JP.555. In ΔABC we know: $m(\widehat{A})=90^\circ, AD\perp BC, D\in(BC), DE\perp AB,$ $E\in(AB), DF\perp AC, F\in(AC), BE=a, CF=b, BC=c, a,b,c>0$

Prove that $c\le 2\sqrt{a^2+b^2}$.

Proposed by Gheorghe Molea – Romania

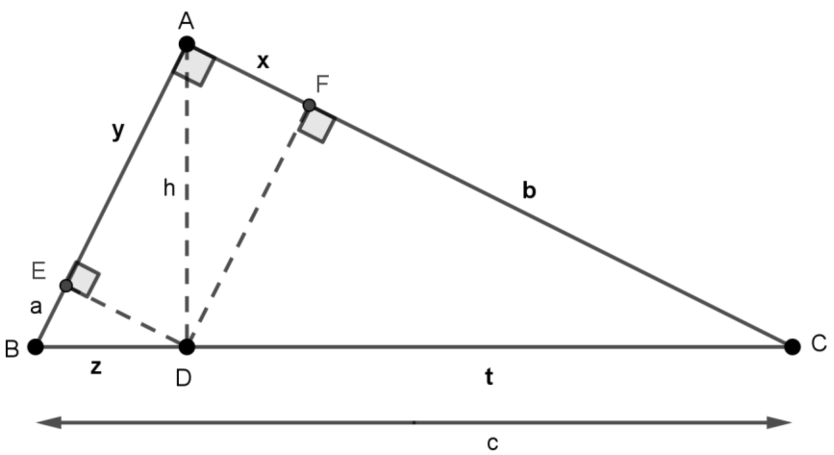
Solution 1 by proposer

We denote: $AD=h, AE=y, AF=x, BD=z, CD=t$.

We have:

$$h^2=x^2+y^2$$
$$x^2=z^2-a^2$$
$$y^2=t^2-b^2$$
$$c=z+t$$

$$\Rightarrow h^2=x^2+y^2=z^2-a^2+t^2-b^2=z^2+t^2-(a^2+b^2) \quad (*)$$



But $c^2=(z+t)^2=z^2+t^2+2zt\Rightarrow c^2=z^2+t^2+2h^2\Rightarrow z^2+t^2=c^2-2h^2$

The relationship (*) becomes:

$$h^2=c^2-2h^2-(a^2+b^2)\Rightarrow h^2=\frac{c^2-(a^2+b^2)}{3}$$

But $h\le$ the median for $A\Rightarrow h^2\le(\text{the median from } A)^2\Rightarrow\frac{c^2-(a^2+b^2)}{3}\le\left(\frac{c}{2}\right)^2$

$$\Rightarrow 4c^2-4(a^2+b^2)\le 3c^2\Rightarrow c^2\le 4(a^2+b^2)$$

$$\Rightarrow c \leq 2\sqrt{a^2+b^2}$$

We have equality $\Leftrightarrow a = b \Leftrightarrow \Delta ABC$ is an right isoscel triangle.

Solution 2 by Marin Chirciu-Romania

We denote $AF = x, AE = y \Rightarrow DE = x, DF = y$.

Using the height theorem in right triangles ABD and ADC we obtain:

$$x^2 = ay, y^2 = bx \Rightarrow x^2 + y^2 = ay + bx$$

Using Pythagoras's theorem in $\Delta ABC \Rightarrow c^2(a+y)^2 + (b+x)^2$.

The inequality we must prove $c \leq 2\sqrt{a^2+b^2} \Leftrightarrow c^2 \leq 4(a^2+b^2) \Leftrightarrow$

$$\Leftrightarrow (a+y)^2 + (b+x)^2 \leq 4(a^2+b^2) \Leftrightarrow a^2 + y^2 + 2ay + b^2 + x^2 + 2bx \leq 4(a^2+b^2)$$

$$\Leftrightarrow (x^2+y^2) + 2ay + 2bx \leq 3(a^2+b^2) \overset{x^2+y^2=ay+bx}{\Leftrightarrow}$$

$$\Leftrightarrow (ay+bx) + 2ay + 2bx \leq 3(a^2+b^2) \Leftrightarrow$$

$$\Leftrightarrow 3(ay+bx) \leq 3(a^2+b^2) \Leftrightarrow (ay+bx) \leq (a^2+b^2) \overset{x^2+y^2=ay+bx}{\Leftrightarrow}$$

$$\Leftrightarrow (ay+bx)^2 \leq (a^2+b^2)(y^2+x^2)$$

see CBS inequality, with equality for $ax = by$.