ROMANIAN MATHEMATICAL MAGAZINE

JP.550 The non-coplanar points are given: A, B,C and D. If K is the middle of the segment $[BD]$, $(KM$ = bisector $\widehat{AKB}$, $M \in (AB)$, $(KP$ = bisector $\widehat{AKD}$, $P \in (AD)$, and $N \in (AC)$, such that $\frac{AC}{AN} - \frac{BD}{2AK} = 1$ then prove that: $AN \cdot NC + AM \cdot MB \geq 2PD \cdot (AN + AM - AP)$

Proposed by Gheorghe Molea – Romania

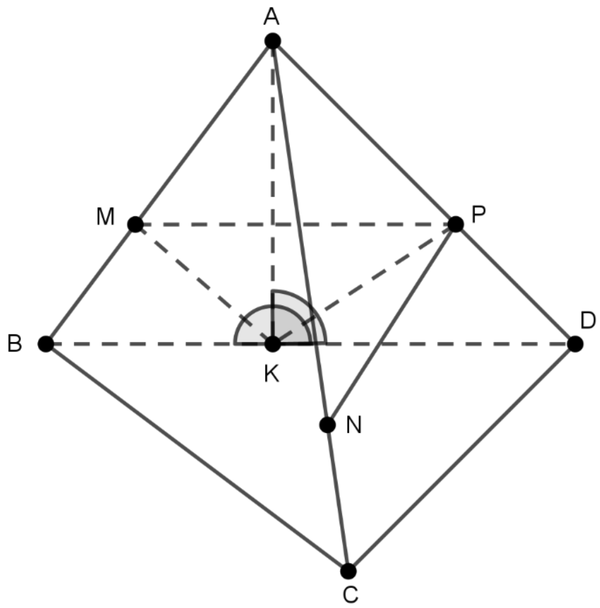
Solution 1 by proposer

$(KM$ = bisector $\widehat{AKB} \Rightarrow \frac{BK}{AK} = \frac{BM}{AM}$

$(KP$ = bisector $AKD \Rightarrow \frac{KD}{KA} = \frac{PD}{PA}$

But $[BK] \equiv [KD]$

$$\Rightarrow \frac{BM}{AM} = \frac{PD}{PA} \overset{RTT}{\Rightarrow} MP \parallel BD$$



$$\frac{AC}{AN} - \frac{BD}{2AK} = 1 \Leftrightarrow \frac{AC}{AN} - \frac{KD}{AK} = 1 \Leftrightarrow$$

$$\frac{AC}{AN} = 1 + \frac{KD}{AK} \Leftrightarrow \frac{AC}{AN} = 1 + \frac{PD}{PA} \Leftrightarrow \frac{AC}{AN} = \frac{AD}{PA} \overset{RTT}{\Rightarrow} PN \parallel CD$$

From $MP \parallel BD \Rightarrow \frac{AP}{PD} = \frac{AM}{MB} \Leftrightarrow AP \cdot MB = AM \cdot PD$

Using means inequality $m_a \geq m_g \Rightarrow \frac{AP}{AM} + \frac{MB}{PD} \geq 2\sqrt{\frac{AP \cdot MB}{AM \cdot PD}} = 2 \Rightarrow$

$$\frac{AP}{AM} + \frac{MB}{PD} \geq 2 \Rightarrow AP \cdot PD + AM \cdot MB \geq 2AM \cdot PD \quad (*)$$

From $PN \parallel DC \Rightarrow \frac{AP}{PD} = \frac{AN}{NC} \Leftrightarrow AP \cdot NC = PD \cdot AN$

From means inequality $m_a \geq m_g$ it follows: $\frac{AP}{AN} + \frac{NC}{PD} \geq 2\sqrt{\frac{AP \cdot NC}{AN \cdot PD}} = 2 \Rightarrow$

$$\frac{AP}{AN} + \frac{NC}{PD} \geq 2 \Rightarrow AP \cdot PD + AN \cdot NC \geq 2AN \cdot PD \quad (**)$$

Adding (*) and (**) we obtain:

$$2AP \cdot PD + AN \cdot NC + AM \cdot MB \geq 2PD(AN + AM)$$

$$\Leftrightarrow AN \cdot NC + AM \cdot MB \geq 2PD(AN + AM + AP)$$

We have equality $\Leftrightarrow [AB] \equiv [AC] \equiv [AD]$

Solution 2 by Marin Chirciu-Romania

Because K is the middle of $[BD] \Rightarrow \frac{BD}{2} = KB = KD \Rightarrow \frac{AC}{AN} - \frac{KB}{AK} = 1$ and $\frac{AC}{AN} - \frac{KD}{AK} = 1$.

