JP.551 Find the real numbers x, y, z knowing that they meet the conditions:

$$x + y + z = 1; xy + (x + y)(z + 1) = \frac{4}{3}$$

Proposed by Gheorghe Molea – Romania

Solution 1 by proposer

From $x + y + z = 1$ it result successively:

$$(x + y + z)^2 = 1 \Leftrightarrow x^2 + y^2 + z^2 + 2(xy + yz + zx) = 1 \Leftrightarrow$$

$x^2 + y^2 + z^2 + 2\left(\frac{4}{3} - x - y\right) = 1$ (we have used also the second condition for the hypothesis) $\Leftrightarrow x^2 + y^2 + z^2 - 2x - 2y = -\frac{5}{3}$

$$\Leftrightarrow (x - 1)^2 + (y - 1)^2 + z^2 = \frac{1}{3} \Leftrightarrow \frac{(x-1)^2 + (y-1)^2 + z^2}{3} = \frac{1}{9} \quad (*)$$

From $x + y + z = 1 \Rightarrow (x - 1) + (y - 1) + z = -1$ (**)

It is obviously that: $(\alpha - \beta)^2 + (\alpha - \gamma)^2 + (\beta - \gamma)^2 \ge 0$

$(\forall)\alpha, \beta, \gamma$, with equality only if $\alpha = \beta = \gamma$, which becomes:

$$3(\alpha^2 + \beta^2 + \gamma^2) \ge (\alpha + \beta + \gamma)^2$$

If in this last inequality, we make the substitutions:

$\alpha = x - 1, \beta = y - 1, \gamma = z$ and we consider the relationships (*) and (**) we obtain:

$1 = 9 \cdot \frac{1}{9} = 3[(x - 1)^2 + (y - 1)^2 + z^2] \ge [(x - 1) + (y - 1) + z]^2 = (-1)^2 = 1$, from

$$x - 1 = y - 1 = z \overset{notation}{=} t \Rightarrow$$

$x = t + 1, y = t + 1, z = t$, and from the first relationship of the hypothesis it follows:

$t = -\frac{1}{3}$, so $x = \frac{2}{3}, y = \frac{2}{3}, z = -\frac{1}{3}$

Solution 2 by Marin Chirciu-Romania

Denoting $x + y = S$ and $xy = P$ we have $\begin{cases} S = 1 - z \\ P + S(z + 1) = \frac{4}{3} \end{cases} \Leftrightarrow \begin{cases} S = 1 - z \\ P = \frac{1}{3} + z^2 \end{cases}$

We form the equation of the second degree knowing the sum $S = 1 - z$ and the product

$$P = \frac{1}{3} + z^2$$

We obtain $t^2 - St + P = 0 \Leftrightarrow t^2 - (1 - z)t + \frac{1}{3} + z^2 = 0 \Leftrightarrow$

$\Leftrightarrow 3t^2 - 3(1 - z)t + 3z^2 + 1 = 0$, with

$$\Delta = 9(z-1)^2 - 12(3z^2+1) = -27t^2 - 18z - 3 = -3(3z+1)^2.$$

Because the solutions x, y of the second-degree equation must be real we must meet the condition $\Delta = -3(3z+1)^2 \geq 0 \Leftrightarrow z = \frac{-1}{3}$

Putting $z = \frac{-1}{3}$ in the equation $3t^2 - 3(1-z)t + 3z^2 + 1 = 0$ we obtain

$9t^2 - 12t + 4 = 0 \Leftrightarrow (3t-2)^2 = 0 \Leftrightarrow t = \frac{2}{3}$, wherefrom $x = y = \frac{2}{3}$.

We deduce that $(x, y, z) = \left(\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}\right)$ is the unique real solution of the system.

Remark: The problem can be developed.

Let $\lambda > 0$ fixed. Solve in real numbers x, y, z the system:

$$\begin{cases} x + y + z = \lambda \\ xy + (x+y)(z+\lambda) = \dfrac{4\lambda^2}{3} \end{cases}$$

Marin Chirciu

Solution

Denoting $x + y = S$ and $xy = P$ we have $\begin{cases} S = \lambda - z \\ P + S(z+\lambda) = \frac{4\lambda^2}{3} \end{cases} \Leftrightarrow \begin{cases} S = \lambda - z \\ P = \frac{\lambda^2}{3} + z^2 \end{cases}$

Forming the equation of second degree knowing the sum $S = \lambda - z$ and the product $P = \frac{\lambda^2}{3} + z^2$.

We obtain $t^2 - St + P = 0 \Leftrightarrow t^2 - (\lambda - z)t + \frac{\lambda^2}{3} + z^2 = 0 \Leftrightarrow$

$\Leftrightarrow 3t^2 - 3(\lambda - z)t + 3z^2 + \lambda^2 = 0$ with

$$\Delta = 9(z-\lambda)^2 - 12(3z^2+\lambda)^2 = -27t^2 - 18\lambda z - 3\lambda^2 = -3(3z+\lambda)^2$$

Because the solutions x, y of the second-degree equation must be real we put the condition

$$\Delta = -3(3z+\lambda)^2 \geq 0 \Leftrightarrow z = \frac{-\lambda}{3}$$

Putting $z = \frac{-\lambda}{3}$ in equation $3t^2 - 3(\lambda - z)t + 3z^2 + \lambda^2 = 0$ we obtain

$9t^2 - 12\lambda t + 4\lambda^2 = 0 \Leftrightarrow (3t - 2\lambda)^2 = 0 \Leftrightarrow t = \frac{2\lambda}{3}$, wherefrom $x = y = \frac{2\lambda}{3}$

We deduce that $(x, y, z) = \left(\frac{2\lambda}{3}, \frac{2\lambda}{3}, -\frac{\lambda}{3}\right)$ is the unique real solution of the system.

Note: For $\lambda = 1$ we obtain Problem JP.551 from RMM Nr.37 – Summer 2025.