JP.549 Let ABC be a triangle with inradius r and circumradius R. Prove that:

$$\sum \frac{\sin^3 A+\sin^3 B}{\sin^5 A+\sin^5 B}\le \left(\frac{R}{r}\right)^2$$

The sum is over all cyclic permutations of (A,B,C).

Proposed by George Apostolopoulos –Greece

Solution 1 by proposer

We will prove that $\frac{a^3+b^3}{a^5+b^5}\le\frac{1}{ab}$. We have

$$a^5+b^5-ab(a^3+b^3)=a^5+b^5-a^4b-ab^4=(a-b)^2(a+b)(a^2+b^2)\ge 0$$

Similarly $\frac{b^3+c^3}{b^5+c^5}\le\frac{1}{bc}$, and $\frac{c^3+a^3}{c^5+a^5}\le\frac{1}{ca}$, where a,b,c be the lengths of the sides of the ΔABC.

Adding up the (3) inequalities, we get:

$$\sum_{cyc}\frac{a^3+b^3}{a^5+b^5}\le\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}$$

It is well-known the inequality $x^2+y^2+z^2\ge xy+yz+zx$ for $x,y,z\in\mathbb{R}$. So,

$$\sum_{cyc}\frac{a^3+b^3}{a^5+b^5}\le\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\quad (*)$$

Now, will prove that $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\le\frac{1}{4r^2}$. We have

$$(b-c)^2\ge 0\Leftrightarrow a^2-(b-c)^2\le a^2\Leftrightarrow\frac{1}{a^2}\le\frac{1}{a^2-(b-c)^2}=\frac{1}{(a+b-c)(a-b+c)}$$

Let $2s=a+b+c$, then $\frac{1}{a^2}\le\frac{1}{4(s-c)(s-b)}$. Similarly

$\frac{1}{b^2}\le\frac{1}{4(s-a)(s-c)}$, and $\frac{1}{c^2}\le\frac{1}{4(s-b)(s-a)}$.

So $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\le\frac{1}{4}\left(\frac{1}{(s-c)(s-b)}+\frac{1}{(s-a)(s-c)}+\frac{1}{(s-b)(s-a)}\right)=$

$$=\frac{1}{4}\cdot\frac{s-a+s-b+s-c}{(s-a)(s-b)(s-c)}=\frac{1}{4}\cdot\frac{3s-2s}{(s-a)(s-b)(s-c)}=\frac{1}{4}\cdot\frac{s^2}{s(s-a)(s-b)(s-c)}$$

We know (Heron) that $F=r\cdot s=\sqrt{s(s-a)(s-b)(s-c)}$ where F is the area of ΔABC.

So, $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\le\frac{s^2}{4F^2}=\frac{s^2}{4r^2s^2}=\frac{1}{4r^2}$. Namely $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\le\frac{1}{4r^2}$.

Using the law of Sines, we get from (*)

$$\sum_{cyc}\frac{(2R\sin A)^3+(2R\sin B)^3}{(2R\sin A)^5+(2R\sin B)^5}\le\frac{1}{4r^2}$$

or

$$\sum_{cyc}\frac{\sin^3 A+\sin^3 B}{\sin^5 A+\sin^5 b}\le\left(\frac{R}{r}\right)^2$$

Equality holds if and only if the triangle ABC is equilateral.

Solution 2 by Marin Chirciu-Romania

Using sine theorem we obtain:

$$LHS=\sum\frac{\sin^3 A+\sin^3 B}{\sin^5 A+\sin^5 B}=\sum\frac{\left(\frac{a}{2R}\right)^3+\left(\frac{b}{2R}\right)^3}{\left(\frac{a}{2R}\right)^5+\left(\frac{b}{2R}\right)^5}=$$

$$=4R^2\sum\frac{a^3+b^3}{a^5+b^5}\overset{(1)}{\le}4R^2\sum\frac{1}{ab}=4R^2\cdot\frac{1}{2Rr}=\frac{2R}{r}\overset{Euler}{\le}\left(\frac{R}{r}\right)^2=RHS,$$

where (1) $\Leftrightarrow\frac{a^3+b^3}{a^5+b^5}\le\frac{1}{ab}\Leftrightarrow a^5+b^5\ge ab(a^3+b^3)\Leftrightarrow(a-b)(a^4-b^4)\ge 0$, because the factors have the same sign.

Equality holds if and only if the triangle is equilateral.

Remark.

The inequality can be strengthened.