Using bisector's theorem in ΔAKB and $\Delta AKD \Rightarrow \frac{AM}{MB} = \frac{AK}{KB}$ and $\frac{AP}{PD} = \frac{AK}{KD}$.

From $\frac{AC}{AN} - \frac{KB}{AK} = 1 \Leftrightarrow \frac{AN+NC}{AN} - \frac{KB}{AK} = 1 \Leftrightarrow 1 + \frac{NC}{AN} - \frac{KB}{AK} = 1 \Leftrightarrow \frac{NC}{AN} = \frac{KB}{AK} \Rightarrow$

$$\Rightarrow \frac{NC}{AN} = \frac{KB}{AK} \overset{T-bis}{\Rightarrow} \frac{MB}{AM} \Rightarrow \frac{NC}{AN} = \frac{MB}{AM} \overset{R-Thales}{\Rightarrow} MN \parallel BC \quad (1)$$

Analogous:

From $\frac{AC}{AN} - \frac{KD}{AK} = 1 \Leftrightarrow \frac{AN+NC}{AN} - \frac{KD}{AK} = 1 \Leftrightarrow 1 + \frac{NC}{AN} - \frac{KD}{AK} = 1 \Leftrightarrow \frac{NC}{AN} = \frac{KD}{AK} \Rightarrow$

$$\Rightarrow \frac{NC}{AN} = \frac{KD}{AK} \overset{T-bis}{\Rightarrow} \frac{PD}{AP} \Rightarrow \frac{NC}{AN} = \frac{PD}{AP} \overset{R-Thales}{\Rightarrow} PN \parallel DC \quad (2).$$

From $\frac{NC}{AN} = \frac{MB}{AM}$ and $\frac{NC}{AN} = \frac{PD}{AP} \Rightarrow \frac{MB}{AM} = \frac{PD}{AP} \overset{R-Thals}{\Rightarrow} PM \parallel BD$ (3).

From (1), (2) and (3) $\Rightarrow \Delta MNP \simeq \Delta BCD$.

Let's get to the main problem.

Dividing both members by $AM \cdot MB$ we obtain:

$$\frac{AN \cdot NC + AM \cdot MB}{AM \cdot MB} \geq \frac{2PD \cdot (AN + AM - AP)}{AM \cdot MB} \Leftrightarrow$$

$$\frac{AN}{AM}\cdot\frac{NC}{MB}+1\geq\frac{2PD}{MB}\left(\frac{AN}{AM}+1-\frac{AP}{AM}\right)$$

Using $\Delta MNP \simeq BCD \Rightarrow \frac{AN}{AM}=\frac{AC}{AB},\frac{NC}{MB}=\frac{AC}{AB},\frac{PD}{MB}=\frac{AD}{AB}$ and $\frac{AP}{AM}=\frac{AD}{AB}$

The above equalities substituted into the required inequality lead to:

$$\frac{AC}{AB}\cdot\frac{AC}{AB}+1\geq\frac{2AD}{AB}\left(\frac{AC}{AB}+1-\frac{AD}{AB}\right)\Leftrightarrow AB^2+AC^2\geq 2AD(AB+AC-AD)\Leftrightarrow$$

$$\Leftrightarrow AB^2+AC^2+2AD^2\geq 2AD(AB+AC)\Leftrightarrow$$

$$\Leftrightarrow (AB^2+AD^2)+(AC^2+AD^2)\geq 2AD(AB+AC)\Leftrightarrow (AB-AD)^2+(AC-AD)^2\geq 0,$$

with equality for $AB=AC=AD$.