inequality, which is as follows:

$$\left(\sum_{i=1}^{n} u_i v_i\right)^2 \leq (\sum_{i=1}^{n} u_i^2)(\sum_{i=1}^{n} v_i^2)$$

Example : For every positive integer n, show that

$$\int_0^1 \frac{1}{x^{n-1} + \cdots + x + 1} \, dx \geq \frac{1}{1 + \frac{1}{2} + \cdots + \frac{1}{n}}.$$

Solution :

Let $1 + \frac{1}{2} + \cdots + \frac{1}{n} = X_n = \int_0^1 (x^{n-1} + \cdots + x + 1) \, dx$.

Also let $f(x) = 1 + x^2 + \cdots + x^{n-1}$. So the inequality is equivalent to :

$$\int_0^1 f(x)dx \int_0^1 \frac{1}{f(x)} dx \geq 1$$

The above inequality is self-evident and can be easily proven using the Cauchy-Schwarz inequality.

Hermite–Hadamard's inequalities

In inequalities, sometimes finding a phrase between expressions greatly aids us. For instance, if we aim to demonstrate that A is greater than B, one approach is to identify a phrase, like C, that is smaller than A and subsequently prove that it is greater than B. Here, we are delving into a stronger form of inequality known as Jensen's inequality, which states:

if $f : I \to R$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then :

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}$$

Simply put, this inequality says that if a continuous function is on a finite interval, its limit value on the interval is less than or equal to its maximum value on the interval.

Example : For all $x > 0$ Prove that:

$$\frac{2x}{x+2} < \ln(x+1)$$

.

Solution :

We want to prove $\frac{2x}{x+2} < \ln(x+1)$ Now let

$$f(x) = \frac{1}{x+1}$$

To complete the proof of this inequality, it is sufficient to substitute $a = 0$ and $b = x$ into the condition of the Hermite- Hadamard's inequality. The desired result is easily obtained.

By considering these two inequalities and the ideas presented in the first part of the article, we can solve a wide range of problems. To demonstrate this, we have gathered four examples for your attention!

Problem 1 : If $0 < a \leq b$, then prove :

$$\int_a^b e^{x^2} dx \geq (b-a).\sqrt[3]{a^2+ab+b^2}$$

Solution: It is easy to see $e^x \geq x+1$. We can put $x^2 = x$ in this inequality. Then $e^{x^2} \geq x^2+1$. By theorem 1 we have :

$$\int_a^b e^{x^2} dx \geq \int_a^b (x^2+1)dx = \frac{b^3-a^3}{3} + b - a$$

$$= \frac{(b-a)(b^2+ab+a^2)}{3} + b - a = (b-a)\cdot(\frac{a^2+ab+b^2}{3}+1)$$

Now only we need to prove $(\frac{a^2+ab+b^2}{3}+1) \geq \sqrt[3]{a^2+ab+b^2}$, which is easy.

As you can see, the main point of this problem was that:

For all integrable functions $f, g : [a,b] \to R$ such as $f \leq g$ we have :

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

Problem 2 : Let $f : [a,b] \to R$ be a differentiable function with $f(a) = 0$ and so that $f^{'}$ is continuous. Show that :

$$\int_a^b |f(x)f'(x)|dx \leq \frac{b-a}{2}\int_a^b f'(x)^2 dx$$

Solution : Let $f(x) = \int_a^b f^{'}(t)dt$. Now from Cauchy-Schwartz one have :

$$(\int_a^x |f'(t)|dt)^2 \leq (\int_a^x |f'(t)|^2 dt)(\int_a^x 1^2 dt)$$

Since $f(x)^2 = (\int_a^x f'(t)dt)^2 \leq (\int_a^x |f'(t)|dt)^2$ we have :

$$f^2(x) = (\int_a^x f'(t)dt)^2 \leq (\int_a^b |f'(t)|^2 dt)(x-a)$$

$$\Rightarrow \int_a^b f(x)^2 dx \leq (\int_a^b |f'(t)|^2 dt)\int_a^b (t-a)dt$$

$$\Rightarrow \int_a^b |f(t)|^2 dt \leq \frac{(b-a)^2}{2}(\int_a^b |f'(t)|^2 dt)$$