In ΔABC:

$$\sum\frac{\sin^3 A+\sin^3 B}{\sin^5 A+\sin^5 B}\le\frac{2R}{r}$$

Using the sine theorem we obtain:

$$LHS=\sum\frac{\sin^3 A+\sin^3 B}{\sin^5 A+\sin^5 B}=\sum\frac{\left(\frac{a}{2R}\right)^3+\left(\frac{b}{2R}\right)^3}{\left(\frac{a}{2R}\right)^5+\left(\frac{b}{2R}\right)^5}=4R^2\sum\frac{a^3+b^3}{a^5+b^5}\overset{(1)}{\le}4R^2\sum\frac{1}{ab}=$$

$$=4R^2\cdot\frac{1}{2Rr}=\frac{2R}{r}\overset{Euler}{\le}\left(\frac{R}{r}\right)^2=RHS,$$

where (1) $\Leftrightarrow \frac{a^3+b^3}{a^5+b^5} \le \frac{1}{ab} \Leftrightarrow a^5+b^5 \ge ab(a^3+b^3) \Leftrightarrow (a-b)(a^4-b^4) \ge 0$, because the factors have the same sign.

Equality holds if and only if the triangle is equilateral.

Remark.

We can write the inequalities:

In ΔABC:

$$\sum \frac{\sin^3 A + \sin^3 B}{\sin^5 A + \sin^5 B} \le \frac{2R}{r} \le \left(\frac{R}{r}\right)^2.$$

Using the sine theorem we obtain:

$$LHS = \sum \frac{\sin^3 A + \sin^3 B}{\sin^5 A + \sin^5 B} = \sum \frac{\left(\frac{a}{2R}\right)^3 + \left(\frac{b}{2R}\right)^3}{\left(\frac{a}{2R}\right)^5 + \left(\frac{b}{2R}\right)^5} = 4R^2 \sum \frac{a^3+b^3}{a^5+b^5} \overset{(1)}{\le} 4R^2 \sum \frac{1}{ab} =$$

$$= 4R^2 \cdot \frac{1}{2Rr} = \frac{2R}{r} \overset{Euler}{\le} \left(\frac{R}{r}\right)^2 = RHS,$$

where (1) $\Leftrightarrow \frac{a^3+b^3}{a^5+b^5} \le \frac{1}{ab} \Leftrightarrow a^5+b^5 \ge ab(a^3+b^3) \Leftrightarrow (a-b)(a^4-b^4) \ge 0$, because the factors have the same sign.

Equality hold if and only if the triangle is equilateral.

Remark.

The problem can be developed and strengthened.

If $n \in \mathbb{N}$ then in ΔABC holds:

$$\sum \frac{\sin^n A + \sin^n B}{\sin^{n+2} A + \sin^{n+2} B} \le \frac{2R}{r}$$

Marin Chirciu

Using the sine theorem:

$$LHS = \sum \frac{\sin^n A + \sin^n B}{\sin^{n+2} A + \sin^{n+2} B} = \sum \frac{\left(\frac{a}{2R}\right)^n + \left(\frac{b}{2R}\right)^n}{\left(\frac{a}{2R}\right)^{n+2} + \left(\frac{b}{2R}\right)^{n+2}} = 4R^2 \sum \frac{a^n+b^n}{a^{n+2}+b^{n+2}} \overset{(1)}{\le}$$

$$\le 4R^2 \sum \frac{1}{ab} = 4R^2 \cdot \frac{1}{2Rr} = \frac{2R}{r} = RHS,$$

where (1)

$$\Leftrightarrow \frac{a^n+b^n}{a^{n+2}+b^{n+2}} \le \frac{1}{ab} \Leftrightarrow a^{n+2}+b^{n+2} \ge ab(a^n+b^n) \Leftrightarrow (a-b)(a^{n+1}-b^{n+1}) \ge 0$$

because the factors have the same sign.

Equality holds if and only if the triangle is equilateral.

Solution 3 by Tapas Das-India

$$\sum \frac{\sin^3 A+\sin^3 B}{\sin^5 A+\sin^5 B} \overset{CEBYSHEV}{\le} \sum \frac{\sin^3 A+\sin^3 B}{\frac{1}{2}(\sin^3 A+\sin^3 B)(\sin^2 A+\sin^2 B)} =$$

$$= \sum \frac{2}{\sin^2 A+\sin^2 B} = 8R^2 \sum \frac{1}{a^2+b^2} \overset{AM-HM}{\le} 2R^2 \sum \left(\frac{1}{a^2}+\frac{1}{b^2}\right) =$$

$$= 4R^2 \sum \frac{1}{a^2} \overset{STEINING}{\le} \frac{4R^2}{4r^2} = \left(\frac{R}{r}\right)^2$$

Equality holds if and only if the triangle is equilateral.