RMM - Geometry Marathon 1601 - 1700

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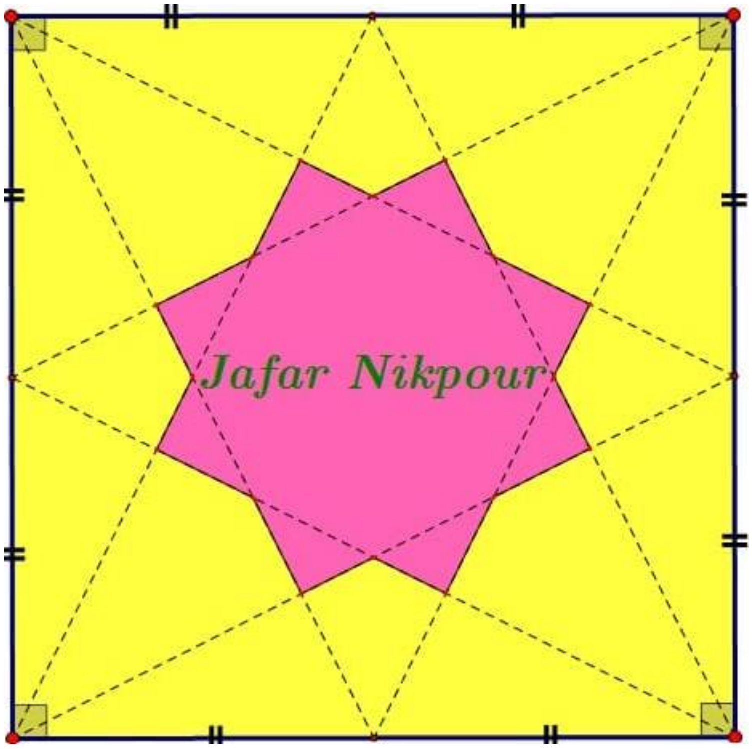
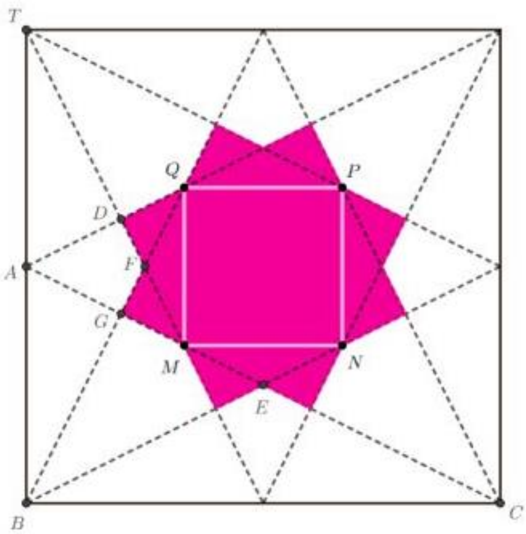
1601.



Prove that 7yellow=23pink

Proposed by Jafar Nikpour-Iran

Solution by Eric-Dimitrie Cismaru-Romania



Let x be the length of the sides of the square. It is sufficient to show that $S=\frac{7x^2}{30}$. Because $MNPQ$ is a square, $S=MN^2+4\cdot(\mathcal{A}_{\Delta AQM}-\mathcal{A}_{ADFG})$. By Menelaus's Theorem in ΔTBS (where $BS=SC$), with $A-M-C$, we have:

$$\frac{TA}{BA}\cdot\frac{BC}{SC}\cdot\frac{SM}{TM}=1\Leftrightarrow\frac{AM}{CM}=\frac{1}{2}\Rightarrow AM=AQ=\frac{AC}{3}=\frac{x\sqrt{5}}{6}$$

Then, $GM=AM-AG=\frac{x\sqrt{5}}{6}-\frac{x\sqrt{5}}{10}=\frac{x\sqrt{5}}{15}$, and $GM+GQ=\frac{x\sqrt{5}}{5}\Rightarrow GQ=\frac{2x\sqrt{5}}{15}$, so we obtain that $MQ=\sqrt{GM^2+GQ^2}=\frac{\pi}{3}$

We also have $\frac{AG}{AM}=\frac{d(A;GD)}{d(A;MQ)}=\frac{3}{5}\Rightarrow d(A;GD)=\frac{x}{5}\Rightarrow d(G;MQ)=\frac{2x}{15}$, yielding to $GD=\frac{x}{5}$.

Finally, we have:

$$S=MN^2+4\cdot(\mathcal{A}_{\Delta AQM}-\mathcal{A}_{ADFG})\Leftrightarrow S=\frac{x^2}{9}+4\cdot\left(\frac{x^2}{18}-\frac{AF\cdot GD}{2}\right),$$

Leading us to $S=\frac{x^2}{9}+4\cdot\left(\frac{x^2}{18}-\frac{x^2}{40}\right)=\frac{7x^2}{30}$, so the proof is finished.

1602. In any Δ ABC, the following relationship holds:

$$\frac{4a^2+b^2+c^2}{2a^2+bc}\le\frac{R}{r}$$

Proposed by Adil Abdullayev-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$R-2r\overset{?}{\ge}\frac{b^2+c^2}{4R}-\frac{bc}{2R}$$

$$\Leftrightarrow R\left(1-\frac{2r}{R}\right)\overset{?}{\ge}\frac{4R^2(\sin^2B+\sin^2C)}{4R}-\frac{4R^2\sin B\sin C}{2R}$$

$$\Leftrightarrow 1-\frac{8R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{R}\overset{?}{\ge}\sin^2B+\sin^2C-2\sin B\sin C=(\sin B-\sin C)^2$$

$$\Leftrightarrow 1-4\sin\frac{A}{2}\left(2\sin\frac{B}{2}\sin\frac{C}{2}\right)\overset{?}{\ge}\left(2\cos\frac{B+C}{2}\sin\frac{B-C}{2}\right)^2$$

$$\Leftrightarrow 1-4\sin\frac{A}{2}\left(\cos\frac{B-C}{2}-\cos\frac{B+C}{2}\right)\overset{?}{\ge}4\sin^2\frac{A}{2}\left(1-\cos^2\frac{B-C}{2}\right)$$

$$\Leftrightarrow 1-4\sin\frac{A}{2}\cos\frac{B-C}{2}+4\sin^2\frac{A}{2}\overset{?}{\ge}4\sin^2\frac{A}{2}-4\sin^2\frac{A}{2}\cos^2\frac{B-C}{2}$$

$$\Leftrightarrow 4\sin^2\frac{A}{2}\cos^2\frac{B-C}{2}-4\sin\frac{A}{2}\cos\frac{B-C}{2}+1\overset{?}{\ge}0\Leftrightarrow\left(2\sin\frac{A}{2}\cos\frac{B-C}{2}-1\right)^2\overset{?}{\ge}0$$

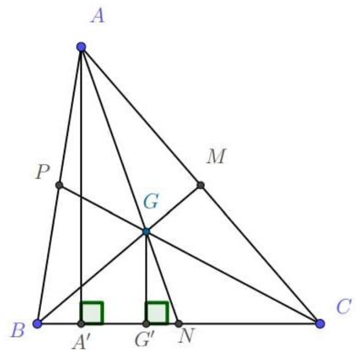
$$\rightarrow \text{true} \Rightarrow (b-c)^2 \le 4R(R-2r) \therefore \frac{4a^2+b^2+c^2-(4a^2+2bc)}{4a^2+2bc} \le \frac{4R(R-2r)}{4a^2+2bc}$$
$$\overset{?}{\le} \frac{R}{2r} - 1 = \frac{R-2r}{2r} \Leftrightarrow 8Rrs \overset{?}{\le} s(4a^2+2bc) \left(\because R-2r \overset{\text{Euler}}{\ge} 0\right)$$
$$\Leftrightarrow 2abc \overset{?}{\le} (a+b+c)(2a^2+bc)$$
$$\Leftrightarrow 2a^3+abc+2a^2b+b^2c+2a^2c+bc^2 \overset{?}{\ge} 2abc$$
$$\Leftrightarrow 2a^3+2a^2b+b^2c+2a^2c+bc^2 \overset{?}{\ge} abc \rightarrow \text{true} \because a^3+b^2c+bc^2 \overset{A-G}{\ge} 3abc$$
$$> abc \Rightarrow 2a^3+2a^2b+b^2c+2a^2c+bc^2 > abc$$
$$\therefore \frac{4a^2+b^2+c^2-(4a^2+2bc)}{4a^2+2bc} \le \frac{R}{2r} - 1 \Rightarrow \frac{4a^2+b^2+c^2}{4a^2+2bc} - 1 \le \frac{R}{2r} - 1$$
$$\Rightarrow \frac{4a^2+b^2+c^2}{2a^2+bc} \le \frac{R}{r} \ \forall \Delta ABC, '' = '' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1603. In any Δ ABC, the following relationship holds :

$$m_b + m_c \ge \sqrt{h_a^2 + \frac{9a^2}{4}}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Eric Cismaru-Romania



Let G be the centroid of triangle ΔABC and let M, N, P be the midpoints of sides AC, BC, AB. Construct $GG' \perp BC$ and $AA' \perp BC$.

Because $AA' \parallel GG' \Leftrightarrow \Delta GG'P \sim \Delta AA'P \Rightarrow GG' = \frac{h_a}{3}$.

Let's apply Pythagoras Theorem in triangles $\Delta BGG'$ and $\Delta GG'C$.

We obtain that $GB^2 = \frac{4m_b^2}{9} = \frac{h_a^2}{9} + BG'^2$ and $GC^2 = \frac{4m_c^2}{9} = \frac{h_a^2}{9} + CG'^2$ and by taking the square root and adding we'll have

$$\frac{2}{3}(m_b+m_c)=\sqrt{\frac{h_a^2}{9}+BG'^2}+\sqrt{\frac{h_a^2}{9}+BG'^2}=\sqrt{\left(\frac{h_a}{3}\right)^2+BG'^2}+\sqrt{\left(\frac{h_a}{3}\right)^2+CG'^2}$$

Using Minkowski's Inequality,

$$\frac{2}{3}(m_b+m_c)\geq\sqrt{\left(\frac{2h_a}{3}\right)^2+(BG'+CG')^2}=\sqrt{\frac{4h_a^2}{9}+a^2},$$

which is equivalent to $m_b+m_c\geq\sqrt{h_a^2+\frac{9a^2}{4}}$.

Equality holds when $BG'=CG'$ or when G' is the midpoint of BC, meaning that

$$G'=N\Leftrightarrow AN\perp BC\Leftrightarrow AB=AC$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$m_b^2+m_c^2=\frac{2c^2+2a^2-b^2+2a^2+2b^2-c^2}{4}=\frac{1}{4}\sum_{cyc}a^2+\frac{3a^2}{4}$$

$$\Rightarrow\frac{9a^2}{4}=3(m_b^2+m_c^2)-\frac{3}{4}\sum_{cyc}a^2=3(m_b^2+m_c^2)-\sum_{cyc}m_a^2$$

$$\Rightarrow\frac{9a^2}{4}=2(m_b^2+m_c^2)-m_a^2\rightarrow(1)$$

$$\therefore m_b+m_c\geq\sqrt{h_a^2+\frac{9a^2}{4}}\overset{\text{via }(1)}{\Leftrightarrow}m_b^2+m_c^2+2m_bm_c\geq h_a^2+2(m_b^2+m_c^2)-m_a^2$$

$$\Leftrightarrow\boxed{m_a^2-h_a^2\overset{(*)}{\geq}(m_b-m_c)^2}$$

$$\text{Now, }m_a^2-h_a^2=s(s-a)+\frac{(b-c)^2}{4}-\frac{4s(s-a)(s-b)(s-c)}{a^2}$$

$$=s(s-a)+\frac{(b-c)^2}{4}-\frac{s(s-a)(a^2-(b-c)^2)}{a^2}=\frac{(b-c)^2}{4}+\frac{s(s-a)(b-c)^2}{a^2}$$

$$=\frac{(b-c)^2}{4a^2}(a^2+4s^2-4sa)=\frac{(b-c)^2(2s-a)^2}{4a^2}=\frac{(b-c)^2(b+c)^2}{4a^2}$$

$$\Rightarrow m_a^2-h_a^2=\frac{(b^2-c^2)^2}{4a^2}\rightarrow(2)\therefore\text{via }(2),(*)\Leftrightarrow\boxed{\frac{(b^2-c^2)^2}{4a^2}\overset{(**)}{\geq}(m_b-m_c)^2}$$

$$\text{Again, }(m_b^2-m_c^2)^2\overset{?}{\geq}\frac{9m_a^2}{4}(b-c)^2$$

$$\Leftrightarrow\left(\frac{(2c^2+2a^2-b^2)-(2a^2+2b^2-c^2)}{4}\right)^2\overset{?}{\geq}\frac{9(2b^2+2c^2-a^2)}{16}(b-c)^2$$

$$\Leftrightarrow 9(b^2-c^2)^2\overset{?}{\geq}9(2b^2+2c^2-a^2)(b-c)^2$$

$$\Leftrightarrow (b-c)^2\left((b+c)^2-\left((b+c)^2+(b-c)^2-a^2\right)\right)\overset{?}{\geq}0$$

$$\Leftrightarrow (b-c)^2\left(a^2-(b-c)^2\right)\overset{?}{\geq}0 \Leftrightarrow (b-c)^2.4(s-b)(s-c)\overset{?}{\geq}0 \rightarrow \text{true}$$

$$\therefore \boxed{\left(m_b^2-m_c^2\right)^2 \geq \frac{9m_a^2}{4}(b-c)^2} \rightarrow (3)$$

Implementing (3) on a triangle with sides $\frac{2m_a}{3},\frac{2m_b}{3},\frac{2m_c}{3}$ whose medians as a consequence of elementary calculations $=\frac{a}{2},\frac{b}{2},\frac{c}{2}$ respectively, we arrive at :

$$\left(\frac{1}{4}\left(b^2-c^2\right)\right)^2 \geq \frac{9.\frac{a^2}{4}}{4}.\left(\frac{2}{3}(m_b-m_c)\right)^2 \Rightarrow \frac{\left(b^2-c^2\right)^2}{4a^2} \geq (m_b-m_c)^2$$

$$\Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore m_b+m_c \geq \sqrt{h_a^2+\frac{9a^2}{4}}\ \forall \Delta ABC, ''='' \text{ iff } b=c \text{ (QED)}$$

Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco

In any ΔABC, we have

$$h_a=\frac{\sqrt{s(s-a)}.2\sqrt{(s-b)(s-c)}}{a}\overset{AM-GM}{\leq}\frac{\sqrt{s(s-a)}.\left((s-b)+(s-c)\right)}{a}=\frac{\sqrt{(b+c)^2-a^2}}{2}$$

$$\Rightarrow b+c \geq \sqrt{4{h_a}^2+a^2}.$$

Using this inequality to the triangle GBC, where G is the centroid of the triangle ABC, we obtain

$$GB+GC \geq \sqrt{4{h_G}^2+BC^2},$$

where h_G is the altitude from G. Since the area of ΔGBC is equal to $\frac{F}{3}$, then we have

$$h_G=\frac{h_a}{3}.$$

Also, we have $GB=\frac{2m_b}{3}, GC=\frac{2m_c}{3}$, then we get

$$m_b+m_c \geq \sqrt{{h_a}^2+\frac{9a^2}{4}}.$$

Equality holds iff $b=c$.