Problem 3 : Prove that if f is continuous nonnegative on $[0,1]$, we have

$$\int_0^1 f(x)^3\ dx \geq 4\left(\int_0^1 xf(x)^2\ dx\right)\left(\int_0^1 x^2 f(x)\ dx\right)$$

Solution : Apply Holder's inequality twice and multiply the positive numbers involved, using that $\int_0^1 x^3\ dx = \frac{1}{4}$:

$$\left(\int_0^1 x^3\ dx\right)^{\frac{1}{3}}\left(\int_0^1 (f(x)^2)^{\frac{3}{2}}\ dx\right)^{\frac{2}{3}} \geq \left(\int_0^1 xf(x)^2\ dx\right)$$

and

$$\left(\int_0^1 (x^2)^{\frac{3}{2}}\ dx\right)^{\frac{2}{3}}\left(\int_0^1 f(x)^3\ dx\right)^{\frac{1}{3}} \geq \left(\int_0^1 x^2 f(x)\ dx\right)$$

problem 4 : Let $b > a > 1$ and n be a positive integer. Prove that:

$$\int_{log(a)}^{log(b)} \frac{\sqrt{e^{nx}}}{e^{nx}+e^{(n-1)x}+\cdots+e^{2x}+e^{x}+1}dx \leq \log(\sqrt[n+1]{\frac{b}{a}})$$

Solution :

$$n^{nx}+e^{(n-1)x}+\cdots+e^{2x}+e^{x}+1 \geq (n+1)\sqrt[n+1]{e^{nx+(n-1)x+\cdots+x+0}}$$

$$= (n+1)\sqrt[n+1]{e^{\frac{nx(n+1)}{2}}} = (n+1)e^{\frac{nx}{2}} = (n+1)\sqrt{e^{nx}} \Rightarrow$$

$$\int_{log\ a}^{log\ b} \frac{\sqrt{e^{nx}}}{e^{nx}+e^{(n-1)x}+\cdots+e^{2x}+e^{x}+1}dx \leq \frac{1}{n+1}\int_{log\ a}^{log\ b} \frac{\sqrt{e^{nx}}}{\sqrt{e^{nx}}}dx$$

$$= \frac{1}{n+1}x|_{log\ a}^{log\ b} = \frac{1}{n+1}(log\ b - log\ a) = \frac{1}{n+1}log(\frac{b}{a}) = \log(\sqrt[n+1]{\frac{b}{a}})$$

As you observed in most of the problems we examined, they could be solved using the ideas presented without needing to look elsewhere! (Of course, most mathematical problems only require step-by-step thinking and, more importantly, correct thinking).

At the end of this note, several questions have been provided to help gain a deeper understanding of the presented concepts. Working through these questions would aid in fully comprehending the topics that have been covered.

Problem 1 : If $0 < a \leq b$ then:

$$6\int_{a}^{b}\int_{a}^{b}(x^3+y^3)^2 dxdy \geq (a^4+a^2b^2+b^4)(b^2-a^2)^2$$

Problem 2 : Show that the inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i-x_j|} \leq \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i+x_j|}$$

holds for all real numbers $x_1, \ldots x_n$.

Problem 3 : Let f be a continuously differentiable function on $[0, a]$ and $f(0) = 0$. Then prove that inequality

$$\int_{0}^{a}(f(x))^2\,|f'(x)|dx \leq \frac{a^2}{3}\int_{0}^{a}|f'(x)|^3 dx$$

Problem 4 : Let $a < b \in R$ and $f : [a, b] \to (0, \infty)$ be continuous. Show :

$$(\int_{a}^{b} f(x)dx)(\int_{a}^{b}\frac{1}{f(y)}dy) \geq (b-a)^2$$

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