JP.544 If $x, y \in \mathbb{R}$ then:

$$\log(1+3\sin^2 x)\cdot\log(1+3\cos^2 x\sin^2 y)\cdot\log(1+3\cos^2 x\cos^2 y)\le\log^3 2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\log(1+3\sin^2 x)\cdot\log(1+3\cos^2 x\sin^2 y)\cdot\log(1+3\cos^2 x\cos^2 y)\le$$

$$\overset{AM-GM}{\le}\left(\frac{\log(1+3\sin^2 x)+\log(1+3\cos^2 x\sin^2 y)+\log(1+3\cos^2 x\cos^2 y)}{3}\right)^3=$$

$$=\frac{1}{27}\cdot\left(\log\left((1+3\sin^2 x)\cdot(1+3\cos^2 x\sin^2 y)\cdot(1+3\cos^2 x\cos^2 y)\right)\right)^3\le$$

$$\overset{AM-GM}{\le}\frac{1}{27}\cdot\left(\log\left(\frac{1+3\sin^2 x+1+3\cos^2 x\sin^2 y+1+3\cos^2 x\cos^2 y}{3}\right)^3\right)^3=$$

$$=\frac{1}{27}\cdot\left(3\log\frac{3+3\sin^2 x+3\cos^2 x(\sin^2 y+\cos^2 y)}{3}\right)^3=$$

$$=\frac{1}{27}\cdot 27\cdot\left(\log\frac{3+3(\sin^2 x+\cos^2 x)}{3}\right)^3=$$

$$=\left(\log\frac{3+3}{3}\right)^3=\log^3 2$$

Equality holds for $x=\pm\arctan\frac{1}{2}; y=\frac{\pi}{4}$

Solution 2 by Marin Chirciu-Romania

$$\text{Using } \log a\cdot\log b\cdot\log c\overset{AM-GM}{\le}\left(\frac{\log a+\log b+\log c}{3}\right)^3\overset{(1)}{\le}\log^3 2,$$

for $(a,b,c)=(1+3\sin^2 x, 1+3\cos^2 x\sin^2 y, 1+3\cos^2 x\cos^2 y)$ and

$$(1)\Leftrightarrow\frac{\log a+\log b+\log c}{3}\le\log 2\Leftrightarrow\log a+\log b+\log c\le\log 8\Leftrightarrow$$

$$\Leftrightarrow\log(abc)\le\log 8\Leftrightarrow abc\le 8,\text{ see } abc\overset{AM-GM}{\le}\left(\frac{a+b+c}{3}\right)^3\overset{(2)}{\le}8\Leftrightarrow$$

$$\Leftrightarrow\frac{a+b+c}{3}\le 2\Leftrightarrow a+b+c\le 16\Leftrightarrow 1+3\sin^2 x+1+3\cos^2 x\sin^2 y+1+$$

$$+3\cos^2 x\cos^2 y\le 6\Leftrightarrow$$

$$\Leftrightarrow 3\sin^2 x+3\cos^2 x\sin^2 y+3\cos^2 x\cos^2 y\le 3\Leftrightarrow$$

$$\Leftrightarrow \sin^2 x \cos^2 x \sin^2 y + \cos^2 x \cos^2 y \le 1 \Leftrightarrow$$

$\Leftrightarrow \sin^2 x + \cos^2 x\,(\sin^2 y + \cos^2 y) \le 1 \Leftrightarrow \sin^2 x + \cos^2 x \cdot 1 \le 1 \Leftrightarrow 1 \le 1$, **obviously.**

Equality holds if and only if:

$$a = b = c \Leftrightarrow 1 + 3\sin^2 x = 1 + 3\cos^2 x \sin^2 y = 1 + 3\cos^2 x \cos^2 y \Leftrightarrow$$

$$\Leftrightarrow 3\sin^2 x = 3\cos^2 x \sin^2 y = 3\cos^2 x \cos^2 y \Leftrightarrow$$

$$\sin^2 x = \cos^2 x \sin^2 y = \cos^2 x \cos^2 y$$

From $\cos^2 x \sin^2 y = \cos^2 x \cos^2 y$ **we obtain** $\cos^2 x = 0$ **or** $\sin^2 y = \cos^2 y \Leftrightarrow$

$$\tan^2 y = 1$$