1604. In ΔABC the following relationship holds:

$$\left(\frac{m_a}{h_a}\right)^{\frac{m_a}{h_a}}\cdot\left(\frac{w_a}{h_a}\right)^{\frac{w_a}{h_a}}+\frac{w_a}{h_a}\geq\frac{m_a}{h_a}+1$$

Proposed by Daniel Sitaru – Romania

Solution by Tapas Das – India

$$\left(\frac{m_a}{h_a}\right)^{\frac{m_a}{h_a}} \cdot \left(\frac{w_a}{h_a}\right)^{-\frac{w_a}{h_a}} = \left(\frac{m_a}{h_a}\right)^{\frac{m_a}{h_a}} \cdot \frac{1}{\left(\frac{w_a}{h_a}\right)^{\frac{w_a}{h_a}}}$$

$$\geq \left(\frac{m_a}{h_a}\right)^{\frac{w_a}{h_a}} \cdot \frac{1}{\left(\frac{w_a}{h_a}\right)^{\frac{w_a}{h_a}}} \quad (\because w_a \leq m_a; w_a \geq h_a)$$

$$\left(\frac{m_a}{w_a}\right)^{\frac{w_a}{h_a}} = \left[1 + \left(\frac{m_a}{w_a} - 1\right)\right]^{\frac{w_a}{h_a}}$$

$$\underset{Bernoulli}{\geq} 1 + \frac{w_a}{h_a}\left(\frac{m_a}{w_a} - 1\right) = 1 + \frac{m_a}{h_a} - \frac{w_a}{h_a}$$

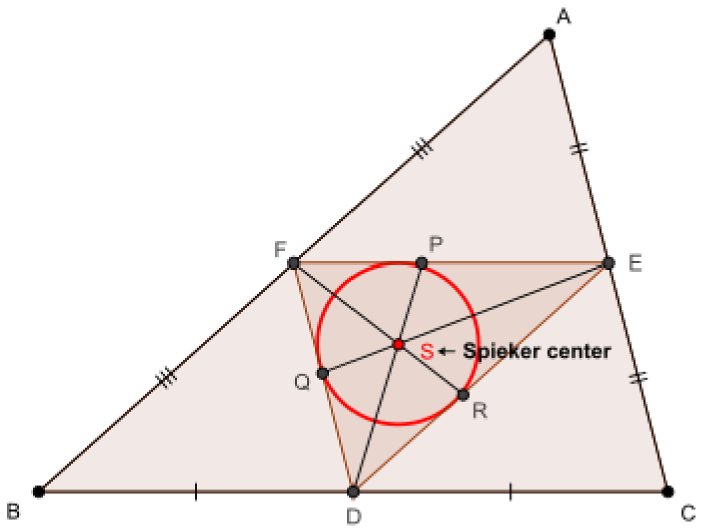
$$\therefore \left(\frac{m_a}{h_a}\right)^{\frac{m_a}{h_a}} \cdot \left(\frac{w_a}{h_a}\right)^{-\frac{w_a}{h_a}} + \frac{w_a}{h_a} \geq 1 + \frac{m_a}{h_a} - \frac{w_a}{h_a} + \frac{w_a}{h_a} = 1 + \frac{m_a}{h_a}$$

1605. In any $\Delta\, ABC$ with $p_a, p_b, p_c \to$ Spieker cevians, the following relationship holds:

$$\frac{r_b + r_c}{r_a + p_a} + \frac{r_c + r_a}{r_b + p_b} + \frac{r_a + r_b}{r_c + p_c} \geq 3$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of $\Delta\, DEF = r'$ (say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{ Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via $(1), (2)$ and using cosine law on $\Delta\, AFS$ and $\Delta\, AES$, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$

$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$

$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a(b^2+c^2) - (b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2} - a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr$$

$$\text{Again}, \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)}\right)$$

$$= \frac{r^2}{4r^2s}\big(ca(s-b) + ab(s-c)\big) = \frac{ab+ca}{4} - 2Rr \overset{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(\mathrm{i}), (*), (**) \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s}$$

$$= \frac{(a+b+c)(b^2+c^2+ab+ca) - (2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$= \frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s} \Rightarrow 2AS^2 \overset{(\mathrm{ii})}{=} \frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on } \Delta\, AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha \overset{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \Delta\, AES, b\sin\beta \overset{(****)}{=} \frac{r(a+c)}{2AS}$$

$$\text{Now}, [BAX] + [BAX] = [ABC] \Rightarrow \frac{1}{2}p_a c\sin\alpha + \frac{1}{2}p_a b\sin\beta = rs$$

$$\overset{\text{via } (***) \text{ and } (****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\blacksquare)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{Now}, b^3+c^3-abc+a(4m_a^2) = b^3+c^3+a^3-abc+a(2b^2+2c^2-a^2)-a^3$$

$$= \sum_{\text{cyc}} a^3 - abc + 2a.2bc\cos A = 2s(s^2-6Rr-3r^2) - 4Rrs + 16Rrs\cos A$$

$$\Rightarrow b^3+c^3-abc+a(4m_a^2) \overset{(\blacksquare\blacksquare)}{=} 2s(s^2-8Rr-3r^2+8Rr\cos A) \therefore (\blacksquare), (\blacksquare\blacksquare)$$

$$\Rightarrow p_a = \frac{2s}{2s+a}.\sqrt{s^2-8Rr-3r^2+8Rr\cos A} \rightarrow (\mathrm{m})$$

$$\text{Also}, r_b + r_c = s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} + \frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right) = \frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)} = 4R\cos^2\frac{A}{2}$$

$$\therefore r_b + r_c \overset{(\mathrm{iii})}{=} 4R\cos^2\frac{A}{2}$$

$$\text{We have : } \frac{r_b + r_c}{r_a + p_a} + \frac{r_c + r_a}{r_b + p_b} + \frac{r_a + r_b}{r_c + p_c}$$

$$= \sum_{cyc} \frac{(r_b + r_c)^2}{r_a(r_b + r_c) + p_a(r_b + r_c)} \overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{cyc}(r_b + r_c)\right)^2}{\sum_{cyc} r_a(r_b + r_c) + \sum_{cyc} p_a(r_b + r_c)}$$

$$= \frac{4(4R + r)^2}{2s^2 + \sum_{cyc} p_a(r_b + r_c)} \overset{?}{\geq} 3 \Leftrightarrow \boxed{4(4R + r)^2 - 6s^2 \underset{(\bullet)}{\overset{?}{\geq}} 3 \sum_{cyc} p_a(r_b + r_c)}$$

$$\text{We also have : } \prod_{cyc}(2s + a) = 8s^3 + 4s^2 \sum_{cyc} a + 2s \sum_{cyc} ab + 4Rrs$$

$$= 8s^3 + 4s^2.2s + 2s(s^2 + 4Rr + r^2) + 4Rrs$$

$$\Rightarrow \prod_{cyc}(2s + a) \overset{(\blacksquare\blacksquare\blacksquare)}{=} 2s(9s^2 + 6Rr + r^2) \text{ and}$$

$$\sum_{cyc} \frac{a(s - a)}{2s + a} = \frac{1}{\prod_{cyc}(2s + a)} \cdot \sum_{cyc} a(s - a)(2s + b)(2s + c) \overset{\text{via }(\blacksquare\blacksquare\blacksquare)}{=}$$

$$\frac{1}{2s(9s^2 + 6Rr + r^2)} \cdot \sum_{cyc} a(s - a)(8s^2 - 2sa + bc)$$

$$= \frac{1}{2s(9s^2 + 6Rr + r^2)} \cdot \left(\begin{array}{c} 8s^2\left(s(2s) - 2(s^2 - 4Rr - r^2)\right) \\ -2s\left(2s(s^2 - 4Rr - r^2) - 2s(s^2 - 6Rr - 3r^2)\right) + 4Rrs \sum_{cyc}(s - a) \end{array} \right)$$

$$\Rightarrow \sum_{cyc} \frac{a(s - a)}{2s + a} \overset{(\blacksquare\blacksquare\blacksquare\blacksquare)}{=} \frac{2rs(15R + 2r)}{9s^2 + 6Rr + r^2}$$

$$\text{Moreover, } \sum_{cyc} \frac{a^2}{2s + a} \overset{\text{via }(\blacksquare\blacksquare\blacksquare)}{=} \frac{1}{2s(9s^2 + 6Rr + r^2)} \cdot \sum_{cyc} a^2(2s + b)(2s + c)$$

$$= \frac{1}{2s(9s^2 + 6Rr + r^2)} \cdot \sum_{cyc} a^2(8s^2 - 2sa + bc)$$

$$= \frac{1}{2s(9s^2 + 6Rr + r^2)} \cdot \left(8s^2.2(s^2 - 4Rr - r^2) - 4s^2(s^2 - 6Rr - 3r^2) + 4Rrs(2s)\right)$$

$$\Rightarrow \sum_{cyc} \frac{a^2}{2s + a} \overset{(\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare)}{=} \frac{2s(3s^2 - 8Rr - r^2)}{9s^2 + 6Rr + r^2}$$

$$\text{Now, } \sum_{cyc} p_a(r_b + r_c) \overset{\text{via (m) and (iii)}}{=}$$

$$\sum_{cyc} \left(\frac{2s.4R}{2s + a} \cdot \sqrt{s^2 - 8Rr - 3r^2 + 8Rr\cos A} \cdot \frac{sa(s - a)}{abc} \right)$$

$$= \frac{8Rs^2}{4Rrs}\cdot\sum_{cyc}\left(\sqrt{\frac{\left(s^2-3r^2-16Rr\sin^2\frac{A}{2}\right)a(s-a)}{2s+a}}\cdot\sqrt{\frac{a(s-a)}{2s+a}}\right)$$

$$\overset{CBS}{\leq}\frac{2s}{r}\cdot\sqrt{\sum_{cyc}\frac{\left(s^2-3r^2-16Rr\sin^2\frac{A}{2}\right)a(s-a)}{2s+a}}\cdot\sqrt{\sum_{cyc}\frac{a(s-a)}{2s+a}}$$

$$=\frac{2s}{r}\cdot\sqrt{(s^2-3r^2).\sum_{cyc}\frac{a(s-a)}{2s+a}-\frac{16Rr(s-b)(s-c)(s-a)}{abc}\cdot\sum_{cyc}\frac{a^2}{2s+a}}\cdot\sqrt{\sum_{cyc}\frac{a(s-a)}{2s+a}}$$

$$\overset{via\,(■■■■)\,and\,(■■■■■)}{=}$$

$$\frac{2s}{r}\cdot\sqrt{(s^2-3r^2).\frac{2rs(15R+2r)}{9s^2+6Rr+r^2}-\frac{16Rr.r^2s}{4Rrs}\cdot\frac{2s(3s^2-8Rr-r^2)}{9s^2+6Rr+r^2}}\cdot\sqrt{\frac{2rs(15R+2r)}{9s^2+6Rr+r^2}}$$

$$=\frac{4s^2}{9s^2+6Rr+r^2}\cdot\sqrt{(225R^2-120Rr-20r^2)s^2-r^2(195R^2+56Rr+4r^2)}$$

$$\Rightarrow \boxed{3\sum_{cyc}p_a(r_b+r_c)\leq\frac{12s^2}{9s^2+6Rr+r^2}\cdot\sqrt{\begin{matrix}(225R^2-120Rr-20r^2)s^2\\-r^2(195R^2+56Rr+4r^2)\end{matrix}}}$$

$$\overset{?}{\leq} 4(4R+r)^2-6s^2 \Leftrightarrow (2(4R+r)^2-3s^2)^2(9s^2+6Rr+r^2)^2$$

$$\overset{?}{\geq} 36s^4\left((225R^2-120Rr-20r^2)s^2-r^2(195R^2+56Rr+4r^2)\right)$$

$$\Leftrightarrow 729s^8-(23652R^2+2484Rr+90r^2)s^6$$

$$+(82944R^4+62208R^3r+24624R^2r^2+4284Rr^3+261r^4)s^4$$

$$+r(110592R^5+122112R^4r+54144R^3r^2+12048R^2r^3+1344Rr^4+60r^5)s^2$$

$$+r^2\begin{pmatrix}36864R^6+49152R^5r+27136R^4r^2+7936R^3r^3\\+1296R^2r^4+112Rr^5+4r^6\end{pmatrix}\boxed{\overset{?}{\geq}}\,0 \quad (\bullet)$$

Now, Rouche $\Rightarrow s^2-(m-n)\geq 0$ and $s^2-(m+n)\leq 0$, where

$m = 2R^2+10Rr-r^2$ and $n = 2(R-2r).\sqrt{R^2-2Rr}$

$$\therefore \left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\leq 0$$

$$\Rightarrow s^4-s^2(2m)+m^2-n^2\leq 0\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\overset{(\blacklozenge)}{\leq} 0$$

