ROMANIAN MATHEMATICAL MAGAZINE

JP.541 If a, b, c sides in acute ΔABC with s – semiperimeter; r – inradii and $x, y, z \in \left(0, \frac{\pi}{2}\right)$ are such that:

$$\cos x = \frac{a}{b+c}; \cos y = \frac{b}{c+a}; \cos z = \frac{c}{a+b}$$

then:

$$\left(\tan^2\frac{x}{2} + \tan^2\frac{y}{2} + \tan^2\frac{z}{2}\right)\tan^2\frac{x}{2}\tan^2\frac{y}{2}\tan^2\frac{z}{2} = \frac{r^2}{s^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\tan^2\frac{x}{2} = \frac{1-\cos x}{1+\cos x} = \frac{1-\frac{a}{b+c}}{1+\frac{a}{b+c}} = \frac{b+c-a}{a+b+c} = \frac{a+b+c-2a}{a+b+c} = \frac{2s-2a}{2s} = \frac{s-a}{s}$$

Analogous:

$$\tan^2\frac{y}{2} = \frac{s-b}{s}; \tan^2\frac{z}{2} = \frac{s-c}{s}$$

$$\tan^2\frac{x}{2} + \tan^2\frac{y}{2} + \tan^2\frac{z}{2} = \frac{s-a}{s} + \frac{s-b}{s} + \frac{s-c}{s} =$$

$$= \frac{3s-(a+b+c)}{s} = \frac{3s-2s}{s} = \frac{s}{s} = 1$$

$$\tan^2\frac{x}{2}\tan^2\frac{y}{2}\tan^2\frac{z}{2} = \frac{s-a}{s}\cdot\frac{s-b}{s}\cdot\frac{s-c}{s} =$$

$$= \frac{(s-a)(s-b)(s-c)}{s^3} = \frac{s(s-a)(s-b)(s-c)}{s^4} = \frac{F^2}{s^4} = \frac{r^2s^2}{s^4} = \frac{r^2}{s^2}$$

$$\left(\tan^2\frac{x}{2} + \tan^2\frac{y}{2} + \tan^2\frac{z}{2}\right)\tan^2\frac{x}{2}\tan^2\frac{y}{2}\tan^2\frac{z}{2} = 1\cdot\frac{r^2}{s^2} = \frac{r^2}{s^2}$$

Equality holds for $a = b = c$.

Solution 2 by Marin Chirciu-Romania

Using $\tan^2\frac{x}{2} = \frac{1-\cos x}{1+\cos x} = \frac{1-\frac{a}{b+c}}{1+\frac{a}{b+c}} = \frac{b+c-a}{b+c+a} = \frac{2s-2a}{2s} = \frac{s-a}{s}$ we obtain:

$$LHS = \sum\tan^2\frac{x}{2}\prod\tan^2\frac{x}{2} = \sum\frac{s-a}{s}\prod\frac{s-a}{s} = \frac{s}{s}\cdot\frac{r^2s}{s^3} = \frac{r^2}{s^2} = RHS$$

Remark.

In the same way.

If a, b, c sides in acute ΔABC with s – semiperimeter; r – inradii, R – circumradii and $x, y, z \in \left(0, \frac{\pi}{2}\right)$ are such that $\cos x = \frac{a}{b+c}, \cos y = \frac{b}{c+a}, \cos z = \frac{c}{a+b}$ then

$$\left(\cot^2\frac{x}{2} + \cot^2\frac{y}{2} + \cot^2\frac{z}{2}\right)\cot^2\frac{x}{2}\cot^2\frac{y}{2}\cot^2\frac{z}{2} = \frac{s^2(4R+r)}{r^3}$$

Marin Chirciu

Solution

Using $\cot^2\frac{x}{2} = \frac{1+\cos x}{1-\cos x} = \frac{1+\frac{a}{b+c}}{1-\frac{a}{b+c}} = \frac{b+c+a}{b+c-a} = \frac{2s}{2s-2a} = \frac{s}{s-a}$ we obtain:

$$LHS = \sum \cot^2\frac{x}{2}\prod \cot^2\frac{x}{2} = \sum \frac{s}{s-a}\prod \frac{s}{s-a} =$$

$$= \frac{s(4R+r)}{sr}\cdot\frac{s^3}{r^2 s} = \frac{s^2(4R+r)}{r^3} = RHS$$