ROMANIAN MATHEMATICAL MAGAZINE

JP.553 Prove that for any $a, b, c \in \mathbb{R}$, we have the inequality:

$$\frac{a^3+a}{a^4+a^2+1}+\frac{b^3+b}{b^4+b^2+1}+\frac{c^3+c}{c^4+c^2+1}\le 2$$

Proposed by Laura Molea and Gheorghe Molea – Romania

Solution 1 by proposers

If $a=b=c=0 \Rightarrow 0+0+0\le 2$ (True)

We suppose $a,b,c\ne 0$ and we will prove that $\frac{x^3+x}{x^4+x^2+1}\le\frac{2}{3}, (\forall)x\in\mathbb{R}$

We obtain: $2x^4+2x^2+2-3x^3-3x\ge 0 \Leftrightarrow$

$$2x^4+x^3+2x^2-4x^3-2x^2-4x+2x^2+x+2\ge 0\Leftrightarrow$$

$$\Leftrightarrow x^2(2x^2+x+2)-2x(2x^2+x+2)+(2x^2+x+2)\ge 0\Leftrightarrow$$

$$\Leftrightarrow (2x^2+x+2)(x-1)^2\ge 0\Leftrightarrow$$

$$\Leftrightarrow (x-1)^2\left(x^2+\frac{1}{2}x+1\right)\ge 0\Leftrightarrow$$

$$\Leftrightarrow (x-1)^2\cdot\left(x^2+\frac{1}{2}x+\frac{1}{16}+\frac{15}{16}\right)\ge 0\Leftrightarrow$$

$\Leftrightarrow (x-1)^2\cdot\left[\left(x+\frac{1}{4}\right)^2+\frac{15}{16}\right]\ge 0$, true, $(\forall)x\in\mathbb{R}$ with equality if $x=1$.

We have $\frac{a^3+a}{a^4+a^2+1}+\frac{b^3+b}{b^4+b^2+1}+\frac{c^3+c}{c^4+c^2+1}\le\frac{2}{3}\cdot 3=2$, with equality $\Leftrightarrow a=b=c=1$.

Solution 2 by Marin Chirciu-Romania

$$LHS=\sum\frac{a^3+a}{a^4+a^2+1}\overset{(1)}{\le}\sum\frac{2}{3}=2=RHS,$$

where (1) $\Leftrightarrow \frac{a^3+a}{a^4+a^2+1}\le\frac{2}{3}\Leftrightarrow 2a^4-3a^3+2a^2-3a+2\ge 0\Leftrightarrow$

$\Leftrightarrow (a-1)^2(2a^2+a+2)\ge 0$ with equality for $a=1$.

Equality holds if and only if $a=b=c=1$.

Remark: The inequality can be developed.

If $a,b,c\in\mathbb{R}$ and $-2<\lambda\le 2$ then

$$\sum\frac{a^3+a}{a^4+\lambda a^2+1}\le\frac{6}{\lambda+2}$$

Marin Chirciu

Solution:

$$LHS = \sum \frac{a^3 + a}{a^4 + \lambda a^2 + 1} \overset{(1)}{\leq} \sum \frac{2}{\lambda + 2} = \frac{6}{\lambda + 2} = RHS,$$

Where (1) $\Leftrightarrow \frac{a^3+a}{a^4+\lambda a^2+1} \leq \frac{2}{\lambda+2} \Leftrightarrow 2a^4 - (\lambda + 2)a^3 + 2\lambda a^2 - (\lambda + 2)a + 2 \geq 0 \Leftrightarrow$

$\Leftrightarrow (a - 1)^2(2a^2 + (2 - \lambda)a + 2) \geq 0$**, with equality for** $a = 1$**, see** $-2 \leq \lambda \leq 2$**.**

Equality holds if and only if $a = b = c = 1$**.**

Note: For $\lambda = 1$ **we obtain Problem JP.553 from RMM Nr.37 – Summer 2025**