ROMANIAN MATHEMATICAL MAGAZINE

JP.552 Justify if there are non-zero natural numbers a,b,c,d, different two by two, so that we have:

$$a(b+c-a)=b(a+c-b)=c(a+b-c)=\frac{a+b+c}{d}$$

Proposed by Gheorghe Molea – Romania

Solution 1 by proposer

We have $\frac{a+b+c}{d}=a(b+c-a)=b(a+c-b)=c(a+b-c)=$

$$=\frac{b+c-a}{\frac{1}{a}}=\frac{a+c-b}{\frac{1}{b}}=\frac{a+b-c}{\frac{1}{c}}=\frac{b+c-a+a+c-b+a+b-c}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}=$$

$$=\frac{a+b+c}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}\Rightarrow\frac{a+b+c}{d}=\frac{a+b+c}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}\Rightarrow\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=d$$

where $a,b,c,d\in\mathbb{N}^*$, different two by two.

We have: $d=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}<\frac{1}{1}+\frac{1}{2}+\frac{1}{3}=\frac{11}{6}<2$ but as $d>0$, it follows $d=1$. Let be $a<b<c$ but

$1=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}<\frac{1}{a}+\frac{1}{a}+\frac{1}{a}=\frac{3}{a}$ wherefrom $a<3$, but as $a>1\Rightarrow a=2$

We have: $1=\frac{1}{2}+\frac{1}{b}+\frac{1}{c}$, where $b,c\in\mathbb{N}^*, b<c$, wherefrom

$$\frac{1}{b}+\frac{1}{c}=\frac{1}{2}\Leftrightarrow bc-2b-2c=0\Leftrightarrow$$

$$\Leftrightarrow bc-2b-2c+4=4\Leftrightarrow b(c-2)-2(c-2)=4\Leftrightarrow(b-2)(c-2)=4$$

As $b<c\Rightarrow b-2=1$ and $c-2=4$, wherefrom $b=3, c=6$

We have obtained the values: $a=2,b=3,c=6,d=1$ which substituted in the relationship in the statement lead to: $14=15=-6=11$ (F), do $(\nexists)\ a,b,c,d$.

Solution 2 by Marin Chirciu-Romania

We have $a(b+c-a)=b(a+c-b)=c(a+b-c)=\frac{b+c-a}{\frac{1}{a}}=\frac{a+c-b}{\frac{1}{b}}=\frac{a+b-c}{\frac{1}{c}}=$

$$=\frac{(a+b-c)+(a+c-b)+(a+b-c)}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}=\frac{a+b+c}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}=\frac{a+b+c}{d}, \text{ if } \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=d,$$

see $(a,b,c,d)=(2,3,6,1)$, because $\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1$.

Remark: The problem can be developed.

Justify if exists non-zero natural numbers a, b, c, d, e different in pairs, such that we have:

$$a(b+c+d-a)=b(a+c+d-b)=c(a+b+d-c)=$$
$$=d(a+b+c-d)=\frac{2(a+b+c+d)}{e}$$

Marin Chirciu

Solution: **We have**

$$a(b+c+d-a)=b(a+c+d-b)=c(a+b+d-c)=d(a+b+c-d)=$$
$$=\frac{b+c+d-a}{\frac{1}{a}}=\frac{a+c+d-b}{\frac{1}{b}}=\frac{a+b+d-c}{\frac{1}{c}}=\frac{a+b+c-d}{\frac{1}{d}}=$$
$$=\frac{(b+c+d-a)+(a+c+d-b)+(a+b+d-c)+(a+b+c-d)}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}}=$$
$$=\frac{2(a+b+c+d)}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}}=\frac{2(a+b+c+d)}{e}, \text{ if } \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=e,$$

See $(a,b,c,d,e)=(2,4,6,12,1)$**, because** $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{12}=1$