ROMANIAN MATHEMATICAL MAGAZINE

JP.548 If $x, y, z \in [0, \infty)$ solve the system:

$$\begin{cases} x^2 = y(y+z) \\ y^2 = z(z+x) \end{cases}$$

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Solution by proposer

If $x = 0$ or $y = 0$, or $z = 0$ it is easy to see that $x = y = z = 0$. How we can assume that $x > 0, y > 0$ and $z > 0$.

From $x^2 = y(y+z)$ we obtain that $x > y$ and from $y^2 = z(z+x)$ we obtain that $y > z$.

So $x > y > z$.

But $x < y + z$ because $x < y + z \leftrightarrow x^2 < y^2 + z^2 + 2yz \leftrightarrow y^2 + yz < y^2 + z^2 + 2yz \leftrightarrow$

$\leftrightarrow 0 < z^2 + yz$. So x, y, z are the sides of a triangle XYZ.

Let $YZ = x, XZ = y$ and $XY = z$.

$$x^2 = y^2 + yz \leftrightarrow \sin^2 X = \sin^2 Y + \sin Y \sin Z \leftrightarrow$$

$$\frac{1-\cos 2X}{2} = \frac{1-\cos 2Y}{2} + \sin Y \sin Z \leftrightarrow \cos 2Y - \cos 2X = 2\sin Y \sin Z \leftrightarrow$$

$$\leftrightarrow 2\sin(y+X)\sin(X-Y) = 2\sin Y \sin Z \leftrightarrow \sin(X-Y)\sin Y \leftrightarrow$$

$$\leftrightarrow 2\sin\frac{X-2Y}{2}\cos\frac{X}{2} = 0 \leftrightarrow \sin\left(\frac{X-2Y}{2}\right) = 0 \leftrightarrow X = 2Y$$

Because $-\frac{\pi}{2} < \frac{X-2Y}{2} < \frac{\pi}{2}$

In the same way from

$\sin^2 Y = \sin^2 Z + \sin Z \sin X$ we obtain $Y = 2Z$

So $X = 2Y, Y = 2Z$ and $X + Y + Z = \pi$. We obtain

$$X = \frac{4\pi}{7}, Y = \frac{2\pi}{7}, z = \frac{\pi}{7}$$

So $\frac{x}{\sin\frac{4\pi}{z}} = \frac{y}{\sin\frac{2\pi}{z}} = \frac{z}{\sin\frac{\pi}{z}}$ and from here

$$x = k\sin\frac{4\pi}{7}, y = k\sin\frac{2\pi}{7}, z = k\sin\frac{\pi}{7}$$