JP.543 Find $x,y,z>0$ such that $x+y+z=1$ and

$$\left(\frac{x^5}{yz+1}+\frac{y^5}{zx+1}+\frac{z^5}{xy+1}\right)\left(\frac{x^7}{yz+1}+\frac{y^7}{zx+1}+\frac{z^7}{xy+1}\right)=\frac{1}{72900}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\sum_{cyc}\frac{x^5}{yz+1}=\sum_{cyc}\frac{x^6}{xyz+x}=\sum_{cyc}\frac{(x^3)^2}{xyz+x}\ge$$

$$\overset{BERGSTROM}{\ge}\frac{(x^3+y^3+z^3)^2}{3xyz+x+y+z}\overset{AM-GM}{\ge}\frac{(x^3+y^3+z^3)^2}{3\cdot\left(\frac{x+y+z}{3}\right)^3+1}=$$

$$=\frac{(x^3+y^3+z^3)^2}{3\cdot\frac{1}{27}+1}\overset{JENSEN}{\ge}\frac{1}{\frac{1}{9}+1}\cdot\left(3\cdot\left(\frac{x+y+z}{3}\right)^3\right)^2=$$

$$=\frac{9}{10}\cdot 9\cdot\frac{1}{3^6}=\frac{3^4}{3^6\cdot 10}=\frac{1}{90}$$

Analogous:

$$\sum_{cyc}\frac{x^7}{yz+1}\ge\frac{1}{810}$$

By multiplying:

$$\left(\sum_{cyc}\frac{x^5}{yz+1}\right)\cdot\left(\sum_{cyc}\frac{x^7}{yz+1}\right)\ge\frac{1}{72900}$$

Equality holds for $x=y=z=\frac{1}{3}$.

Solution 2 by Marin Chirciu-Romania

$$\sum\frac{x^5}{yz+1}\overset{Holder}{\ge}\frac{(\sum x)^5}{3^3\sum(yz+1)}=\frac{1^5}{3^3(\sum yz+3)}\overset{SOS}{\ge}\frac{1}{3^3\left(\frac{1}{3}+3\right)}=\frac{1}{90},$$

with equality for $x=y=z=\frac{1}{3}$.

We have used $\sum yz\le\frac{1}{3}$, see: $1=(x+y+z)^2\ge 3(xy+yz+zx)\Rightarrow xy+yz+zx\le\frac{1}{3}$

Analogous:

$$\sum\frac{x^7}{yz+1}\overset{Holder}{\geq}\frac{\left(\sum x\right)^7}{3^5\sum(yz+1)}=\frac{1^7}{3^5\left(\sum yz+3\right)}\overset{SOS}{\geq}\frac{1}{3^5\left(\frac{1}{3}+3\right)}=\frac{1}{810}$$

with equality for $x=y=z=\frac{1}{3}$

It follows:

$\sum\frac{x^5}{yz+1}\cdot\sum\frac{x^7}{yz+1}\geq\frac{1}{90}\cdot\frac{1}{810}=\frac{1}{72900}$, with equality for $x=y=z=\frac{1}{3}$.

We deduce that $x=y=z=\frac{1}{3}$ is the solution of the problem.

Remark.

The inequality can be developed.

Let $n\in\mathbb{N}$. Find $x,y,z>0$ such that $x+y+z=1$ and

$$\sum\frac{x^{n+2}}{yz+1}\cdot\sum\frac{x^{n+4}}{yz+1}=\frac{1}{9^n\cdot100}$$

Marin Chirciu

Solution

$$\sum\frac{x^{n+2}}{yz+1}\overset{Holder}{\geq}\frac{\left(\sum x\right)^{n+2}}{3^n\sum(yz+1)}=\frac{1^{n+2}}{3^n\left(\sum yz+3\right)}\overset{SOS}{\geq}\frac{1}{3^n\left(\frac{1}{3}+3\right)}=\frac{1}{3^{n-1}\cdot10},$$

with equality for $x=y=z=\frac{1}{3}$

We have used $\sum yz\leq\frac{1}{3}$, see: $1=(x+y+z)^2\geq3(xy+yz+zx)\Rightarrow xy+yz+zx\leq\frac{1}{3}$.

Analogous:

$$\sum\frac{x^{n+4}}{yz+1}\overset{Holder}{\geq}\frac{\left(\sum x\right)^{n+4}}{3^{n+2}\sum(yz+1)}=\frac{1^{n+4}}{3^{n+2}\left(\sum yz+3\right)}\overset{SOS}{\geq}\frac{1}{3^{n+2}\left(\frac{1}{3}+3\right)}=\frac{1}{3^{n+1}\cdot10},$$

with equality for $x=y=z=\frac{1}{3}$.

It follows:

$\sum\frac{x^{n+2}}{yz+1}\cdot\sum\frac{x^{n+4}}{yz+1}\geq\frac{1}{3^{n-1}\cdot10}\cdot\frac{1}{3^{n+1}\cdot10}=\frac{1}{3^{2n}\cdot100}$, with equality for $x=y=z=\frac{1}{3}$.

We deduce that $x=y=z=\frac{1}{3}$ is the solution of the problem.

Remark: For $n=3$ we obtain Problem JP.543 from RMM Nr. 37 – Summer 2025