$\therefore 729(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)^2\geq 0 \therefore$ in order to prove $(\bullet)$,

it suffices to prove : LHS of $(\bullet)\geq$

$$729(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)^2$$

$$\Leftrightarrow -(17820R^2-26676Rr+3006r^2)s^6$$

$$+(71280R^4-147744R^3r-325296R^2r^2+45108Rr^3-4113r^4)s^4$$

$$+r\begin{pmatrix}483840R^5+2268288R^4r+1337184R^3r^2\\+227832R^2r^3-4488Rr^4-2856r^5\end{pmatrix}s^2$$

$$-r^2\begin{pmatrix}2949120R^6+4429824R^5r+2772224R^4r^2+925184R^3r^3\\+173664R^2r^4+17384Rr^5+725r^6\end{pmatrix}\boxed{\overset{(\bullet\bullet)}{\geq}}\ 0\ and$$

$$\because -(17820R^2-26676Rr+3006r^2)s^2\begin{pmatrix}s^4-s^2(4R^2+20Rr-2r^2)\\+r(4R+r)^3\end{pmatrix}\overset{via\ (\blacklozenge)}{\geq}0$$

$\therefore$ in order to prove $(\bullet\bullet)$, it suffices to prove : LHS of $(\bullet\bullet)\geq$

$$-(17820R^2-26676Rr+3006r^2)s^2(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)$$

$$\Leftrightarrow -(397440R^3-231840R^2r+68364Rr^2-1899r^3)s^4$$

$$+\begin{pmatrix}1624320R^5+1416384R^4r+462960R^3r^2\\+69828R^2r^3+4908Rr^4+150r^5\end{pmatrix}s^2$$

$$-r\begin{pmatrix}2949120R^6+4429824R^5r+2772224R^4r^2+925184R^3r^3\\+173664R^2r^4+17384Rr^5+725r^6\end{pmatrix}\boxed{\overset{(\bullet\bullet\bullet)}{\geq}}\ 0\ and\ \because$$

$$-(397440R^3-231840R^2r+68364Rr^2-1899r^3)\begin{pmatrix}s^4-s^2(4R^2+20Rr-2r^2)\\+r(4R+r)^3\end{pmatrix}$$

$\overset{via\ (\blacklozenge)}{\geq}\ 0\ \therefore$ in order to prove $(\bullet\bullet)$, it suffices to prove : LHS of $(\bullet\bullet\bullet)\geq$

$$-(397440R^3-231840R^2r+68364Rr^2-1899r^3)\begin{pmatrix}s^4-s^2(4R^2+20Rr-2r^2)\\+r(4R+r)^3\end{pmatrix}$$

$$\Leftrightarrow (1080R^5-175158R^4r+175662R^3r^2-54798R^2r^3+5613Rr^4-114r^5)s^2$$

$$+r\begin{pmatrix}702720R^6-5952R^5r-148624R^4r^2-4684R^3r^3\\+10116R^2r^4+881Rr^5-82r^6\end{pmatrix}\boxed{\overset{(\bullet\bullet\bullet\bullet)}{\geq}}\ 0$$

Case 1 $1080R^5-175158R^4r+175662R^3r^2-54798R^2r^3+5613Rr^4-114r^5$ ≥ 0 *and* then : LHS of $(\bullet\bullet\bullet\bullet)\geq$

$$r\begin{pmatrix}702720R^6-5952R^5r-148624R^4r^2-4684R^3r^3+\\10116R^2r^4+881Rr^5-82r^6\end{pmatrix}$$

$$=r\left((R-2r)\begin{pmatrix}702720R^5+1399488R^4r+2650352R^3r^2\\+5296020R^2r^3+10602156Rr^4+21205193r^5\end{pmatrix}+42410304r^6\right)$$

$$\overset{Euler}{\geq}42410304r^7>0\Rightarrow(\bullet\bullet\bullet\bullet)\text{ is true (strict inequality)}$$

Case 2 $1080R^5-175158R^4r+175662R^3r^2-54798R^2r^3+5613Rr^4-114r^5$ <0 *and* then : LHS of $(\bullet\bullet\bullet\bullet)$

$$=-\left(-\begin{pmatrix}1080R^5-175158R^4r+175662R^3r^2-54798R^2r^3\\+5613Rr^4-114r^5\end{pmatrix}\right)s^2$$

$$+r\begin{pmatrix}702720R^6-5952R^5r-148624R^4r^2-4684R^3r^3\\+10116R^2r^4+881Rr^5-82r^6\end{pmatrix}\overset{Gerretsen}{\geq}$$

$$-\left(-\begin{pmatrix}1080R^5-175158R^4r+175662R^3r^2-54798R^2r^3\\+5613Rr^4-114r^5\end{pmatrix}\right)(4R^2+4Rr+3r^2)$$

$$+r\begin{pmatrix}702720R^6-5952R^5r-148624R^4r^2-4684R^3r^3\\+10116R^2r^4+881Rr^5-82r^6\end{pmatrix}\overset{?}{\geq}0$$

$$\Leftrightarrow 2160t^7+3204t^6-348t^5-95321t^4+162781t^3$$

$$-66141t^2+8632t-212\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)\left(\begin{matrix}2160t^5+11844t^4+38388t^3+10855t^2\\+52649t+101035\end{matrix}\right)+202176\right)\overset{?}{\geq}0$$

$$\to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (\bullet\bullet\bullet\bullet) \text{ is true} \therefore \text{combining both cases}, (\bullet\bullet\bullet\bullet)$$

$$\Rightarrow (\bullet\bullet\bullet) \Rightarrow (\bullet\bullet) \Rightarrow (\bullet) \text{ is true } \forall \Delta ABC \therefore \frac{r_b+r_c}{r_a+p_a}+\frac{r_c+r_a}{r_b+p_b}+\frac{r_a+r_b}{r_c+p_c}\geq 3$$

$$\forall \Delta ABC, ''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1606. In any ΔABC, the following relationship holds:

$$\sum_{cyc}\sqrt{\frac{w_a}{g_a+g_b}}+\frac{R^3}{8r^3}\geq 1+\sum_{cyc}\sqrt{\frac{g_a}{w_b+w_c}}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Triangle inequality} \Rightarrow g_a \leq AI + r \overset{?}{\leq} w_a \Leftrightarrow \frac{r}{\sin\frac{A}{2}}+r\overset{?}{\leq}\frac{2abc\cos\frac{A}{2}}{a(b+c)}$$

$$\Leftrightarrow \frac{r}{\sin\frac{A}{2}}+r\overset{?}{\leq}\frac{8Rrs\cos\frac{A}{2}}{4R(b+c)\sin\frac{A}{2}\cos\frac{A}{2}} \Leftrightarrow \frac{1}{\sin\frac{A}{2}}+1\overset{?}{\leq}\frac{a+b+c}{(b+c)\sin\frac{A}{2}}$$

$$\Leftrightarrow \frac{1}{\sin\frac{A}{2}}+1\overset{?}{\leq}\frac{a}{(b+c)\sin\frac{A}{2}}+\frac{1}{\sin\frac{A}{2}} \Leftrightarrow (b+c)\sin\frac{A}{2}\overset{?}{\leq} a$$

$$\Leftrightarrow 4R\cos\frac{A}{2}\cos\frac{B-C}{2}\sin\frac{A}{2}\overset{?}{\leq}4R\sin\frac{A}{2}\cos\frac{A}{2} \Leftrightarrow \cos\frac{B-C}{2}\overset{?}{\leq}1 \to \text{true}$$

$$\therefore g_a \leq w_a \text{ and analogs} \to (1)$$

$$\text{Now, } \sum_{cyc}\frac{a}{b}\overset{CBS}{\leq}\sqrt{\sum_{cyc}a^2}.\sqrt{\sum_{cyc}\frac{b^2c^2}{16R^2r^2s^2}}\overset{\substack{\text{Leibnitz}\\+\\\text{Goldstone}}}{\leq}\sqrt{9R^2}.\sqrt{\frac{4R^2s^2}{16R^2r^2s^2}} \Rightarrow \sum_{cyc}\frac{a}{b}\leq\frac{3R}{2r}\to(2)$$

$$\text{Also, } \sum_{cyc}\frac{1}{\sin\frac{A}{2}}=\sum_{cyc}\sqrt{\frac{bc(s-a)}{r^2s}}\overset{CBS}{\leq}\frac{1}{r.\sqrt{s}}.\sqrt{\sum_{cyc}(s-a)}.\sqrt{s^2+4Rr+r^2}\overset{\text{Geretsen}}{\leq}$$

$$\frac{1}{r}.\sqrt{4R^2+4Rr+3r^2+4Rr+r^2} \Rightarrow \sum_{cyc}\frac{1}{\sin\frac{A}{2}}\leq\frac{2(R+r)}{r}\to(3)$$

$$\text{Now, via (1), } \sum_{cyc}\sqrt{\frac{w_a}{g_a+g_b}}\geq\sum_{cyc}\sqrt{\frac{w_a}{w_a+w_b}}\geq\sum_{cyc}\sqrt{\frac{h_a}{m_a+m_b}}\overset{\text{Panaitopol}}{\geq}$$

$$\sum_{cyc}\sqrt{\frac{h_a}{\frac{R}{2r}(h_a+h_b)}}=\sqrt{\frac{2r}{R}}\cdot\sum_{cyc}\sqrt{\frac{\frac{bc}{2R}}{\frac{bc+ca}{2R}}}=\sqrt{\frac{2r}{R}}\cdot\sum_{cyc}\frac{1}{\sqrt{1+\frac{a}{b}}}\overset{\substack{\text{Bergstrom}\\+\\\text{CBS}\\\text{and via (2)}}}{\geq}\sqrt{\frac{2r}{R}}\cdot\frac{9}{\sqrt{3}.\sqrt{3+\frac{3R}{2r}}}$$

$$\therefore \boxed{\sum_{cyc}\sqrt{\frac{w_a}{g_a+g_b}}\geq\frac{3}{\sqrt{t(t+1)}}}\to(\text{i})\ \left(\text{where } t=\frac{R}{2r}\overset{\text{Euler}}{\geq}1\right)$$

$$\text{Again, via (1)},\ \sum_{cyc}\sqrt{\frac{g_a}{w_b+w_c}}\leq\sum_{cyc}\sqrt{\frac{w_a}{w_b+w_c}}\leq\sum_{cyc}\sqrt{\frac{w_a}{h_b+h_c}}$$

$$=\sum_{cyc}\sqrt{\frac{2bc.\cos\frac{A}{2}}{(b+c)\frac{(b+c).4R\cos\frac{A}{2}\sin\frac{A}{2}}{2R}}}=\frac{1}{2}\sum_{cyc}\sqrt{\frac{4bc}{(b+c)^2}\cdot\frac{1}{\sin\frac{A}{2}}}\overset{\substack{\text{A-G}\\\text{and}\\\text{CBS}}}{\leq}\frac{\sqrt{3}}{2}\cdot\sqrt{\sum_{cyc}\frac{1}{\sin\frac{A}{2}}}$$

$$\overset{\text{via (3)}}{\leq}\frac{\sqrt{3}}{2}\cdot\sqrt{\frac{2(R+r)}{r}}\ \therefore\ \boxed{\sum_{cyc}\sqrt{\frac{g_a}{w_b+w_c}}\leq\sqrt{3}.\sqrt{t+\frac{1}{2}}}\to(\text{ii})$$

$$\therefore(\text{i}),(\text{ii})\Rightarrow\text{in order to prove}:\sum_{cyc}\sqrt{\frac{w_a}{g_a+g_b}}+\frac{R^3}{8r^3}\geq1+\sum_{cyc}\sqrt{\frac{g_a}{w_b+w_c}},$$

$$\text{it suffices to prove}:\boxed{\frac{3}{\sqrt{t(t+1)}}+t^3-1\overset{(*)}{\geq}\sqrt{3}.\sqrt{t+\frac{1}{2}}}$$

$$\text{Let } f(t)=\frac{3}{\sqrt{t(t+1)}}+t^3-1-\sqrt{3}.\sqrt{t+\frac{1}{2}}\ \forall t\geq1$$

$$\therefore f'(t)=3t^2-\frac{\sqrt{3}}{2.\sqrt{t+\frac{1}{2}}}-\frac{3(2t+1)}{2.(t(t+1))^{\frac{3}{2}}}\ \text{and}$$

$$f''(t)=6t+\frac{\sqrt{3}}{4(t+1)^{\frac{3}{2}}}+\frac{9(2t+1)^2}{4t(t+1)(t(t+1))^{\frac{3}{2}}}-\frac{3}{(t(t+1))^{\frac{3}{2}}}$$

$$=6t+\frac{\sqrt{3}}{4(t+1)^{\frac{3}{2}}}+\frac{3}{(t(t+1))^{\frac{3}{2}}}\cdot\frac{3(2t+1)^2-4t(t+1)}{4t(t+1)}$$

$$=6t+\frac{\sqrt{3}}{4(t+1)^{\frac{3}{2}}}+\frac{3}{(t(t+1))^{\frac{3}{2}}}\cdot\frac{8t^2+8t+3}{4t(t+1)}>0\Rightarrow f''(t)>0\Rightarrow f'(t)\text{ is}\uparrow$$

$$\text{on }[1,\infty)\Rightarrow f'(t)\geq f'(0)\approx0.701903>0\Rightarrow f(t)\text{ is}\uparrow\text{ on }[1,\infty)$$

$$\Rightarrow f(t)\geq f(1)=0\Rightarrow\frac{3}{\sqrt{t(t+1)}}+t^3-1-\sqrt{3}.\sqrt{t+\frac{1}{2}}\geq0\ \forall t\geq1\Rightarrow(*)\text{ is true}$$

$$\therefore \sum_{cyc}\sqrt{\frac{w_a}{g_a+g_b}}+\frac{R^3}{8r^3}\ge 1+\sum_{cyc}\sqrt{\frac{g_a}{w_b+w_c}}\ \forall\,\Delta\,ABC,$$
$$''=''\ \text{iff}\ \Delta\,ABC\ \text{is equilateral (QED)}$$

1607.

