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NAGEL'S AND GERGONNE'S CEVIANS - APPLICATIONS AND RESULTS

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Extended Abstract
This paper is a new study in the field of geometry of triangle involving Nagel and Gergonne cevians. We obtain new identities and inequalities involving those two types of cevians using very well-known relationships in triangle geometry involving other elements of a triangle. After obtaining new identities involving those type of cevians, we obtain new inequalities using well known inequalitys both geometric and algebraic.
Those type of cevians as we will see are very close related with very well-known elements of triangle, which helps us to manipulate the expressions we obtain for a better form.
This paper is not the first written by me involving Nagel and Gergonne's cevians, but the results are new, we also use previous results in papers written by me with this topic for keeping fresh the interest of the readers, sometimes the expressions obtain are not friendly and they must be manipulated to obtain a better form, for using in future research. This paper like others was born from passion and curiosity for obscure elements which in school of math from Europe and USA was not studied so much. This topic is very rich and as we will see, we can obtain hundreds of results, even more.
I hope this topic will be researched in the future and new results will be obtained.

Keywords: Nagel's and Gergonne's cevians, geometric inequalities, identities in triangle

1. Research methodology

In this paper we will present applications and results with Nagel and Gergonne cevians. Results about those type of cevians are very rare in Europe and USA math school. We will present the new connections of those cevians with very well-known elements of a triangle. These new results will lead to obtain new geometric inequalities in triangle and inequalities related with already very well-known.

We consider triangle ABC with sides $BC = a$, $AC = b$, $AB = c$ and $\mathrm{p} = \frac{1}{2}(a+b+c)$ circumradius **R**, inradius r, l_a, l_b, l_c: the angle-bisectors; r_a, r_b, r_c the radii of excircles; m_a, m_b, m_c: the medians; h_a, h_b, h_c: the altitudes; n_a, g_a - cevians of Nagel and Gergonne from A to BC (and analogs);

We know that: $\mathrm{g}_\mathrm{a}^2 = (\mathrm{p}-\mathrm{a})\left[\mathrm{p} - \frac{(\mathrm{b}-\mathrm{c})^2}{\mathrm{a}}\right]$ (and analogs) [1;3]

$4r_a r = 4(p-b)(p-c) = a^2 - (b-c)^2$ (and analogs), we will obtain:

$\mathrm{g}_\mathrm{a}^2 - (\mathrm{p}-\mathrm{a})^2 = \frac{4\mathrm{r}_\mathrm{a}\mathrm{r}(\mathrm{p}-\mathrm{a})}{\mathrm{a}}$ (and analogs), but we know that $\mathrm{r}_\mathrm{a} = \frac{\mathrm{S}}{\mathrm{p}-\mathrm{a}}$ (and analogs),
$2\mathrm{S} = \mathrm{a}\mathrm{h}_\mathrm{a} = 2\mathrm{pr}$ and after banal computations we obtain a new formula:

$$\mathbf{g}_\mathbf{a}^2 = (\mathbf{p}-\mathbf{a})^2 + 2\mathbf{r}\mathbf{h}_\mathbf{a} \text{ (and analogs)} \qquad (1)$$

From $\mathrm{r}_\mathrm{a} = \frac{\mathrm{S}}{\mathrm{p}-\mathrm{a}}$ (and analogs) $\rightarrow \frac{\mathrm{p}-\mathrm{a}}{\mathrm{r}} = \frac{\mathrm{p}}{\mathrm{r}_\mathrm{a}}$ (and analogs)

From S^2 =p(p-a)(p-b)(p-c) (Heron) and $\mathrm{ctg}\frac{\mathrm{A}}{2} = \sqrt{\frac{\mathrm{p}(\mathrm{p}-\mathrm{a})}{(\mathrm{p}-\mathrm{b})(\mathrm{p}-\mathrm{c})}}$ (and analogs) we will obtain
$\mathrm{ctg}\frac{\mathrm{A}}{2} = \frac{\mathrm{p}-\mathrm{a}}{\mathrm{r}} = \frac{\mathrm{p}}{\mathrm{r}_\mathrm{a}}$ (and analogs); From (1) and this identity we obtain:

$$\frac{g_a^2}{r^2}=ctg^2\frac{A}{2}+\frac{2h_a}{r}\text{ (and analogs)}\qquad(2)$$

