ROMANIAN MATHEMATICAL MAGAZINE

JP.554 Solve the equation in integers:

$$x(x-1)^2+y(y-1)^2=x(3x+7y)$$

Proposed by Laura Molea and Gheorghe Molea – Romania

Solutions 1 by proposers

The equation becomes:

$$x^3+y^3-5x^2-2y^2+x+y-7xy=0 \Leftrightarrow$$

$$(x+y)(x^2+y^2-xy)+(x+y)-5x(x+y)-2y(x+y)=0 \Leftrightarrow$$

$$\Leftrightarrow (x+y)(x^2+y^2-xy+1-5x-2y)=0$$

We have the situations:

1. $x+y=0 \Rightarrow x=k, y=-k$, where $k\in\mathbb{Z}$ and

2. $x^2+y^2-xy+1-5x-2y=0 \Leftrightarrow$

$$\Leftrightarrow x^2-6x-xy+x+y^2-2y+1=0 \Leftrightarrow$$

$$\Leftrightarrow x(x-6)-x(y-1)+(y-1)^2=0$$

We denote $z=y-1 \Rightarrow z^2-xz+x(x-6)=0$

$$\Delta=x^2-4x(x-6)=3x(8-x)$$

$\Delta\geq 0 \Leftrightarrow x\in[0;8]$. As $x\in\mathbb{Z}\Rightarrow x\in\{0,1,2,\dots,8\}$ and we will check, which of these values leads to $z\in\mathbb{Z}$, where $z=\frac{x\pm\sqrt{3x(8-x)}}{2}$. We obtain: $x=0\Rightarrow$

$\Rightarrow z=0\Rightarrow y=1; x=2\Rightarrow z=4$ or $z=-2$

$\Rightarrow y=5$ or $y=-1; x=6\Rightarrow z=6$ or $z=0$

$\Rightarrow y=7$ or $y=1; x=8\Rightarrow z=4\Rightarrow y=5$

$\Rightarrow$

x	0	2	2	6	6	8
y	1	5	-1	7	1	5

and from 1. $x=k, y=-k$ where $k\in\mathbb{Z}$

Solution 2 by Marin Chirciu-Romania

If $x = 0 \Rightarrow y(y-1)^2 = 0 \Rightarrow y \in \{0,1\} \Rightarrow (x,y) \in \{(0,0),(0,1)\}$.

If $x \neq 0 \Rightarrow x(x-1)^2 + y(y-1)^2 = x(3x^2 + 7y) \Leftrightarrow$

$$\Leftrightarrow x^3 - 5x^2 + x(1-7y) + y(y-1)^2 = 0$$

As $x \in \mathbb{Z}$, I am looking for solutions among the dividers of the free term $y(y-1)^2$

Solution $x = -y$.

It follows that $(x,y) = (k,-k), k \in \mathbb{Z}$ are integer solutions for the equation from the enunciation.

Remark: In the same way.

Prove that the equation:

$$x(x-1)^2 + y(y-1)^2 + 4x^2 = 0,$$

has an infinity of real solutions.

Marin Chirciu

Solution:

We prove that $(x,y) = (\lambda,-\lambda), \lambda \in \mathbb{R}$ are solutions for the equation from enunciation.

The conclusion follows.