In any Δ ABC and for all $n\ge 2$, the following relationship holds :

$$\sum_{cyc}\sqrt[2011]{\frac{a(b+c)}{bc}}+\frac{R^n}{r^n}\ge 2^n+\sum_{cyc}\sqrt[2011]{\frac{m_a(m_b+m_c)}{m_bm_c}}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\sqrt[2011]{\frac{a(b+c)}{bc}}\ge\sum_{cyc}\sqrt[2011]{\frac{2\sqrt{(s-b)(s-c)}.(b+c)}{bc}}\overset{A-G}{\ge}$$
$$\sum_{cyc}\sqrt[2011]{\frac{4.\sqrt{(s-b)(s-c)}}{\sqrt{bc}}}=\sum_{cyc}\sqrt[2011]{4\sin\frac{A}{2}}\overset{A-G}{\ge}3\sqrt[6033]{64\prod_{cyc}\sin\frac{A}{2}}=3.\sqrt[6033]{64.\frac{r}{4R}}$$
$$\therefore \boxed{\sum_{cyc}\sqrt[2011]{\frac{a(b+c)}{bc}}\ge 3.\sqrt[6033]{\frac{16r}{R}}}\to(1)$$

$$\text{Again, }\sum_{cyc}\sqrt[2011]{\frac{m_a(m_b+m_c)}{m_bm_c}}=\sum_{cyc}\sqrt[2011]{\frac{m_a}{m_c}+\frac{m_a}{m_b}}\overset{Panaitopol}{\le}\sum_{cyc}\sqrt[2011]{\frac{\frac{Rs}{a}}{\frac{2rs}{c}}+\frac{\frac{Rs}{a}}{\frac{2rs}{b}}}$$
$$=\sum_{cyc}\sqrt[2011]{\frac{R}{2r}\cdot\frac{b+c}{a}}=\sum_{cyc}\sqrt[2011]{\frac{R}{2r}\cdot\frac{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{4R\sin\frac{A}{2}\cos\frac{A}{2}}}\overset{0<\cos\frac{B-C}{2}\le 1}{\le}$$
$$\sqrt[2011]{\frac{R}{r}}\cdot\sum_{cyc}\sqrt[2011]{\frac{1}{2\sin\frac{A}{2}}}\overset{A-G}{\le}\sqrt[2011]{\frac{R}{r}}\cdot\sum_{cyc}\frac{\frac{1}{2\sin\frac{A}{2}}+2010}{2011}$$
$$=\sqrt[2011]{\frac{R}{r}}\cdot\left(\frac{6030}{2011}+\frac{1}{4022}\cdot\sum_{cyc}\sqrt{\frac{bc(s-a)}{(s-a)(s-b)(s-c)}}\right)\overset{CBS}{\le}$$
$$\sqrt[2011]{\frac{R}{r}}\cdot\left(\frac{6030}{2011}+\frac{1}{4022}\cdot\frac{1}{r.\sqrt{s}}\cdot\sqrt{s^2+4Rr+r^2}.\sqrt{s}\right)\overset{Gerretsen}{\le}$$

$$\sqrt[2011]{\frac{R}{r}}\cdot\left(\frac{6030}{2011}+\frac{1}{4022}\cdot\frac{1}{r}\cdot\sqrt{4R^2+8Rr+4r^2}\right)$$

$$\therefore \boxed{\sum_{cyc}\sqrt[2011]{\frac{m_a(m_b+m_c)}{m_bm_c}}\le\sqrt[2011]{\frac{R}{r}}\cdot\left(\frac{6031}{2011}+\frac{1}{2011}\cdot\frac{R}{r}\right)}\to(2)$$

$\therefore (1),(2)\Rightarrow$ **in order to prove :**

$$\sum_{cyc}\sqrt[2011]{\frac{a(b+c)}{bc}}+\frac{R^2}{r^2}\ge 4+\sum_{cyc}\sqrt[2011]{\frac{m_a(m_b+m_c)}{m_bm_c}},\text{ it suffices to prove :}$$

$$3.\sqrt[6033]{\frac{16r}{R}}+\frac{R^2}{r^2}\ge 4+\sqrt[2011]{\frac{R}{r}}\cdot\left(\frac{6031}{2011}+\frac{1}{2011}\cdot\frac{R}{r}\right)$$

$$\Leftrightarrow \boxed{3.\sqrt[6033]{\frac{16}{t}}+t^2\overset{(*)}{\ge}4+\sqrt[2011]{t}.\left(\frac{6031}{2011}+\frac{t}{2011}\right)}\left(t=\frac{R}{r}\overset{\text{Euler}}{\ge}2\right)$$

$$\text{Let } f(t)=3.\sqrt[6033]{\frac{16}{t}}+t^2-4-\sqrt[2011]{t}.\left(\frac{6031}{2011}+\frac{t}{2011}\right)\ \forall\, t\ge 2\ \textit{and}\text{ then :}$$

$$f'(t)=2t-\frac{\sqrt[2011]{t}}{2011}-\frac{\frac{t}{2011}+\frac{6031}{2011}}{2011.t^{\frac{2010}{2011}}}-\frac{\sqrt[6033]{16}}{2011.t^{\frac{6034}{6033}}}\ \textit{and}$$

$$f''(t)=2\left(1-\frac{1}{4044121.t^{\frac{2010}{2011}}}\right)+\frac{2010\left(\frac{t}{2011}+\frac{6031}{2011}\right)}{4044121.t^{\frac{4021}{2011}}}+\frac{6034.\sqrt[6033]{16}}{12132363.t^{\frac{12067}{6033}}}$$

$$\text{Now, }\because t\ge 2\ \therefore \frac{2010}{2011}.\ln t>-\ln 2\Rightarrow t^{\frac{2010}{2011}}>\frac{1}{2}\ \therefore 4044121.t^{\frac{2010}{2011}}>4044121.\frac{1}{2}>1$$

$$\Rightarrow 1-\frac{1}{4044121.t^{\frac{2010}{2011}}}>0\ \therefore f''(t)>0\ \forall\, t\ge 2\Rightarrow f'(t)\text{ is }\uparrow\text{ on }[2,\infty)$$

$$\Rightarrow f'(t)\ge f'(2)\approx 3.998508>0\ \forall\, t\ge 2\Rightarrow f(t)\ge f(2)=0$$

$$\Rightarrow 3.\sqrt[6033]{\frac{16}{t}}+t^2-4-\sqrt[2011]{t}.\left(\frac{6031}{2011}+\frac{t}{2011}\right)\ge 0\Rightarrow(*)\text{ is true}$$

$$\Rightarrow\frac{R^2}{r^2}-4\overset{(\bullet)}{\ge}\sum_{cyc}\sqrt[2011]{\frac{m_a(m_b+m_c)}{m_bm_c}}-\sum_{cyc}\sqrt[2011]{\frac{a(b+c)}{bc}}$$

$$\text{Let } F(n)=t^n-2^n\ \forall\, t=\frac{R}{r}\ge 2\ (t\to\text{fixed})\ \textit{and}\ \forall\, n\ge 2\ \textit{and}\text{ then :}$$

$$F'(n)=t^n.(\ln t)-2^n.(\ln 2)\ge 0$$

$$(\because t^n\ge 2^n\ \textit{and}\ \ln t\ge\ln 2\Rightarrow t^n.(\ln t)-2^n.(\ln 2)\ge 0)$$

$$\therefore F(n)\text{ is }\uparrow\ \forall\, n\ge 2\Rightarrow F(n)\ge F(2)\Rightarrow\left(\frac{R}{r}\right)^n-2^n$$

$$\ge\frac{R^2}{r^2}-4\overset{\text{via }(\bullet)}{\ge}\sum_{cyc}\sqrt[2011]{\frac{m_a(m_b+m_c)}{m_bm_c}}-\sum_{cyc}\sqrt[2011]{\frac{a(b+c)}{bc}}$$

$$\therefore \sum_{cyc} \sqrt[2011]{\frac{a(b+c)}{bc}} + \frac{R^n}{r^n} \geq 2^n + \sum_{cyc} \sqrt[2011]{\frac{m_a(m_b+m_c)}{m_b m_c}}$$

$\forall \Delta ABC$ and $\forall n \geq 2$, $'' = ''$ iff ΔABC is equilateral (QED)

1608. In any ΔABC, the following relationship holds :

$$\sum_{cyc} \frac{g_a^2}{w_b^2 + w_c^2} + \frac{R^3 - 8r^3}{2r^3} \geq \sum_{cyc} \frac{m_a^2}{h_b^2 + h_c^2}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Firstly, } \sum_{cyc} m_a^4 = \left(\sum_{cyc} m_a^2\right)^2 - 2\sum_{cyc} m_a^2 m_b^2$$

$$= \frac{9}{16}\left(\sum_{cyc} a^2\right)^2 - 2.\frac{9}{16}.\sum_{cyc} a^2b^2 = \frac{9}{16}.\sum_{cyc} a^4 = \frac{9}{16}.\left(2\sum_{cyc} a^2b^2 - 16r^2s^2\right)$$

$$\overset{\text{Goldstone}}{\leq} \frac{9}{16}.(8R^2s^2 - 16r^2s^2) \Rightarrow \sum_{cyc} m_a^4 \leq \frac{9s^2}{2}.(R^2 - 2r^2) \rightarrow (1)$$

$$w_a \leq w_b \Leftrightarrow \frac{2bc\cos\frac{A}{2}}{b+c} \leq \frac{2ca\cos\frac{B}{2}}{c+a} \Leftrightarrow \frac{4R\cos\frac{B}{2}\sin\frac{B}{2}}{\cos\frac{B}{2}}.\frac{1}{b+c} \leq \frac{4R\cos\frac{A}{2}\sin\frac{A}{2}}{\cos\frac{A}{2}}.\frac{1}{c+a}$$

$$\Leftrightarrow \frac{(b+c)^2(s-b)(s-c)}{bc} \geq \frac{(c+a)^2(s-c)(s-a)}{ca}$$

$$\Leftrightarrow a(b+c)^2(c+a-b) \geq b(c+a)^2(b+c-a)$$

$$\Leftrightarrow ab(a^2-b^2) + c^3(a-b) + 3abc(a-b) + c^2(a^2-b^2) \geq 0$$

$$\Leftrightarrow (a-b)\left((ab+c^2)(a+b) + c^3 + 3abc\right) \geq 0 \rightarrow \text{true for } a \geq b$$

$$\Rightarrow w_a \leq w_b \text{ for } a \geq b \text{ and analogs} \rightarrow (2)$$

$$\text{Now, WLOG assuming } a \geq b \geq c \Rightarrow h_a^2 \leq h_b^2 \leq h_c^2 \text{ and}$$

$$\frac{1}{w_b^2+w_c^2} \leq \frac{1}{w_c^2+w_a^2} \leq \frac{1}{w_a^2+w_b^2}\ (\text{via } (2)) \therefore \sum_{cyc} \frac{g_a^2}{w_b^2+w_c^2} \geq \sum_{cyc} \frac{h_a^2}{w_b^2+w_c^2}$$

$$\overset{\text{Chebyshev}}{\geq} \frac{1}{3}.\frac{\sum_{cyc} a^2b^2}{4R^2}.\sum_{cyc} \frac{1}{w_b^2+w_c^2} \geq \frac{1}{36R^2}.\left(\sum_{cyc} ab\right)^2.\sum_{cyc} \frac{1}{s(s-b+s-c)}$$

$$\geq \frac{1}{36R^2}.3abc\left(\sum_{cyc} a\right).\frac{1}{4Rrs^2}.\left(\sum_{cyc} ab\right) \geq \frac{1}{36R^2}.24Rrs^2.\frac{1}{4Rrs^2}.18Rr$$

$$\left(\because s^2 - 14Rr + r^2 = s^2 - 16Rr + 5r^2 + 2r(R-2r) \overset{\text{Gerretsen + Euler}}{\geq} 0 \right.$$
$$\left. \Rightarrow s^2 + 4Rr + r^2 \geq 18Rr\right)$$

$$\therefore \boxed{\sum_{cyc} \frac{g_a^2}{w_b^2 + w_c^2} \ge \frac{3r}{R}} \to (3)$$

$$\text{Again}, \sum_{cyc} \frac{m_a^2}{h_b^2 + h_c^2} \overset{CBS}{\le} \sqrt{\sum_{cyc} m_a^4} \cdot \sqrt{\sum_{cyc} \left(\frac{1}{h_b^2 + h_c^2}\right)^2} \overset{A-G}{\le} \sqrt{\sum_{cyc} m_a^4} \cdot \sqrt{\sum_{cyc} \frac{h_a^2}{4h_a^2 h_b^2 h_c^2}}$$

$$\overset{\text{via }(1)}{\le} \sqrt{\frac{9s^2}{2} \cdot (R^2 - 2r^2)} \cdot \sqrt{\frac{\sum_{cyc} a^2 b^2}{16R^2 \cdot \frac{4r^4 s^4}{R^2}}} \overset{\text{Goldstone}}{\le} \sqrt{\frac{9s^2}{2} \cdot (R^2 - 2r^2)} \cdot \sqrt{\frac{4R^2 s^2}{16R^2 \cdot \frac{4r^4 s^4}{R^2}}}$$

$$\therefore \boxed{\sum_{cyc} \frac{m_a^2}{h_b^2 + h_c^2} \le \sqrt{\frac{9R^2(R^2 - 2r^2)}{32r^4}}} \to (4) \therefore (3), (4) \Rightarrow \text{it suffices to prove :}$$

$$\frac{3r}{R} + \frac{R^3 - 8r^3}{2r^3} \ge \sqrt{\frac{9R^2(R^2 - 2r^2)}{32r^4}} \Leftrightarrow \left(\frac{R(R^3 - 8r^3) + 6r^4}{2Rr^3}\right)^2 \ge \frac{9R^2(R^2 - 2r^2)}{32r^4}$$

$$\Leftrightarrow 8t^8 - 9t^6 - 128t^5 + 114t^4 + 512t^2 - 768t + 288 \ge 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t - 2)\Big((t - 2)(8t^6 + 32t^5 + 87t^4 + 92t^3 + 134t^2 + 168t + 648) + 1152\Big) \ge 0$$

$$\to \text{true} \because t \overset{\text{Euler}}{\ge} 2 \therefore \sum_{cyc} \frac{g_a^2}{w_b^2 + w_c^2} + \frac{R^3 - 8r^3}{2r^3} \ge \sum_{cyc} \frac{m_a^2}{h_b^2 + h_c^2}$$

$$\forall \Delta ABC, '' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1609. In any acute triangle ABC holds:

$$\frac{1}{h_a}\sqrt{\tan A} + \frac{1}{h_b}\sqrt{\tan B} + \frac{1}{h_c}\sqrt{\tan C} \ge \frac{2}{R}\sqrt[4]{3}$$