Now we use $p^2=n_a^2+2r_ah_a$ (and analogs) [2] and we obtain:

$$\left(\frac{g_a}{r}\right)^2=\left(\frac{n_a}{r_a}\right)^2+\frac{2h_a}{r}+\frac{2h_a}{r_a}\text{ (and analogs)}\qquad(3)$$

From $r_a=\frac{2S}{2(p-a)}=\frac{ah_a}{b+c-a}\to\frac{r_a}{h_a}=\frac{a}{b+c-a}\to\frac{h_a}{r_a}=\frac{b+c}{a}-1$ (and analogs)

$ah_a=2pr=(a+b+c)r\to\frac{h_a}{r}=1+\frac{b+c}{a}$ (and analogs)

$\frac{2h_a}{r}+\frac{2h_a}{r_a}=2\left(\frac{b+c}{a}-1+1+\frac{b+c}{a}\right)=4\frac{b+c}{a}$ (and analogs)

From (3) we obtain a new result:

$$\left(\frac{g_a}{r}\right)^2=\left(\frac{n_a}{r_a}\right)^2+4\frac{b+c}{a}\text{ (and analogs)}\qquad(4)$$

After summation we obtain new identities:

$$\frac{g_a^2+g_b^2+g_c^2}{r^2}=\sum ctg^2\frac{A}{2}+\frac{2(h_a+h_b+h_c)}{r}\qquad(5)$$

$$\frac{g_a^2+g_b^2+g_c^2}{r^2}=\sum\left(\frac{n_a}{r_a}\right)^2+4\sum\frac{b+c}{a}\qquad(6)$$

From $ctg^2\frac{A}{2}=\left(\frac{p}{r_a}\right)^2\frac{2}{2}=\frac{n_b^2+n_c^2+2h_br_c+2h_cr_c}{2r_a^2}\geq\frac{2(n_bn_c+h_br_c+h_cr_c)}{2r_a^2}$ we obtain:

$$ctg^2\frac{A}{2}\geq\frac{n_bn_c+h_br_c+h_cr_c}{r_a^2}\text{ (and analogs)}\qquad(7)$$

From (2) and (7) we obtain: $\left(\frac{g_a}{r}\right)^2\geq\frac{n_bn_c+h_br_c+h_cr_c}{r_a^2}+\frac{2h_a}{r}$ (and analogs) (8)

From $2rh_a=g_a^2-(p-a)^2=(g_a+p-a)(g_a+a-p)$ we obtain:

$\frac{h_a}{g_a+a-p}=\frac{g_a+p-a}{2r}$ (and analogs), and after summation we obtain:

$$\sum\frac{h_a}{g_a+a-p}=\frac{g_a+g_b+g_c+p}{2r}\qquad(9)$$

$\frac{h_a}{g_a+p-a}=\frac{g_a+a-p}{2r}$ (and analogs), after summation we obtain:

$$\sum\frac{h_a}{g_a+p-a}=\frac{g_a+g_b+g_c-p}{2r}\qquad(10)$$

From [4] we have: $\frac{p}{r}=\sum\frac{n_a}{h_a}+2\sum\frac{r_a}{n_a+p}$ and $\frac{3p}{r}=\frac{n_a+n_b+n_c}{r}+2\sum\frac{2r_a+h_a}{n_a+p}$ and using (9) and (10) we obtain new results:

$$2\sum\frac{h_a}{g_a+p-a}=\frac{g_a+g_b+g_c}{r}-\left(\sum\frac{n_a}{h_a}+2\sum\frac{r_a}{n_a+p}\right)\qquad(11)$$

$$2\sum\frac{h_a}{g_a+p-a}=\frac{g_a+g_b+g_c}{r}-\left(\frac{n_a+n_b+n_c}{3r}+\frac{2}{3}\sum\frac{2r_a+h_a}{n_a+p}\right)\qquad(12)$$

$$2\sum\frac{h_a}{g_a+a-p}=\frac{g_a+g_b+g_c}{r}+\sum\frac{n_a}{h_a}+2\sum\frac{r_a}{n_a+p}\qquad(13)$$

$$2\sum\frac{h_a}{g_a+a-p}=\frac{g_a+g_b+g_c}{r}+\frac{n_a+n_b+n_c}{3r}+\frac{2}{3}\sum\frac{2r_a+h_a}{n_a+p}\qquad(14)$$

