ROMANIAN MATHEMATICAL MAGAZINE

**JP. 545 If $x, y, z > 0$ then:**

$$\frac{x}{7x+5y+5z}+\frac{y}{5x+7y+5z}+\frac{z}{5x+5y+7z}\le\frac{3}{17}$$

*Proposed by Daniel Sitaru – Romania*

*Solution 1 by proposer*

Denote: $\begin{cases}7x+5y+5z=u\\5x+7y+5z=v\\5x+5y+7z=w\end{cases}\Rightarrow\begin{cases}x=\frac{1}{34}(12u-5v-5w)\\y=\frac{1}{34}(-5u+12v-5w)\\z=\frac{1}{34}(-5u-5v+12w)\end{cases}$

$$\sum_{cyc}\frac{x}{7x+5y+5z}=\sum_{cyc}\frac{12u-5v-5w}{34u}=\frac{1}{34}\sum_{cyc}\left(12-5\left(\frac{v}{u}+\frac{w}{v}\right)\right)=$$

$$=\frac{12\cdot 3}{34}-\frac{5}{34}\sum_{cyc}\left(\frac{v}{u}+\frac{w}{v}\right)\le\frac{36}{34}-\frac{5\cdot 6}{34}=\frac{6}{34}=\frac{3}{17}$$

Equality holds for $x=y=z=1$.

*Solution 2 by Marin Chirciu-Romania*

Return the inequality: $\sum\frac{x}{7x+5y+5z}\le\frac{3}{17}\Leftrightarrow\sum\frac{y+z}{7x+5y+5z}\ge\frac{6}{17}$, which follows from:

$$\sum\frac{y+z}{7x+5y+5z}=\sum\frac{(y+z)^2}{(y+z)(7x+5y+5z)}\overset{CS}{\ge}\frac{(\sum(y+z))^2}{\sum(y+z)(7x+5y+5z)}=$$

$$=\frac{4(\sum x^2+2\sum yz)}{10\sum x^2+24\sum yz}\overset{(1)}{\ge}\frac{6}{17},$$

where (1) $\Leftrightarrow\frac{4(\sum x^2+2\sum yz)}{10\sum x^2+24\sum yz}\ge\frac{6}{17}\Leftrightarrow\sum x^2\ge\sum yz\Leftrightarrow\sum(x-y)^2\ge 0$.

Equality holds if and only if $x=y=z$.

Remark.

Inequality can be developed.

If $x, y, z > 0$ and $\lambda\ge n>0$ then

$$\sum\frac{x}{\lambda x+ny+nz}\le\frac{3}{\lambda+2n}$$

*Marin Chirciu*

*Solution*

Return the inequality: $\sum\frac{x}{\lambda+ny+nz}\le\frac{3}{\lambda+2n}\Leftrightarrow\sum\frac{y+z}{\lambda x+ny+nz}\ge\frac{6}{\lambda+2n}$, which follows from:

$$\sum\frac{y+z}{\lambda x+ny+nz}=\sum\frac{(y+z)^2}{(y+z)(\lambda x+ny+nz)}\overset{CS}{\geq}\frac{(\sum(y+z))^2}{\sum(y+z)(\lambda x+ny+nz)}=$$

$$=\frac{4(\sum x^2+2\sum yz)}{2n\sum x^2+2(\lambda+n)\sum yz}\overset{(1)}{\geq}\frac{6}{2\lambda+n}$$

**where (1) $\Leftrightarrow \frac{4(\sum x^2+2\sum yz)}{2n\sum x^2+2(\lambda+n)\sum yz}\geq\frac{6}{2\lambda+n}\Leftrightarrow(\lambda-n)\sum x^2\geq(\lambda-n)\sum yz$, which follows from the condition for hypothesis $\lambda\geq n\geq 0$ and $\sum x^2\geq\sum yz\Leftrightarrow\sum(x-y)^2\geq 0$**

**Equality holds if and only if $x=y=z$.**

**Note.**

**For $\lambda=7, n=5$ we obtain Problem JP.545 from RMM Nr. 37 – Summer 2025**

***Solution 3 by Tapas Das-India***

$$let\ x+y+z=p, then\ the\ given\ problem\ can\ be\ written\ as$$

$$\frac{x}{2x+5p}+\frac{y}{2y+5p}+\frac{z}{2z+5p}\leq\frac{3}{17}, let\ f(a)=\frac{a}{2a+5p}$$

$$where\ a>0, f'(a)=\frac{5p}{(2a+5p)^2}, f''(a)=\frac{-10p}{(2a+5p)^3}$$
$$so\ f\ is\ concave\ \ in\ a\in(0,\infty).$$

$$Now\ using\ Jensen\ f(x)+f(y)+f(z)\leq 3f\left(\frac{x+y+z}{3}\right)=3f\left(\frac{p}{3}\right)\ or,$$

$$\sum\frac{x}{2x+5p}=\sum\frac{x}{7x+5y+5z}\leq\ 3\frac{\frac{p}{3}}{\frac{2p}{3}+5p}=\frac{3}{17}$$