Proposed by Vasile Mircea Popa – Romania

Solution by Tapas Das – India

Note 1: $A + B + C = \pi$

$$\therefore A + B = \pi - C$$

$$\tan(A + B) = \tan(\pi - C) \Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C \Rightarrow \sum \tan A$$
$$= \tan A \cdot \tan B \cdot \tan C$$

Note 2: In any acute triangle : $\tan A + \tan B + \tan C \ge 3\sqrt{3}$

Proof: Since $f(x) = \tan x$ is convex on $\left(0, \frac{\pi}{2}\right)$. So by Jensen's inequality

$$\tan A+\tan B+\tan C\ge 3\tan\frac{A+B+C}{3}=3\tan\frac{\pi}{3}=3\sqrt{3}$$

$$\therefore \sum\frac{1}{h_a}\sqrt{\tan A}\overset{Chebyshev}{\ge}\frac{1}{3}\cdot\sum\frac{1}{h_a}\cdot\sum\sqrt{\tan A}$$

$$[\text{WLOG } a\ge b\ge c\quad \therefore h_a\le h_b\le h_c\Rightarrow\frac{1}{h_a}\ge\frac{1}{h_b}\ge\frac{1}{h_c};\tan A\ge\tan B\ge\tan C]$$

$$\therefore \sum\frac{1}{h_a}\sqrt{\tan A}\ge\frac{1}{3}\cdot\frac{1}{r}\cdot\sum\sqrt{\tan A}\overset{AM-GM}{\ge}\frac{1}{3r}\cdot 3(\tan A\tan B\tan C)^{\frac{1}{6}}=$$

$$=\frac{1}{r}(3\sqrt{3})^{\frac{1}{6}}\overset{Euler}{\ge}\frac{2}{R}\cdot\left(3^{\frac{3}{2}}\right)^{\frac{1}{6}}=\frac{2}{R}\sqrt[4]{3}$$

1610. In any Δ ABC, the following relationship holds :

$$\frac{6r}{R}\le\frac{m_a}{m_b}+\frac{w_b}{w_c}+\frac{h_c}{h_a}\le\frac{3}{8}\left(9\left(\frac{R}{r}\right)^3-64\right)$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{am_a}\sum_{cyc}a^2\ge 2\sqrt{3}\Leftrightarrow\frac{1}{a^2m_a^2}\ge\frac{12}{\left(\sum_{cyc}a^2\right)^2}\Leftrightarrow$$

$$\left(\sum_{cyc}a^2\right)^2-3a^2(2b^2+2c^2-a^2)\ge 0\Leftrightarrow\left(\sum_{cyc}a^2\right)^2-3a^2\left(2\sum_{cyc}a^2-3a^2\right)\ge 0$$

$$\Leftrightarrow\left(\sum_{cyc}a^2\right)^2-6a^2\sum_{cyc}a^2+9a^4\ge 0\Leftrightarrow\left(\sum_{cyc}a^2-3a^2\right)^2\ge 0$$

$$\Leftrightarrow\left(b^2+c^2-2a^2\right)^2\ge 0\to\text{true}\Rightarrow m_a\le\frac{\sum_{cyc}a^2}{2\sqrt{3}a}\text{ and analogs}\to(1)$$

$$\text{Now,}\frac{m_a}{m_b}+\frac{w_b}{w_c}+\frac{h_c}{h_a}\le\frac{m_a}{h_b}+\frac{m_b}{h_c}+\frac{m_c}{h_a}=\frac{bm_a+cm_b+am_c}{2rs}$$

$$\overset{\substack{\text{via }(1)\\ \text{and}\\ \text{Mitrinovic}}}{\le}\frac{\sum_{cyc}a^2}{2\sqrt{3}}\cdot\frac{\frac{b}{a}+\frac{c}{b}+\frac{a}{c}}{2r.3\sqrt{3}r}\overset{\substack{\text{Leibnitz}\\ \text{and}\\ \text{CBS}}}{\le}\frac{9R^2}{36r^2}\cdot\sqrt{\sum_{cyc}a^2}\cdot\sqrt{\frac{\sum_{cyc}a^2b^2}{16R^2r^2s^2}}\overset{\substack{\text{Leibnitz}\\ \text{and}\\ \text{Goldstone}}}{\le}\frac{R^2}{4r^2}\cdot\sqrt{\frac{9R^2.4R^2s^2}{16R^2r^2s^2}}$$

$$=\frac{3R^3}{8r^3}=\frac{3}{8}\left(9\left(\frac{R}{r}\right)^3-8\left(\frac{R}{r}\right)^3\right)\overset{Euler}{\le}\frac{3}{8}\left(9\left(\frac{R}{r}\right)^3-8.8\right)$$

$$\therefore\boxed{\frac{m_a}{m_b}+\frac{w_b}{w_c}+\frac{h_c}{h_a}\le\frac{3}{8}\left(9\left(\frac{R}{r}\right)^3-64\right)}$$

$$\text{Again,}\frac{m_a}{m_b}+\frac{w_b}{w_c}+\frac{h_c}{h_a}\ge\frac{h_a}{m_b}+\frac{h_b}{m_c}+\frac{h_c}{m_a}=\frac{h_a^2}{m_bh_a}+\frac{h_b^2}{h_bm_c}+\frac{h_c^2}{m_ah_c}$$

$$\geq \frac{h_a^2}{m_b m_a}+\frac{h_b^2}{m_b m_c}+\frac{h_c^2}{m_a m_c} \overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{cyc} h_a\right)^2}{\sum_{cyc} m_b m_c} = \frac{\left(\sum_{cyc} ab\right)^2}{4R^2.\frac{\left(\sum_{cyc} m_a\right)^2-\sum_{cyc} m_a^2}{2}}$$

$$\overset{\text{Chu and Yang}}{\geq} \frac{(s^2+4Rr+r^2)^2}{4R^2.\frac{4s^2-16Rr+5r^2-\frac{3}{2}(s^2-4Rr-r^2)}{2}} = \frac{(s^2+4Rr+r^2)^2}{R^2(5s^2-20Rr+13r^2)} \overset{?}{\geq} \frac{6r}{R}$$

$$\Leftrightarrow s^4-(22Rr-2r^2)s^2+r^2(136R^2-70Rr+r^2) \underset{(*)}{\overset{?}{\geq}} 0 \text{ and}$$

$$\because (s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (*), \text{it suffices to prove :}$$

$$\text{LHS of } (*) \geq (s^2-16Rr+5r^2)^2 \Leftrightarrow (5R-4r)s^2 \overset{(**)}{\geq} r(60R^2-45Rr+12r^2)$$

$$\text{Now, } (5R-4r)s^2 \overset{\text{Gerretsen}}{\geq} (5R-4r)(16Rr-5r^2) \overset{?}{\geq} r(60R^2-45Rr+12r^2)$$

$$\Leftrightarrow 4(5R^2-11Rr+2r^2) \overset{?}{\geq} 0 \Leftrightarrow 4(5R-r)(R-2r) \overset{?}{\geq} 0 \to \text{true} \because R \overset{\text{Euler}}{\geq} 2r$$

$$\Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \boxed{\frac{m_a}{m_b}+\frac{w_b}{w_c}+\frac{h_c}{h_a} \geq \frac{6r}{R}} \therefore \frac{6r}{R} \leq \frac{m_a}{m_b}+\frac{w_b}{w_c}+\frac{h_c}{h_a}$$

$$\leq \frac{3}{8}\left(9\left(\frac{R}{r}\right)^3-64\right) \forall \Delta ABC, ''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1611. In any Δ ABC, the following relationship holds :

$$\frac{4r}{R^2} \leq \frac{m_a}{m_b m_c}+\frac{w_b}{w_c w_a}+\frac{h_c}{h_a h_b} \leq \frac{1}{2r}\left(\frac{81}{16}\left(\frac{R}{r}\right)^5-160\right)$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{am_a}\sum_{cyc} a^2 \geq 2\sqrt{3} \Leftrightarrow \frac{1}{a^2 m_a^2} \geq \frac{12}{\left(\sum_{cyc} a^2\right)^2} \Leftrightarrow$$

$$\left(\sum_{cyc} a^2\right)^2 - 3a^2(2b^2+2c^2-a^2) \geq 0 \Leftrightarrow \left(\sum_{cyc} a^2\right)^2 - 3a^2\left(2\sum_{cyc} a^2 - 3a^2\right) \geq 0$$

$$\Leftrightarrow \left(\sum_{cyc} a^2\right)^2 - 6a^2\sum_{cyc} a^2 + 9a^4 \geq 0 \Leftrightarrow \left(\sum_{cyc} a^2 - 3a^2\right)^2 \geq 0$$

$$\Leftrightarrow (b^2+c^2-2a^2)^2 \geq 0 \to \text{true} \Rightarrow m_a \leq \frac{\sum_{cyc} a^2}{2\sqrt{3}a} \text{ and analogs} \to (1)$$

$$\text{Now, } \frac{m_a}{m_b m_c}+\frac{w_b}{w_c w_a}+\frac{h_c}{h_a h_b} \leq \frac{m_a}{h_b h_c}+\frac{m_b}{h_c h_a}+\frac{m_c}{h_a h_b} \overset{\text{via (1)}}{\leq} \frac{\sum_{cyc} a^2}{2\sqrt{3}}.\left(\sum_{cyc}\frac{4R^2}{a.ca.ab}\right)$$

$$\le \frac{\sum_{cyc} a^2}{2\sqrt{3}}\cdot\frac{4R^2}{4Rrs}\cdot\sum_{cyc}\frac{1}{4(s-b)(s-c)} = \frac{\sum_{cyc} a^2}{2\sqrt{3}}\cdot\frac{R}{rs}\cdot\frac{s}{4r^2s} \overset{\substack{\text{Leibnitz}\\ \text{and}\\ \text{Mitrinovic}}}{\le} \frac{9R^3}{8\sqrt{3}r^3.3\sqrt{3}r}$$

$$= \frac{R^3}{8r^4} \overset{?}{\le} \frac{1}{2r}\left(\frac{81}{16}\left(\frac{R}{r}\right)^5 - 160\right) = \frac{1}{2r}\cdot\left(\frac{81R^5 - 2560r^5}{16r^5}\right)$$

$$\Leftrightarrow 81t^5 - 4t^3 - 2560 \overset{?}{\ge} 0 \;\left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(81t^4 + 162t^3 + 320t^2 + 640t + 1280) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2$$

$$\therefore \frac{m_a}{m_b m_c} + \frac{w_b}{w_c w_a} + \frac{h_c}{h_a h_b} \le \frac{1}{2r}\left(\frac{81}{16}\left(\frac{R}{r}\right)^5 - 160\right)$$

$$\text{Again}, \frac{m_a}{m_b m_c} + \frac{w_b}{w_c w_a} + \frac{h_c}{h_a h_b} \ge \frac{h_a}{m_b m_c} + \frac{h_b}{m_c m_a} + \frac{h_c}{m_a m_b} = \sum_{cyc}\frac{h_a^2}{h_a m_b m_c}$$

$$\overset{\text{Bergstrom}}{\ge} \frac{\left(\sum_{cyc} h_a\right)^2}{\sum_{cyc} h_a m_b m_c} \overset{\text{Chebyshev}}{\ge} \frac{3\left(\sum_{cyc} h_a\right)^2}{\left(\sum_{cyc} h_a\right)\left(\sum_{cyc} m_b m_c\right)}$$

$(\because$ WLOG *assuming* $a \ge b \ge c \Rightarrow h_a \le h_b \le h_c$ *and* $m_b m_c \ge m_c m_a \ge m_a m_b)$

$$= \frac{3\left(\sum_{cyc} h_a\right)}{\frac{\left(\sum_{cyc} m_a\right)^2 - \sum_{cyc} m_a^2}{2}} \overset{\text{Chu and Yang}}{\ge} \frac{3(s^2 + 4Rr + r^2)}{2R.\frac{4s^2 - 16Rr + 5r^2 - \frac{3}{2}(s^2 - 4Rr - r^2)}{2}}$$

$$= \frac{6(s^2 + 4Rr + r^2)}{R(5s^2 - 20Rr + 13r^2)} \overset{?}{\ge} \frac{4r}{R^2} \Leftrightarrow (3R - 10r)s^2 + r(12R^2 + 43Rr - 26r^2) \underbrace{\overset{?}{\ge}}_{(*)} 0$$

Case 1 $3R - 10r \ge 0$ *and then* : LHS of $(*) \ge r(12R^2 + 43Rr - 26r^2) \overset{\text{Euler}}{\ge}$ $r(12R^2 + 86r^2 - 26r^2) > 0 \Rightarrow (*)$ is true (strict inequ*a*lity)

Case 2 $3R - 10r < 0$ *and then* : LHS of $(*) = -(10r - 3R)s^2 + r\left(12R^2 + 43Rr - 26r^2\right)$

$$\overset{\text{Gerretsen}}{\ge} -(10r - 3R)(4R^2 + 4Rr + 3r^2) + r(12R^2 + 43Rr - 26r^2) \overset{?}{\ge} 0$$

$$\Leftrightarrow 3t^3 - 4t^2 + 3t - 14 \overset{?}{\ge} 0 \Leftrightarrow (t-2)(t-2)(3t^2 + 2t + 7) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2$$

$\Rightarrow (*)$ is true $\therefore$ combining both c*a*ses, $(*)$ is true $\forall\, \Delta$ ABC

$$\therefore \frac{m_a}{m_b m_c} + \frac{w_b}{w_c w_a} + \frac{h_c}{h_a h_b} \ge \frac{4r}{R^2} \text{ and so,}$$

$$\frac{4r}{R^2} \le \frac{m_a}{m_b m_c} + \frac{w_b}{w_c w_a} + \frac{h_c}{h_a h_b} \le \frac{1}{2r}\left(\frac{81}{16}\left(\frac{R}{r}\right)^5 - 160\right)$$