From $p^2=n_a^2+2r_ah_a$ (and analogs) we have: $2r_ah_a=(p+n_a)(p-n_a)$ and

$p-n_a=\frac{2r_ah_a}{p+n_a}\rightarrow\frac{p-n_a}{r_a}=\frac{2h_a}{p+n_a}$ (and analogs) also we use $\sum\frac{1}{r_a}=\frac{1}{r}$ we obtain:

$$\frac{\mathbf{p}}{\mathbf{r}}=\sum\frac{\mathbf{n_a}}{\mathbf{r_a}}+\sum\frac{\mathbf{2h_a}}{\mathbf{p+n_a}} \tag{15}$$

and using (9) and (10) we obtain:

$$2\sum\frac{\mathbf{h_a}}{\mathbf{g_a+p-a}}=\frac{\mathbf{g_a+g_b+g_c}}{\mathbf{r}}-\left(\sum\frac{\mathbf{n_a}}{\mathbf{r_a}}+\sum\frac{\mathbf{2h_a}}{\mathbf{p+n_a}}\right) \tag{16}$$

$$2\sum\frac{\mathbf{h_a}}{\mathbf{g_a+a-p}}=\frac{\mathbf{g_a+g_b+g_c}}{\mathbf{r}}+\sum\frac{\mathbf{n_a}}{\mathbf{r_a}}+\sum\frac{\mathbf{2h_a}}{\mathbf{p+n_a}} \tag{17}$$

From (1) we obtain: $\left(\frac{g_a}{h_a}\right)^2=\left(\frac{p-a}{h_a}\right)^2+\frac{2r}{h_a}$ (and analogs); We will use the well-known relations: $\sum\frac{1}{h_a}=\frac{1}{r}$; $r_br_c=p(p-a)$ (and analogs); $2r_br_c=h_a(r_b+r_c)$ (and analogs); $\sum r_br_c=p^2$; and we have: $\frac{p-a}{h_a}=\frac{r_b+r_c}{2p}$ (and analogs) and after simple manipulation we obtain: $\sum\left(\frac{p-a}{h_a}\right)^2=\frac{1}{2}+\frac{r_a^2+r_b^2+r_c^2}{2p^2}$ and $\sum\left(\frac{g_a}{h_a}\right)^2=2+\sum\left(\frac{p-a}{h_a}\right)^2$.

We will obtain:

$$\sum\left(\frac{\mathbf{g_a}}{\mathbf{h_a}}\right)^2=\frac{5}{2}+\frac{\mathbf{r_a^2+r_b^2+r_c^2}}{\mathbf{2p^2}} \tag{18}$$

From (18) we can write: $\sum\left(\frac{g_a}{h_a}\right)^2=\frac{5}{2}+\frac{r_a^2+r_b^2+r_c^2}{2p^2}+\frac{1}{2}-\frac{1}{2}=3+\frac{r_a^2+r_b^2+r_c^2-p^2}{2p^2}$

Is easy to see $r_a^2+r_b^2+r_c^2-p^2=r_a^2+r_b^2+r_c^2-\sum r_br_c$ and we obtain:

$$r_a^2+r_b^2+r_c^2-\sum r_br_c=\frac{1}{2}[(r_a-r_b)^2+(r_c-r_b)^2+(r_c-r_a)^2]$$

$$\sum\left(\frac{\mathbf{g_a}}{\mathbf{h_a}}\right)^2=\mathbf{3}+\frac{\mathbf{1}}{\mathbf{4p^2}}[(\mathbf{r_a}-\mathbf{r_b})^2+(\mathbf{r_c}-\mathbf{r_b})^2+(\mathbf{r_c}-\mathbf{r_a})^2] \tag{19}$$

From well-known relation: $r_ar_br_c=Sp=p^2r$ and $2r_br_c=h_a(r_b+r_c)$ (and analogs) $\rightarrow\frac{r_ah_a(r_b+r_c)}{2r}=p^2$ (and analogs). From $\frac{r_ah_a(r_b+r_c)}{2r}=p^2$ and $p^2=n_a^2+2r_ah_a$ (and analogs) we obtain:

$$\mathbf{n_a^2}=\mathbf{r_ah_a}\left(\frac{\mathbf{r_b+r_c}}{\mathbf{2r}}-\mathbf{2}\right)\ \textbf{(and analogs)} \tag{20}$$

From $r_a+r_b+r_c=4R+r$ and (20) we obtain:

$$\sum\frac{\mathbf{n_a^2}}{\mathbf{r_ah_a}}=\frac{\mathbf{4R}}{\mathbf{r}}-\mathbf{5} \tag{21}$$

