ROMANIAN MATHEMATICAL MAGAZINE

JP. 546 In ΔABC the following relationship holds:

$$\sum \frac{\sin^4\frac{A}{2}+\sin^4\frac{B}{2}}{\sin^2\frac{A}{2}+\sin^2\frac{B}{2}} \geq \frac{3}{4}$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

Lemma.

If $x, y > 0$ then:

$$\sum \frac{x^4+y^4}{x^2+y^2} \geq \sum x^2$$

Proof.

$$\sum \frac{x^4+y^4}{x^2+y^2} \overset{CS}{\geq} \sum \frac{\frac{(x^2+y^2)^2}{2}}{x^2+y^2} = \frac{1}{2}\sum (x^2+y^2) = \frac{1}{2}\cdot 2\sum x^2 = \sum x^2$$

Let's get back to the main problem

Using Lemma for $(x,y)=\left(\sin\frac{A}{2}, \sin\frac{B}{2}\right)$ we obtain:

$$\sum \frac{\sin^4\frac{A}{2}+\sin^4\frac{B}{2}}{\sin^2\frac{A}{2}+\sin^2\frac{B}{2}} \overset{Lemma}{\geq} \sum \sin^2\frac{A}{2} \geq \frac{3}{4}, \text{ see}$$

$$\sum \sin^2\frac{A}{2} = 1 - \frac{r}{2R} \overset{Euler}{\geq} \frac{3}{4}$$

Equality holds if and only if the triangle is equilateral.

Solution 2 by Tapas Das-India

$$\sum \frac{\sin^4\frac{A}{2}+\sin^4\frac{B}{2}}{\sin^2\frac{A}{2}+\sin^2\frac{B}{2}} \geq \sum \frac{\left(\sin^2\frac{A}{2}+\sin^2\frac{B}{2}\right)^2}{2\left(\sin^2\frac{A}{2}+\sin^2\frac{B}{2}\right)} =$$

$$\sum (\sin^2\frac{A}{2}+\sin^2\frac{B}{2})\frac{1}{2} = \sum \sin^2\frac{A}{2} = 1 - \frac{r}{2R} \overset{Euler}{\geq} 1 - \frac{1}{4} \geq \frac{3}{4}$$

Equality holds if and only if the triangle is equilateral.