$\forall\, \Delta$ ABC," =" iff Δ ABC is equil*a*ter*a*l (QED)

1612. In *a*ny Δ ABC, the fo*ll*owing re*la*tionship ho*l*ds :

$$h_a(b+c)^2 + h_b(c+a)^2 + h_c(a+b)^2 \le 54R^3$$

Proposed by Huseyn Elizade, Sultan Cabbarli-Turkiye

Solution by Soumava Chakraborty-Kolkata-India

$$h_a(b+c)^2+h_b(c+a)^2+h_c(a+b)^2=\sum_{cyc}\frac{bc(b^2+c^2+2bc)}{2R}$$

$$\overset{A-G}{\leq}\frac{1}{2R}\left(\sum_{cyc}\frac{(b^2+c^2)^2}{2}+2\sum_{cyc}a^2b^2\right)=\frac{1}{2R}\left(\sum_{cyc}a^4+2\sum_{cyc}a^2b^2+\sum_{cyc}a^2b^2\right)$$

$$=\frac{1}{2R}\left(\left(\sum_{cyc}a^2\right)^2+\sum_{cyc}a^2b^2\right)\overset{\substack{\text{Leibnitz}\\ \text{and}\\ \text{Goldstone}}}{\leq}\frac{1}{2R}(81R^4+4R^2s^2)$$

$$\overset{\text{Mitrinovic}}{\leq}\frac{1}{2R}\left(81R^4+4R^2.\frac{27R^2}{4}\right)\therefore h_a(b+c)^2+h_b(c+a)^2+h_c(a+b)^2\leq 54R^3$$

$$\forall\,\Delta\,ABC,''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

1613. In ΔABC the following relationship holds:

$$\sum_{cyc}\left(\frac{\cos^2\frac{A}{2}+2\sin^2\frac{A}{4}}{3}\right)>\frac{3}{4}$$

Proposed by Khaled Abd Imouti-Syria

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{\cos^2\frac{A}{2}+2\sin^2\frac{A}{4}}{3}=\frac{\frac{1+\cos A}{2}+(1-\cos\frac{A}{2})}{3}=\frac{1+\cos A+2-2\cos\frac{A}{2}}{6}=$$

$$\frac{3+\cos A-2\cos\frac{A}{2}}{6}=\frac{3+2\cos^2\frac{A}{2}-1-2\cos\frac{A}{2}}{6}=\frac{2+2\cos^2\frac{A}{2}-2\cos\frac{A}{2}}{6}=$$

$$\frac{1+\cos^2\frac{A}{2}-\cos\frac{A}{2}+\frac{1}{4}-\frac{1}{4}}{3}=\frac{\frac{3}{4}+(\cos\frac{A}{2}-\frac{1}{2})^2}{3}>\frac{1}{4}$$

1614. In any non − obtuse $\Delta\,ABC$, the following relationship holds :

$$\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}+\frac{2}{3}(\sin^2A+\sin^2B+\sin^2C)\geq 3$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

$\boxed{Case\ 1}$ $\Delta\,ABC$ is right triangle and then : $s=2R+r$ and

$$\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}+\frac{2}{3}(\sin^2 A+\sin^2 B+\sin^2 C)\ge 3$$

$$\Leftrightarrow\left(\sum_{cyc}a^2\right)\cdot\frac{\left(\sum_{cyc}a^2\right)^2+\sum_{cyc}a^2b^2}{\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^2b^2\right)-a^2b^2c^2}+\frac{2}{3.4R^2}\cdot\left(\sum_{cyc}a^2\right)\ge 6$$

$$\Leftrightarrow\frac{(s^2-4Rr-r^2)\left(4(s^2-4Rr-r^2)^2+(s^2+4Rr+r^2)^2-16R^2r^2s^2\right)}{(s^2-4Rr-r^2)((s^2+4Rr+r^2)^2-16R^2r^2s^2)-8R^2r^2s^2}$$

$$+\frac{s^2-4Rr-r^2}{3R^2}\ge 6$$

$$\Leftrightarrow s^8-(3R^2+16Rr)s^6+r(36R^3+37R^2r+16Rr^2-2r^3)s^4$$

$$-r^3(56R^3+69R^2r+16Rr^2)s^2$$

$$+r^3(192R^5+400R^4r+292R^3r^2+99R^2r^3+16Rr^4+r^5)\overset{(1)}{\ge}0$$

a**nd putiing** $s=2R+r$ **in LHS of** (1)**, we have** $(1)\Leftrightarrow R^4-4R^2r^2-4Rr^3-r^4\ge 0$

$$\Leftrightarrow(t^2-2t-1)(t+1)^2\ge 0\ \left(t=\frac{R}{r}\right)\Leftrightarrow t^2-2t-1\overset{(2)}{\ge}0$$

WLOG we may assume $A=90^\circ$ *and* **then :** $a^2=b^2+c^2\overset{A-G}{\ge}\frac{2bca}{a}$

$$\Rightarrow 8R^3\sin^3 90^\circ=8Rrs\Rightarrow R^2\ge r(2R+r)\Rightarrow t^2-2t-1\ge 0\Rightarrow(2)\Rightarrow(1)\text{ is true}$$

$$\therefore\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}+\frac{2}{3}(\sin^2 A+\sin^2 B+\sin^2 C)\ge 3$$

is true $\forall$ **right** $\Delta\,ABC$

Case 2 $\Delta\,ABC$ **is** *a***cute tri***a***ngle** *and* $\because b^2+c^2>a^2$ *and analogs*

$\therefore a^2,b^2,c^2$ **form sides of** *a* **triangle XYZ (***say***)**

$$\Rightarrow\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}+\frac{2}{3}(\sin^2 A+\sin^2 B+\sin^2 C)\ge 3$$

$$\Leftrightarrow\sum_{cyc}\frac{a^2}{b^2+c^2}+\frac{2}{3}\cdot\frac{1}{4.\frac{a^2b^2c^2}{2\sum_{cyc}a^2b^2-\sum_{cyc}a^4}}\cdot\sum_{cyc}a^2$$

$$\Leftrightarrow\sum_{cyc}\frac{x}{y+z}+\frac{2}{3}\cdot\frac{1}{4.\frac{xyz}{2\sum_{yc}xy-\sum_{cyc}x^2}}\cdot\sum_{cyc}x\ge 3$$

$$\Leftrightarrow\sum_{cyc}\frac{x}{y+z}+\frac{1}{6}\cdot\frac{2\sum_{yc}xy-\sum_{cyc}x^2}{xyz}\cdot\sum_{cyc}x\overset{(*)}{\ge}3$$

Now, we sh*all* **prove that** $\forall\,\Delta\,ABC:\sum_{cyc}\frac{a}{b+c}+\frac{1}{6}\cdot\frac{2\sum_{yc}ab-\sum_{cyc}a^2}{abc}\cdot\sum_{cyc}a\overset{(\bullet)}{\ge}3$

$$(\bullet)\Leftrightarrow\left(\sum_{cyc}a\right)\cdot\frac{\sum_{cyc}(c+a)(a+b)}{2s(s^2+2Rr+r^2)}+\frac{1}{6}\cdot\frac{2(s^2+4Rr+r^2)-2(s^2-4Rr-r^2)}{4Rrs}.2s$$

$$\ge 6\Leftrightarrow\frac{2s\left(\left(\sum_{cyc}a^2+2\sum_{cyc}ab\right)+\sum_{cyc}ab\right)}{2s(s^2+2Rr+r^2)}+\frac{(16Rr+4r^2)(2s)}{24Rrs}\ge 6$$

$$\Leftrightarrow \frac{4s^2+s^2+4Rr+r^2}{s^2+2Rr+r^2}+\frac{4R+r}{3R} \geq 6 \Leftrightarrow (R+r)s^2 \overset{(\bullet\bullet)}{\geq} r\left(16R^2+9Rr-r^2\right)$$

We have : $(R+r)s^2 \overset{\text{Gerretsen}}{\geq} (R+r)\left(16Rr-5r^2\right) \overset{?}{\geq} r\left(16R^2+9Rr-r^2\right)$

$\Leftrightarrow 2Rr \overset{?}{\geq} 4r^2 \rightarrow$ true via Euler $\Rightarrow (\bullet\bullet) \Rightarrow (\bullet) \Rightarrow (*)$ is true

$$\therefore \frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}+\frac{2}{3}\left(\sin^2 A+\sin^2 B+\sin^2 C\right) \geq 3\ \forall \text{ acute } \Delta\, ABC$$

$\therefore$ combining both cases, $\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}$

$+\frac{2}{3}\left(\sin^2 A+\sin^2 B+\sin^2 C\right) \geq 3\ \forall$ non − obtuse $\Delta\, ABC$,

" = " iff $\Delta\, ABC$ is right or equilateral (QED)

1615. In any non − obtuse Δ ABC with ω → Brocard's angle,

the following relationship holds :

$$\csc^2\omega+4(\sin^2 A+\sin^2 B+\sin^2 C) \geq 13$$

When does equality hold ?

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

Case 1 $\Delta\, ABC$ is right triangle and then : $s=2R+r$ and

$$\csc^2\omega+4\left(\sin^2 A+\sin^2 B+\sin^2 C\right) \geq 13$$

$$\Leftrightarrow \frac{\left(s^2+4Rr+r^2\right)^2-16Rrs^2}{4r^2s^2}+\frac{s^2-4Rr-r^2}{R^2} \geq 13$$

$$\Leftrightarrow R^2\left(\left((2R+r)^2+4Rr+r^2\right)^2-16Rr(2R+r)^2\right)$$

$$+8r^2(2R+r)^2\left((2R+r)^2-4Rr-r^2\right) \geq 52R^2r^2(2R+r)^2$$

$$\Leftrightarrow R^4-4R^2r^2-4Rr^3-r^4 \geq 0 \Leftrightarrow \left(t^2-2t-1\right)(t+1)^2 \geq 0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow t^2-2t-1 \overset{(1)}{\geq} 0$$

WLOG we may assume $A=90°$ and then : $a^2=b^2+c^2 \overset{A-G}{\geq} \frac{2bca}{a} \Rightarrow 8R^3\sin^3 90°$

$=8Rrs \Rightarrow R^2 \geq r(2R+r) \Rightarrow t^2-2t-1 \geq 0 \Rightarrow (2) \Rightarrow (1)$ is true

$\therefore \csc^2\omega+4\left(\sin^2 A+\sin^2 B+\sin^2 C\right) \geq 13\ \forall$ right $\Delta\, ABC$

Case 2 $\Delta\, ABC$ is acute triangle and $\because b^2+c^2>a^2$ and analogs

$\therefore a^2, b^2, c^2$ form sides of a triangle XYZ (say)

$\Rightarrow \csc^2\omega+4\left(\sin^2 A+\sin^2 B+\sin^2 C\right) \geq 13 \Leftrightarrow$

$$\frac{4\sum_{cyc}a^2b^2}{2\sum_{cyc}a^2b^2-\sum_{cyc}a^4}+\frac{\sum_{cyc}a^2}{\frac{a^2b^2c^2}{2\sum_{cyc}a^2b^2-\sum_{cyc}a^4}} \geq 13$$

$$\Leftrightarrow \boxed{\frac{4\sum_{yc} xy}{2\sum_{yc} xy - \sum_{cyc} x^2} + \frac{\left(\sum_{cyc} x\right)\left(2\sum_{yc} xy - \sum_{cyc} x^2\right)}{xyz} \overset{(*)}{\geq} 13}$$

Now, we shall prove that $\forall \Delta ABC$:

$$\frac{4\sum_{yc} ab}{2\sum_{yc} ab - \sum_{cyc} a^2} + \frac{\left(\sum_{cyc} a\right)\left(2\sum_{yc} ab - \sum_{cyc} a^2\right)}{abc} \overset{(\bullet)}{\geq} 13$$

$$(\bullet) \Leftrightarrow \frac{4(s^2+4Rr+r^2)}{2(s^2+4Rr+r^2) - 2(s^2-4Rr-r^2)}$$

$$+\frac{2s\left(2(s^2+4Rr+r^2) - 2(s^2-4Rr-r^2)\right)}{4Rrs} \geq 13$$

$$\Leftrightarrow \frac{R(s^2+4Rr+r^2) + 2r(4R+r)^2}{Rr(4R+r)} \geq 13 \Leftrightarrow \boxed{Rs^2 \overset{(\bullet\bullet)}{\geq} r(16R^2-4Rr-2r^2)}$$

Now, via Rouche, $Rs^2 \geq R\left(2R^2+10Rr-r^2-2(R-2r).\sqrt{R^2-2Rr}\right)$

$$\overset{?}{\geq} r(16R^2-4Rr-2r^2) \Leftrightarrow 2R^3-6R^2r+3Rr^2+2r^3 \overset{?}{\geq} 2R(R-2r).\sqrt{R^2-2Rr}$$

$$\Leftrightarrow \boxed{(R-2r)(2R^2-2Rr-r^2) \underset{(\bullet\bullet)}{\overset{?}{\geq}} 2R(R-2r).\sqrt{R^2-2Rr}} \text{ and } \because R-2r \overset{\text{Euler}}{\geq} 0$$

$\therefore$ in order to prove $(\bullet\bullet)$, it suffices to prove : $\left(2R^2-2Rr-r^2\right)^2 > 4R^2(R^2-2Rr)$

$\Leftrightarrow 4Rr^3 + r^4 > 0 \rightarrow \text{true} \Rightarrow (\bullet\bullet) \Rightarrow (\bullet) \Rightarrow (*)$ is true

$\therefore \csc^2\omega + 4(\sin^2 A + \sin^2 B + \sin^2 C) \geq 13 \ \forall$ acute ΔABC

$\therefore$ combining both cases, $\csc^2\omega + 4(\sin^2 A + \sin^2 B + \sin^2 C) \geq 13$

$\forall$ non − obtuse ΔABC, " = " iff ΔABC is right or equilateral (QED)

1616. **In any non − obtuse ΔABC, the following relationship holds :**

$$h_a^2 + h_b^2 + h_c^2 + \frac{18F^2}{m_a^2+m_b^2+m_c^2} \geq 8F + 4R^2\cos A\cos B\cos C$$

When does equality hold ?