From $h_a=\left(1+\frac{b+c}{a}\right)r$ (and analogs) and $\frac{h_a}{r_a}=\frac{b+c}{a}-1$ (and analogs) we obtain

$r_ah_a=(2r_a+h_a)r$ (and analogs). From $r_ah_a=(2r_a+h_a)r$ (and analogs) and (20) we obtain:

$$\mathbf{n_a^2}=(\mathbf{2r_a}+\mathbf{h_a})\left(\frac{\mathbf{r_b+r_c}}{\mathbf{2}}-\mathbf{2r}\right)\ \textbf{(and analogs)} \tag{22}$$

From $m_al_a\geq p(p-a)=r_br_c$ (Panaitopol inequality) [5] and $2r_br_c=h_a(r_b+r_c)$ (and analogs) we obtain $\frac{m_al_a}{h_a}\geq\frac{r_b+r_c}{2}$ (and analogs) and from (22) we obtain:

$$(\mathbf{2r_a}+\mathbf{h_a})\left(\frac{\mathbf{m_al_a}}{\mathbf{h_a}}-\mathbf{2r}\right)\geq\mathbf{n_a^2}\ \textbf{(and analogs)} \tag{23}$$

We will use now well-known inequality $\sqrt{\frac{R}{2r}} \geq \frac{l_a}{h_a}$ (and analogs) (for proof see [6]) and using (23) we obtain:

$$(2r_a + h_a)\left(m_a\sqrt{\frac{R}{2r}} - 2r\right) \geq n_a^2 \text{ (and analogs)} \tag{24}$$

From (23) and (24) after summation we obtain two new inequalities:

$$\sum(2r_a + h_a)\left(\frac{m_a l_a}{h_a} - 2r\right) \geq \sum n_a^2 \tag{25}$$

$$\sum(2r_a + h_a)\left(m_a\sqrt{\frac{R}{2r}} - 2r\right) \geq \sum n_a^2 \tag{26}$$

From (1) and $abc = 4RS$; $a^2 + b^2 + c^2 = 2(p^2 - 4Rr - r^2)$;

$a^3 + b^3 + c^3 = 2p(p^2 - 6Rr - 3r^2)$ and $2S = ah_a = 2pr$ (and analogs) we have:

$$\sum \frac{g_a^2}{h_a} = 6r + \sum \frac{(p-a)^2}{h_a} \text{ and } \sum \frac{g_a^2}{h_a} = 2R + 5r \tag{27}$$

From (27) and Bergstrom inequality: if x_k-real numbers and $a_k > 0$, $k \in \{1,2,\dots,n\}$ then $\frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + \dots + \frac{x_n^2}{a_n} \geq \frac{(x_1+x_2+\dots+x_n)^2}{a_1+a_2+\dots+a_n}$ with equality only if $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \dots = \frac{x_n}{a_n}$, we obtain: $2R + 5r \geq \frac{(g_a+g_b+g_c)^2}{h_a+h_b+h_c}$ wich can be written as:

$$\sqrt{(2R + 5r)(h_a + h_b + h_c)} \geq g_a + g_b + g_c \tag{28}$$

From $g_a \geq h_a$(and analogs) and (28) we obtain:

$$2R + 5r \geq g_a + g_b + g_c \tag{29}$$

From (28) and (9), (10) we obtain two new inequalitys:

$$\sum \frac{h_a}{g_a + a - p} \leq \frac{p + \sqrt{(2R+5r)(h_a+h_b+h_c)}}{2r} \tag{30}$$

$$\sum \frac{h_a}{g_a + p - a} \leq \frac{-p + \sqrt{(2R+5r)(h_a+h_b+h_c)}}{2r} \tag{31}$$

From (4), (8) and (28) we obtain another two inequalitys:

$$\sum \sqrt{\left(\frac{n_a}{r_a}\right)^2 + 4\frac{b+c}{a}} \leq \frac{\sqrt{(2R+5r)(h_a+h_b+h_c)}}{r} \tag{32}$$

$$\sum \sqrt{\frac{n_b n_c + h_b r_c + h_c r_c}{r_a^2} + \frac{2h_a}{r}} \leq \frac{\sqrt{(2R+5r)(h_a+h_b+h_c)}}{r} \tag{33}$$

We use now Wolstenholme's inequality: if x, y, z are real numbers, A, B, C angles of triangle ABC then we have:

$x^2 + y^2 + z^2 \geq 2xy\cos C + 2yz\cos A + 2zx\cos B$ with equality if and only if: $\frac{x}{\sin A} = \frac{y}{\sin B} = \frac{z}{\sin C}$ [7] and (25) and (26) and we obtain:

$$\sum(2r_a + h_a)\left(\frac{m_a l_a}{h_a} - 2r\right) \geq 2n_b n_c \cos A + 2n_a n_c \cos B + 2n_a n_b \cos C \tag{34}$$

$$\sum(2r_a + h_a)\left(m_a\sqrt{\frac{R}{2r}} - 2r\right) \geq 2n_b n_c \cos A + 2n_a n_c \cos B + 2n_a n_b \cos C \tag{35}$$

Using Wolstenholme's inequality and $\frac{3p}{r} = \frac{n_a+n_b+n_c}{r} + 2\sum\frac{2r_a+h_a}{n_a+p}$ we obtain:

$$\mathbf{3p \geq 2\sum \cos A\sqrt{n_b n_c} + 2r\sum\frac{2r_a+h_a}{n_a+p}} \quad \textbf{(36)}$$

From $\frac{p}{r} = \sum\frac{n_a}{h_a} + 2\sum\frac{r_a}{n_a+p}$ and Wolstenholme's inequality we obtain:

$$\mathbf{\frac{p}{r} \geq 2\sum \cos A\sqrt{\frac{n_b n_c}{h_b h_c}} + 2\sum\frac{r_a}{n_a+p}} \quad \textbf{(37)}$$

From (6) and Wolstenholme's inequality we obtain:

$$\mathbf{\frac{g_a^2+g_b^2+g_c^2}{r^2} \geq 2\sum\frac{n_b n_c}{r_b r_c}\cos A + 4\sum\frac{b+c}{a}} \quad \textbf{(38)}$$

$$\mathbf{\sum\left(\frac{n_a}{r_a}\right)^2 + 4\sum\frac{b+c}{a} \geq 2\sum\frac{g_b g_c}{r^2}\cos A} \quad \textbf{(39)}$$

From (15) and Wolstenholme's inequality we obtain:

$$\mathbf{\frac{p}{r} \geq 2\sum \cos A\sqrt{\frac{n_b n_c}{r_b r_c}} + 2\sum\frac{h_a}{n_a+p}} \quad \textbf{(40)}$$

From (27) and Wolstenholme's inequality we obtain:

$$\mathbf{2R + 5r \geq 2\sum\frac{g_b g_c}{\sqrt{h_b h_c}}\cos A} \quad \textbf{(41)}$$

From (18) and Wolstenholme's inequality we obtain:

$$\mathbf{\frac{5}{2} + \frac{r_a^2+r_b^2+r_c^2}{2p^2} \geq 2\sum\frac{g_b g_c}{h_b h_c}\cos A} \quad \textbf{(42)}$$

The last result presented is the following:

$\mathbf{l_a \leq m_a \leq p_a \leq n_a}$ **(and analogs)** **(43)**

p_a-Spieker cevian from A to BC

For the demonstration of this result, we will use this theorem: Points I, G, S_P, N_a are colinear, line that passes through these points is called Nagel line.[8]

I (incenter), G (triangle centroid), S_P (Spieker center), N_a (Nagel point)

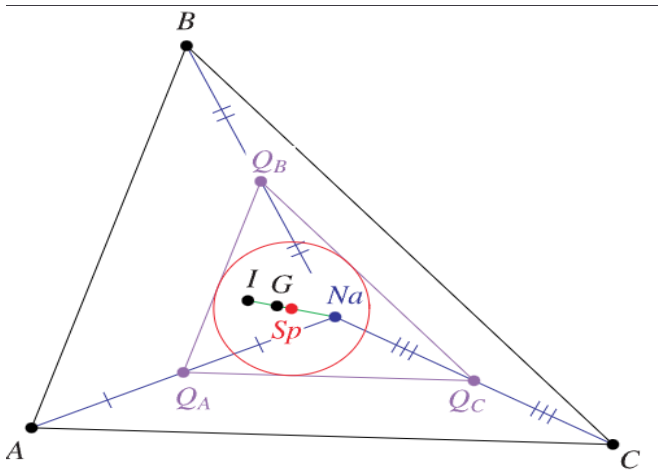

Figure 1.

2. Conclusion

The field of geometry is full of surprises and we can find new connections between well known elements of a triangle.

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