ROMANIAN MATHEMATICAL MAGAZINE

JP.542 Solve for real numbers:

$$\begin{cases} \dfrac{6x+6y}{9+4xy} = z \\ \dfrac{6y+6z}{9+4yz} = x \\ \dfrac{6z+6x}{9+4zx} = y \end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\begin{cases} 6x+6y = 9z+4xyz \\ 6y+6z = 9x+4xyz \\ 6z+6x = 9y+4xyz \end{cases} \Rightarrow \begin{cases} x+y = \dfrac{3z}{2} + \dfrac{2xyz}{3} \\ y+z = \dfrac{3x}{2} + \dfrac{2xyz}{3} \\ z+x = \dfrac{3y}{2} + \dfrac{2xyz}{3} \end{cases}$$

$$\Rightarrow \begin{cases} y+z-(x+y) = \dfrac{3x}{2} + \dfrac{2xyz}{3} - \left(\dfrac{3z}{2} + \dfrac{2xyz}{3}\right) \\ z+x-(y+z) = \dfrac{3y}{2} + \dfrac{2xyz}{3} - \left(\dfrac{3x}{2} + \dfrac{2xyz}{3}\right) \end{cases}$$

$$\Rightarrow \begin{cases} z-x = +\dfrac{3}{2}(x-z) \\ x-y = \dfrac{3}{2}(y-x) \end{cases} \Rightarrow \begin{cases} \dfrac{5}{2}(x-z) = 0 \\ \dfrac{5}{2}(y-x) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x-z = 0 \\ y-x = 0 \end{cases} \Rightarrow x = y = z$$

$$6x+6x = 9x+4x^3$$

$$4x^3-3x = 0 \Rightarrow x(4x^2-3) = 0$$

Solutions: $\begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}$ $\quad \begin{cases} x = \frac{\sqrt{3}}{2} \\ y = \frac{\sqrt{3}}{2} \\ z = \frac{\sqrt{3}}{2} \end{cases}$ $\quad \begin{cases} x = -\frac{\sqrt{3}}{2} \\ y = -\frac{\sqrt{3}}{2} \\ z = -\frac{\sqrt{3}}{2} \end{cases}$

Solution 2 by Marin Chirciu-Romania

$$\begin{cases}\dfrac{6x+6y}{9+4xy}=z\\ \dfrac{6y+6z}{9+4yz}=x\\ \dfrac{6z+6x}{9+4zx}=y\end{cases} \Leftrightarrow \begin{cases}6x+6y=9z+4xyz\\ 6y+6z=9x+4xyz\\ 6z+6x=9y+4xyz\end{cases}$$

Subtract the equations (1) and (2) $\Rightarrow 6(x-z)=9(z-x) \Rightarrow x=z.$

Subtract the equations (2) and (3) $\Rightarrow 6(y-x)=9(x-y) \Rightarrow x=y.$

Using the equation (1) and $x=y=z \Rightarrow 4x^3-3x=0 \Rightarrow x\in\left\{-\frac{\sqrt{3}}{2},0,\frac{\sqrt{3}}{2}\right\}$

The set of solutions of the equation is $S=\left\{\left(-\frac{\sqrt{3}}{2},-\frac{\sqrt{3}}{2},-\frac{\sqrt{3}}{2}\right),(0,0,0),\left(\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}\right)\right\}$

Remark.

The inequality can be developed.

Let $0<\lambda<2n$ **fixed. Solve for real numbers:**

$$\begin{cases}\dfrac{\lambda nx+\lambda ny}{\lambda^2+n^2xy}=z\\ \dfrac{\lambda ny+\lambda nz}{\lambda^2+n^2yz}=x\\ \dfrac{\lambda nz+\lambda nx}{\lambda^2+n^2zx}=y\end{cases}$$

Marin Chirciu

Solution

$$\begin{cases}\dfrac{\lambda nx+\lambda ny}{\lambda^2+n^2xy}=z\\ \dfrac{\lambda ny+\lambda nz}{\lambda^2+n^2yz}=x\\ \dfrac{\lambda nz+\lambda nx}{\lambda^2+n^2zx}=y\end{cases} \Leftrightarrow \begin{cases}\lambda nx+\lambda ny=\lambda^2z++n^2xyz\\ \lambda ny+\lambda nz=\lambda^2x++n^2xyz\\ \lambda nz+\lambda nx=\lambda^2y++n^2xyz\end{cases}$$

Subtract the equations (1) and (2) $\Rightarrow \lambda n(x-z)=\lambda^2(z-x) \Rightarrow x=z.$

Subtract the equations (2) and (3) $\Rightarrow \lambda n(y-x)=\lambda^2(x-y) \Rightarrow x=y.$

Using equation (1)

and $x = y = z \Rightarrow n^2x^3 - (2\lambda n - \lambda^2)x = 0 \Rightarrow x \in \left\{-\frac{\sqrt{2\lambda n-\lambda^2}}{n}, 0, \frac{\sqrt{2\lambda n-\lambda^2}}{n}\right\}$

The set of solutions of the equation is:

$$S = \left\{ \begin{matrix} \left(-\frac{\sqrt{2\lambda n - \lambda^2}}{n}, -\frac{\sqrt{2\lambda n - \lambda^2}}{n}, -\frac{\sqrt{2\lambda n - \lambda^2}}{n}\right), (0, 0, 0), \\ \left(\frac{\sqrt{2\lambda n - \lambda^2}}{n}\right), \frac{\sqrt{2\lambda n - \lambda^2}}{n}, \frac{\sqrt{2\lambda n - \lambda^2}}{n} \end{matrix} \right\}$$

Note.

For $\lambda = 3, n = 2$ **we obtain Problem JP.542 from RMM Nr. 37 – Summer 2025**