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

$\boxed{\text{Case 1}}$ ΔABC is right triangle and WLOG we may assume $A = 90°$; then :

$$a^2 = b^2 + c^2 \overset{A-G}{\geq} \frac{2bca}{a} \Rightarrow 8R^3\sin^3 90° = 8Rrs \Rightarrow R^2 \geq r(2R+r)$$

$\Rightarrow t^2 - 2t - 1 \geq 0$ with equality iff ΔABC is right isosceles $\rightarrow$ (i)

ΔABC is right triangle $\Rightarrow s = 2R + r$ and

$$h_a^2 + h_b^2 + h_c^2 + \frac{18F^2}{m_a^2+m_b^2+m_c^2} \geq 8F + 4R^2\cos A\cos B\cos C$$

$$\Leftrightarrow \frac{\left(s^2+4Rr+r^2\right)^2 - 16Rrs^2}{4R^2} + \frac{18r^2s^2}{\frac{3}{2}(s^2-4Rr-r^2)} \geq 8rs$$

$$\Leftrightarrow \frac{\left((2R+r)^2+4Rr+r^2\right)^2-16Rr(2R+r)^2}{4R^2}+\frac{12r^2(2R+r)^2}{4R^2}\ge 8r(2R+r)$$

$$\Leftrightarrow R^4-4R^3r+2R^2r^2+4Rr^3+r^4\ge 0 \Leftrightarrow \boxed{\left(t^2-2t-1\right)^2\ge 0}\ \left(t=\frac{R}{r}\right)\to \text{true}$$

with equality iff Δ ABC is right isosceles (via (i))

$\boxed{\text{Case 2}}$ Δ ABC is acute $\Rightarrow b^2+c^2>a^2$ and analogs $\therefore a^2,b^2,c^2$ form sides of a triangle XYZ (say) $\therefore h_a^2+h_b^2+h_c^2+\frac{18F^2}{m_a^2+m_b^2+m_c^2}-4R^2\cos A\cos B\cos C$

$$=\frac{\sum_{cyc}a^2b^2}{\frac{4a^2b^2c^2}{2\sum_{cyc}a^2b^2-\sum_{cyc}a^4}}+\frac{\frac{18}{16}\left(2\sum_{cyc}a^2b^2-\sum_{cyc}a^4\right)}{\frac{3}{4}\sum_{cyc}a^2}-\left(\left(s^2-4Rr-r^2\right)-4R^2\right)$$

$$=\frac{\left(\sum_{yc}xy\right)\left(2\sum_{yc}xy-\sum_{cyc}x^2\right)}{4xyz}+\frac{\frac{9}{8}\cdot\frac{4}{3}\cdot\left(2\sum_{yc}xy-\sum_{cyc}x^2\right)}{\sum_{cyc}x}-\frac{\sum_{cyc}a^2}{2}+\frac{4xyz}{2\sum_{yc}xy-\sum_{cyc}x^2}$$

$$=\frac{\left(\sum_{yc}xy\right)\left(2\sum_{yc}xy-\sum_{cyc}x^2\right)}{4xyz}+\frac{\frac{9}{8}\cdot\frac{4}{3}\cdot\left(2\sum_{yc}xy-\sum_{cyc}x^2\right)}{\sum_{cyc}x}-\frac{\sum_{cyc}x}{2}$$

$$+\frac{4xyz}{2\sum_{yc}xy-\sum_{cyc}x^2}\ge 8F=2.\sqrt{2\sum_{cyc}a^2b^2-\sum_{cyc}a^4}=2.\sqrt{2\sum_{yc}xy-\sum_{cyc}x^2}$$

$$\Leftrightarrow\frac{\left(\sum_{yc}xy\right)\left(2\sum_{yc}xy-\sum_{cyc}x^2\right)}{4xyz}+\frac{\frac{9}{8}\cdot\frac{4}{3}\cdot\left(2\sum_{yc}xy-\sum_{cyc}x^2\right)}{\sum_{cyc}x}-\frac{\sum_{cyc}x}{2}$$

$$+\frac{4xyz}{2\sum_{yc}xy-\sum_{cyc}x^2}\ \boxed{\overset{(*)}{\ge}}\ 2.\sqrt{2\sum_{yc}xy-\sum_{cyc}x^2}$$

Now, we shall prove that $\forall\ \Delta$ ABC :

$$\frac{\left(\sum_{yc}ab\right)\left(2\sum_{yc}ab-\sum_{cyc}a^2\right)}{4abc}+\frac{\frac{9}{8}\cdot\frac{4}{3}\cdot\left(2\sum_{yc}ab-\sum_{cyc}a^2\right)}{\sum_{cyc}a}-\frac{\sum_{cyc}a}{2}$$

$$+\frac{4abc}{2\sum_{yc}ab-\sum_{cyc}a^2}\ \boxed{\overset{(\bullet)}{\ge}}\ 2.\sqrt{2\sum_{yc}ab-\sum_{cyc}a^2}$$

$$(\bullet)\Leftrightarrow\frac{\left(s^2+4Rr+r^2\right)\left(2\left(s^2+4Rr+r^2\right)-2\left(s^2-4Rr-r^2\right)\right)}{16Rrs}$$

$$+\frac{\frac{9}{8}\cdot\frac{4}{3}\cdot\left(16Rr+4r^2\right)}{2s}-s+\frac{16Rrs}{16Rr+4r^2}\ge 2.\sqrt{16Rr+4r^2}$$

$$\Leftrightarrow\frac{\left(s^2+16Rr+r^2\right)(4R+r)^2-4R(4R+r)s^2+16R^2s^2}{4Rs(4R+r)}\ge 2.\sqrt{16Rr+4r^2}$$

$$\Leftrightarrow\frac{(4R+r)\left((4R+r)\left(16Rr+r^2\right)+rs^2\right)+16R^2s^2}{4Rs(4R+r)}\ge 2.\sqrt{16Rr+4r^2}\Leftrightarrow$$

$$\left((4R+r)\left((4R+r)(16Rr+r^2)+rs^2\right)+16R^2s^2\right)^2 \ge 256(4Rr+r^2)R^2s^2(4R+r)^2$$
$$\Leftrightarrow (256R^4+128R^3r+48R^2r^2+8Rr^3+r^4)s^4$$
$$-rs^2(8192R^5+5632R^4r+640R^3r^2-256R^2r^3-56Rr^4-2r^5)$$
$$+r^2\begin{pmatrix}65536R^6+73728R^5r+33024R^4r^2+7424R^3r^3+864R^2r^4\\+48R^5r+r^6\end{pmatrix} \overset{(\bullet\bullet)}{\ge} 0$$

$\because$ LHS of $(\bullet\bullet)$ is a quadratic polynomial in s^2 $\therefore$ in order to prove $(\bullet\bullet)$, it suffices to prove : $r^2(8192R^5+5632R^4r+640R^3r^2-256R^2r^3-56Rr^4-2r^5)^2$

$$-4\begin{pmatrix}256R^4+128R^3r+\\48R^2r^2+8Rr^3+r^4\end{pmatrix}.r^2\begin{pmatrix}65536R^6+73728R^5r+33024R^4r^2\\+7424R^3r^3+864R^2r^4+48R^5r+r^6\end{pmatrix}<0$$
(i.e., discriminant <0)
$$\Leftrightarrow -1024R^2r^3\begin{pmatrix}16384R^7+40960R^6r+36864R^5r^2+\\16640R^4r^3+4160R^3r^4+576R^2r^5+40Rr^6+r^7\end{pmatrix}<0 \to true$$
$\Rightarrow (\bullet\bullet) \Rightarrow (\bullet) \Rightarrow (*)$ is true (strict inequality)
$$\therefore h_a^2+h_b^2+h_c^2+\frac{18F^2}{m_a^2+m_b^2+m_c^2} > 8F+4R^2\cos A\cos B\cos C \therefore$$
combining both cases, $h_a^2+h_b^2+h_c^2+\dfrac{18F^2}{m_a^2+m_b^2+m_c^2} \ge 8F+4R^2\cos A\cos B\cos C$

$\forall$ non – obtuse Δ ABC,'' ='' iff Δ ABC is right isosceles (QED)

1617. In any Δ ABC, the following relationship holds :

$$\frac{18r^2}{R} \le \frac{m_am_b}{m_c}+\frac{w_bw_c}{w_a}+\frac{h_ch_a}{h_b} \le \frac{9}{r}\left(3.\left(\frac{R^2}{4r}\right)^2-2r^2\right)$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{m_am_b}{m_c}+\frac{w_bw_c}{w_a}+\frac{h_ch_a}{h_b} \le \frac{m_am_b}{h_c}+\frac{m_bm_c}{h_a}+\frac{m_cm_a}{h_b} = m_am_bm_c.\sum_{cyc}\frac{1}{h_am_a}$$
$$\overset{m_am_bm_c \le \frac{Rs^2}{2}}{\le} \frac{Rs^2}{2}.\sum_{cyc}\frac{1}{h_a^2} = \frac{Rs^2}{2}.\frac{1}{4r^2s^2}.\sum_{cyc}a^2 \overset{\text{Leibnitz}}{\le} \frac{Rs^2}{2}.\frac{9R^2}{4r^2s^2} = \frac{9R^3}{8r^2}$$
$$\overset{?}{\le} \frac{9}{r}\left(3.\left(\frac{R^2}{4r}\right)^2-2r^2\right) \Leftrightarrow 3R^4-32r^4 \overset{?}{\ge} 2R^3r \Leftrightarrow 3t^4-2t^3-32 \overset{?}{\ge} 0 \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)(3t^3+4t^2+8t+16) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2$$
$$\therefore \frac{m_am_b}{m_c}+\frac{w_bw_c}{w_a}+\frac{h_ch_a}{h_b} \le \frac{9}{r}\left(3.\left(\frac{R^2}{4r}\right)^2-2r^2\right)$$
$$\text{Again, } \frac{m_am_b}{m_c}+\frac{w_bw_c}{w_a}+\frac{h_ch_a}{h_b} \ge \frac{h_ah_b}{m_c}+\frac{h_bh_c}{m_a}+\frac{h_ch_a}{m_b} = \sum_{cyc}\frac{ca.ab}{4R^2m_a}$$

$$=\frac{4Rrs}{4R^2}\cdot\sum_{cyc}\frac{a^2}{am_a}\overset{Bergstrom}{\geq}\frac{rs}{R}\cdot\frac{4s^2}{\sum_{cyc}am_a}\overset{Chebyshev}{\geq}\frac{rs}{R}\cdot\frac{4s^2}{\frac{1}{3}\left(\sum_{cyc}a\right)\left(\sum_{cyc}m_a\right)}$$

$$(\because \text{WLOG assuming } a\geq b\geq c\Rightarrow m_a\leq m_b\leq m_c)\overset{Leuenberger}{\geq}\frac{12rs.s^2}{2Rs(4R+r)}$$

$$\overset{Euler}{\geq}\frac{6rs^2}{\frac{9R^2}{2}}=\frac{2r.2s^2}{3R^2}\overset{Gerretsen}{\geq}\frac{2r.\left(27Rr+5r(R-2r)\right)}{3R^2}\overset{Euler}{\geq}\frac{2r.(27Rr)}{3R^2}$$

$$\therefore\frac{m_am_b}{m_c}+\frac{w_bw_c}{w_a}+\frac{h_ch_a}{h_b}\geq\frac{18r^2}{R}$$

$$\therefore\frac{18r^2}{R}\leq\frac{m_am_b}{m_c}+\frac{w_bw_c}{w_a}+\frac{h_ch_a}{h_b}\leq\frac{9}{r}\left(3.\left(\frac{R^2}{4r}\right)^2-2r^2\right)$$

$\forall\,\Delta\,ABC$, " = " iff $\Delta\,ABC$ is equilateral (QED)

Proof of $m_am_bm_c\leq\frac{Rs^2}{2}$

$$m_a^2m_b^2m_c^2=\frac{1}{64}(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)$$

$$\overset{(1)}{=}\frac{1}{64}\left\{-4\sum_{cyc}a^6+6\left(\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4\right)+3a^2b^2c^2\right\}$$

$$\text{Now,}\sum_{cyc}a^6=\left(\sum_{cyc}a^2\right)^3-3(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

$$=\left(\sum_{cyc}a^2\right)^3-3\left(2a^2b^2c^2+\sum_{cyc}\left(a^2b^2\left(\sum_{cyc}a^2-c^2\right)\right)\right)$$

$$=\left(\sum_{cyc}a^2\right)^3+3a^2b^2c^2-3\left(\sum_{cyc}a^2\,b^2\right)\left(\sum_{cyc}a^2\right)$$

$$\therefore\sum_{cyc}a^6\overset{(2)}{=}\left(\sum_{cyc}a^2\right)^3+3a^2b^2c^2-3\left(\sum_{cyc}a^2\,b^2\right)\left(\sum_{cyc}a^2\right)$$

$$\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4=\sum_{cyc}\left(a^2b^2\left(\sum_{cyc}a^2-c^2\right)\right)\overset{(3)}{=}$$

$$\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) - 3a^2b^2c^2 \therefore (1),(2),(3) \Rightarrow m_a^2 m_b^2 m_c^2$$

$$= \frac{1}{64}\left(\begin{array}{c} -4\left(\sum_{cyc} a^2\right)^3 - 12a^2b^2c^2 + 12\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) \\ +6\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) - 18a^2b^2c^2 + 3a^2b^2c^2 \end{array}\right)$$

$$= \frac{1}{64}\left(-4\left(\sum_{cyc} a^2\right)^3 + 18\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) - 27a^2b^2c^2\right)$$

$$= \frac{1}{64}\left(-4\left(\sum_{cyc} a^2\right)^3 + 18\left(\left(\sum_{cyc} ab\right)^2 - 16Rrs^2\right)\left(\sum_{cyc} a^2\right) - 27a^2b^2c^2\right)$$

$$= \frac{1}{64}\left\{\begin{array}{c} -32(s^2-4Rr-r^2)^3 + 36(s^2-4Rr-r^2)(s^2+4Rr+r^2)^2 \\ -576Rrs^2(s^2-4Rr-r^2) - 432R^2r^2s^2 \end{array}\right\}$$

$$= \frac{1}{16}\{s^6 - s^4(12Rr - 33r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4) - r^3(4R+r)^3\}$$

$$\leq \frac{R^2s^4}{4} \Leftrightarrow$$

$$s^6 - s^4(4R^2 + 12Rr - 33r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4) - r^3(4R+r)^3 \overset{(\bullet)}{\leq} 0$$

$$\text{Now, LHS of } (\bullet) \overset{\text{Gerretsen}}{\leq} -s^4(8Rr - 36r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4)$$

$$-r^3(4R+r)^3 \overset{?}{\leq} 0$$

$$\Leftrightarrow s^4(8R - 16r) + s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R+r)^3 \underset{(\bullet\bullet)}{\overset{?}{\geq}} 20rs^4$$

$$\text{Now, LHS of } (\bullet\bullet) \underset{(a)}{\underbrace{\overset{\text{Gerretsen}}{\geq}}} s^2(16Rr - 5r^2)(8R - 16r)$$

$$+s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R+r)^3 \textit{ and}$$

$$\text{RHS of } (\bullet\bullet) \underset{(b)}{\underbrace{\overset{\text{Geretsen}}{\leq}}} 20rs^2(4R^2 + 4Rr + 3r^2)$$

