JP.547 In ΔABC:

$$\sum \frac{\cos^4\frac{A}{2}+\cos^4\frac{B}{2}}{\cos^2\frac{A}{2}+\cos^2\frac{B}{2}} \ge \frac{9r}{2R}$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

Lemma: If $x,y>0$ then:

$$\sum \frac{x^4+y^4}{x^2+y^2} \ge \sum x^2$$

Proof.

$$\sum \frac{x^4+y^4}{x^2+y^2} \overset{CS}{\ge} \sum \frac{\frac{(x^2+y^2)^2}{2}}{x^2+y^2} = \frac{1}{2}\sum (x^2+y^2) = \frac{1}{2}\cdot 2\sum x^2 = \sum x^2$$

Let's get back to the main problem.

Using Lemma for $(x,y)=\left(\cos\frac{A}{2},\cos\frac{B}{2}\right)$ we obtain:

$$\sum \frac{\cos^4\frac{A}{2}+\cos^4\frac{B}{2}}{\cos^2\frac{A}{2}+\cos^2\frac{B}{2}} \overset{Lemma}{\ge} \sum \cos^2\frac{A}{2} \ge \frac{3}{4}, \text{ see}$$

$$\sum \cos^2\frac{A}{2} = 2+\frac{r}{2R} = \frac{4R+r}{2R} \overset{Euler}{\ge} \frac{9r}{2R}$$

Equality holds if and only if the triangle is equilateral.

Solution 2 by Tapas Das-India

$$\sum \frac{\cos^4\frac{A}{2}+\cos^4\frac{B}{2}}{\cos^2\frac{A}{2}+\cos^2\frac{B}{2}} \overset{CBS}{\ge} \sum \frac{\left(\cos^2\frac{A}{2}+\cos^2\frac{B}{2}\right)^2}{2\left(\cos^2\frac{A}{2}+\cos^2\frac{B}{2}\right)} =$$

$$= \sum \cos^2\frac{A}{2} = \left(2+\frac{r}{2R}\right) = \frac{4R+r}{2R} \overset{Euler}{\ge} \frac{9r}{2R}$$

Equality holds if and only if the triangle is equilateral.

Solution 3 by Daniel Sitaru-Romania

$$\sum\frac{\cos^4\frac{A}{2}+\cos^4\frac{B}{2}}{\cos^2\frac{A}{2}+\cos^2\frac{B}{2}}=\sum\frac{\cos^4\frac{A}{2}+\cos^4\frac{B}{2}}{\cos^3\frac{A}{2}+\cos^3\frac{B}{2}}\cdot\frac{\cos^3\frac{A}{2}+\cos^3\frac{B}{2}}{\cos^2\frac{A}{2}+\cos^2\frac{B}{2}}\overset{LEHMER}{\geq}$$

$$\geq\sum_{cyc}\left(\frac{\cos^3\frac{A}{2}+\cos^3\frac{B}{2}}{\cos^2\frac{A}{2}+\cos^2\frac{B}{2}}\right)^2\overset{LEHMER}{\geq}\sum_{cyc}\left(\frac{\cos^2\frac{A}{2}+\cos^2\frac{B}{2}}{\cos\frac{A}{2}+\cos\frac{B}{2}}\right)^2\overset{LEHMER}{\geq}$$

$$\geq\sum_{cyc}\left(\frac{\cos\frac{A}{2}+\cos\frac{B}{2}}{1+1}\right)^2\overset{AM-GM}{\geq}\sum_{cyc}\cos\frac{A}{2}\cos\frac{B}{2}\overset{AM-GM}{\geq}$$

$$\geq 3\cdot\sqrt[3]{\left(\prod_{cyc}\cos\frac{A}{2}\right)^2}=3\sqrt[3]{\left(\frac{s}{4R}\right)^2}=\frac{3}{2R}\sqrt[3]{\frac{1}{2}\cdot Rs^2}\overset{EULER}{\geq}$$

$$\geq\frac{3}{2R}\sqrt[3]{rs^2}\overset{MITRINOVIC}{\geq}\frac{3}{2R}\sqrt[3]{r\cdot 27r^2}=\frac{9r}{2R}$$

Equality holds for $a=b=c$.