$(a),(b) \Rightarrow$ **in order to prove $(\bullet\bullet)$, it suffices to prove :**

$$s^2(16Rr - 5r^2)(8R - 16r) + s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R+r)^3$$

$$\geq 20rs^2(4R^2 + 4Rr + 3r^2)$$

$$\Leftrightarrow s^2(108R^2 - 256Rr + 53r^2) + r(4R+r)^3 \geq 0$$

$$\Leftrightarrow s^2(108R^2-256Rr+80r^2)+r(4R+r)^3 \overset{(\bullet\bullet\bullet)}{\geq} 27r^2s^2$$

$$\text{Now, LHS of }(\bullet\bullet\bullet)\underbrace{\overset{\text{Gerretsen}}{\geq}}_{(c)}(108R^2-256Rr+80r^2)(16Rr-5r^2)+r(4R+r)^3$$

$$\text{and RHS of }(\bullet\bullet\bullet)\underbrace{\overset{\text{Geretsen}}{\leq}}_{(d)}27r^2(4R^2+4Rr+3r^2)$$

$$(c),(d)\Rightarrow \text{in order to prove }(\bullet\bullet\bullet),\text{it suffices to prove :}$$

$$(108R^2-256Rr+80r^2)(16Rr-5r^2)+r(4R+r)^3 \geq 27r^2(4R^2+4Rr+3r^2)$$

$$\Leftrightarrow 224t^3-587t^2+308t-60\geq 0\left(\text{where } t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\big((t-2)(224t+309)+648\big)\geq 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (\bullet\bullet\bullet)\Rightarrow(\bullet\bullet)$$

$$\Rightarrow (\bullet)\text{ is true} \Rightarrow m_a^2m_b^2m_c^2\leq \frac{R^2s^4}{4}\Rightarrow m_am_bm_c\leq\frac{Rs^2}{2}\text{ (QED)}$$

1618. In any Δ ABC, the following relationship holds :

$$\frac{48r^3}{28R^3-192r^3}\leq\frac{h_a}{h_b+h_c}+\frac{w_b}{w_c+w_a}+\frac{m_c}{m_a+m_b}\leq 3.\left(28.\left(\frac{R}{4r}\right)^3-3\right)$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{h_a}{h_b+h_c}+\frac{w_b}{w_c+w_a}+\frac{m_c}{m_a+m_b}\leq\frac{m_a}{h_b+h_c}+\frac{m_b}{h_c+h_a}+\frac{m_c}{h_a+h_b}$$

$$\overset{\text{Reverse Bergstrom}}{\leq}\frac{1}{4}\sum_{cyc}\left(\frac{m_a}{h_b}+\frac{m_a}{h_c}\right)\overset{\text{CBS}}{\leq}\frac{1}{4}.\sqrt{2\sum_{cyc}m_a^2}.\sqrt{2\sum_{cyc}\frac{1}{h_a^2}}$$

$$=\frac{1}{4}.\sqrt{4.\frac{3}{4}.\sum_{cyc}a^2.\frac{1}{4r^2s^2}.\sum_{cyc}a^2}\overset{\substack{\text{Leibnitz}\\ \text{and}\\ \text{Mitrinovic}}}{\leq}\frac{1}{4}.\sqrt{\frac{3.81R^4}{4r^2.27r^2}}=\sqrt{\frac{9R^4}{64r^4}}$$

$$\overset{?}{\leq}3.\left(28.\left(\frac{R}{4r}\right)^3-3\right)=\sqrt{\frac{9(7R^3-48r^3)^2}{256r^6}}\Leftrightarrow(7t^3-48)^2\overset{?}{\geq}4t^4\left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow 49t^6-4t^4-672t^3+2304\overset{?}{\geq}0$$

$$\Leftrightarrow (t-2)\left((t-2)(49t^4+196t^3+584t^2+880t+1184)+1216\right)\overset{?}{\geq}0$$

$$\to \text{true}\because t\overset{\text{Euler}}{\geq}2 \therefore \boxed{\frac{h_a}{h_b+h_c}+\frac{w_b}{w_c+w_a}+\frac{m_c}{m_a+m_b}\leq 3.\left(28.\left(\frac{R}{4r}\right)^3-3\right)}$$

$$\text{Again,}\frac{h_a}{h_b+h_c}+\frac{w_b}{w_c+w_a}+\frac{m_c}{m_a+m_b}\geq\frac{h_a}{m_b+m_c}+\frac{h_b}{m_c+m_a}+\frac{h_c}{m_a+m_b}$$

$$\overset{\text{Chebyshev}}{\geq}\frac{1}{3}\left(\sum_{cyc}h_a\right)\left(\sum_{cyc}\frac{1}{m_b+m_c}\right)\left(\begin{array}{c}\because \text{WLOG } assuming\ a\geq b\geq c\Rightarrow h_a\leq h_b\leq h_c\ and\\ \frac{1}{m_b+m_c}\leq\frac{1}{m_c+m_a}\leq\frac{1}{m_a+m_b}\end{array}\right)$$

$$=\frac{1}{3}\left(2rs\sum_{cyc}\frac{1}{a}\right)\left(\sum_{cyc}\frac{1}{m_b+m_c}\right)\overset{\text{Bergstrom}}{\geq}\frac{1}{3}.2rs.\frac{9}{2s}.\frac{9}{2\sum_{cyc}m_a}\overset{\text{Leuenberger + Euler}}{\geq}3r.\frac{9}{2.\frac{9R}{2}}$$

$$=\frac{3r}{R}\overset{?}{\geq}\frac{48r^3}{28R^3-192r^3}\Leftrightarrow 28t^3-16t-192\overset{?}{\geq}0\Leftrightarrow(t-2)(28t^2+56t+96)\overset{?}{\geq}0$$

$$\rightarrow\text{true}\because t\overset{\text{Euler}}{\geq}2\therefore\boxed{\frac{h_a}{h_b+h_c}+\frac{w_b}{w_c+w_a}+\frac{m_c}{m_a+m_b}\geq\frac{48r^3}{28R^3-192r^3}}$$

$$and\ so,\frac{48r^3}{28R^3-192r^3}\leq\frac{h_a}{h_b+h_c}+\frac{w_b}{w_c+w_a}+\frac{m_c}{m_a+m_b}\leq 3.\left(28.\left(\frac{R}{4r}\right)^3-3\right)$$

$\forall\,\Delta\,ABC,''=''$ iff $\Delta\,ABC$ is equilateral (QED)

1619. In $\Delta ABC, O$ –circumcenter, r_1,r_2,r_3 –inradii of $\Delta OBC, \Delta OCA, \Delta OAB$. Prove that:

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\geq\frac{6+4\sqrt{3}}{R}$$

Proposed by Mehmet Şahin-Turkiye

Solution by Daniel Sitaru-Romania

Known results:

$$\sum_{cyc}\frac{1}{sin2A}\geq\frac{9R^2}{2F}\geq\frac{s^2+(R+r)^2}{2F}\geq\frac{2(4R+r)}{s}\geq 2\sqrt{3}\qquad(1)$$

$$\sum_{cyc}\frac{1}{cosA}=\frac{s^2+r^2-4R^2}{s^2-(2R+r)^2}\geq 6\qquad(2)$$

Back to the problem:

$$\sum_{cyc}\frac{1}{r_1}=\sum_{cyc}\frac{\frac{OB+OC+BC}{2}}{[OBC]}=\sum_{cyc}\frac{\frac{1}{2}(R+R+a)}{\frac{1}{2}\cdot R\cdot R\cdot sin2A}=\sum_{cyc}\frac{2R+a}{R^2sin2A}=$$

$$=\sum_{cyc}\frac{2R}{R^2sin2A}+\sum_{cyc}\frac{a}{R^2sin2A}=\frac{2}{R}\sum_{cyc}\frac{1}{sin2A}+\sum_{cyc}\frac{2RsinA}{2R^2sinAcosA}\geq$$

$$\overset{(1)}{\geq}\frac{2}{R}\cdot 2\sqrt{3}+\frac{1}{R}\sum_{cyc}\frac{1}{cosA}\overset{(2)}{\geq}\frac{4\sqrt{3}}{R}+\frac{1}{R}\cdot 6=\frac{6+4\sqrt{3}}{R}$$

Equality holds for $a=b=c$.

1620. In any Δ ABC, the fol$l$owing rel$a$tionship holds :

$$1+\frac{4R}{r}\geq\sqrt{\cot^2\frac{A}{2}+\cot^2\frac{B}{2}+3}+\sqrt{\cot^2\frac{B}{2}+\cot^2\frac{C}{2}+3}+\sqrt{\cot^2\frac{C}{2}+\cot^2\frac{A}{2}+3}\geq\frac{5\sqrt{3}s}{6r}+\frac{3}{2}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\cot\frac{A}{2}=\sum_{cyc}\frac{s}{r_a}=\frac{s}{r}\Rightarrow\sum_{cyc}\cot\frac{A}{2}=\frac{s}{r}\to(1)\ and\ \sum_{cyc}\cot\frac{A}{2}\cot\frac{B}{2}$$

$$=\sum_{cyc}\frac{s}{r_a}\cdot\frac{s}{r_b}=\frac{s^2}{rs^2}\cdot\sum_{cyc}r_a\Rightarrow\sum_{cyc}\cot\frac{A}{2}\cot\frac{B}{2}=\frac{4R+r}{r}\to(2)\ and\ also,$$

$$\prod_{cyc}\cot\frac{A}{2}=\prod_{cyc}\frac{s}{r_a}=\frac{s^3}{rs^2}\therefore\prod_{cyc}\cot\frac{A}{2}=\frac{s}{r}\to(3)$$

$$\text{Now, }\sqrt{\cot^2\frac{A}{2}+\cot^2\frac{B}{2}+3}+\sqrt{\cot^2\frac{B}{2}+\cot^2\frac{C}{2}+3}+\sqrt{\cot^2\frac{C}{2}+\cot^2\frac{A}{2}+3}$$

$$=\sum_{cyc}\left(\sqrt{\left(\cot^2\frac{B}{2}+\cot^2\frac{C}{2}+3\right)\cot\frac{A}{2}}\cdot\sqrt{\tan\frac{A}{2}}\right)$$

$$\overset{CBS}{\leq}\sqrt{\sum_{cyc}\left(\left(\cot^2\frac{B}{2}+\cot^2\frac{C}{2}+3\right)\cot\frac{A}{2}\right)}\cdot\sqrt{\sum_{cyc}\tan\frac{A}{2}}$$

$$=\sqrt{3\sum_{cyc}\cot\frac{A}{2}+\left(\sum_{cyc}\cot\frac{A}{2}\right)\left(\sum_{cyc}\cot\frac{A}{2}\cot\frac{B}{2}\right)-3\prod_{cyc}\cot\frac{A}{2}}\cdot\sqrt{\frac{4R+r}{s}}$$

$$\overset{via\ (1),(2)\ and\ (3)}{=}\sqrt{\frac{3s}{r}+\frac{s}{r}\cdot\frac{4R+r}{r}-\frac{3s}{r}}\cdot\sqrt{\frac{4R+r}{s}}$$

$$\therefore\boxed{\sqrt{\cot^2\frac{A}{2}+\cot^2\frac{B}{2}+3}+\sqrt{\cot^2\frac{B}{2}+\cot^2\frac{C}{2}+3}+\sqrt{\cot^2\frac{C}{2}+\cot^2\frac{A}{2}+3}\leq\frac{4R+r}{r}}$$

$$\text{Again, }\sqrt{\cot^2\frac{A}{2}+\cot^2\frac{B}{2}+3}+\sqrt{\cot^2\frac{B}{2}+\cot^2\frac{C}{2}+3}+\sqrt{\cot^2\frac{C}{2}+\cot^2\frac{A}{2}+3}$$

$$\geq\frac{5\sqrt{3}s}{6r}+\frac{3}{2}\Leftrightarrow 2\sum_{cyc}\cot^2\frac{A}{2}+9+$$

$$2\sum_{cyc}\sqrt{\left(\cot^2\frac{A}{2}+\cot^2\frac{B}{2}+3\right)\left(\cot^2\frac{A}{2}+\cot^2\frac{C}{2}+3\right)}\overset{(*)}{\geq}\left(\frac{5\sqrt{3}s}{6r}+\frac{3}{2}\right)^2$$

Now, via Reverse CBS, $2\sum_{cyc}\sqrt{\left(\cot^2\frac{A}{2}+\cot^2\frac{B}{2}+3\right)\left(\cot^2\frac{A}{2}+\cot^2\frac{C}{2}+3\right)}$

$$\geq 2\sum_{cyc}\left(\cot^2\frac{A}{2}+\cot\frac{B}{2}\cot\frac{C}{2}+3\right)\Rightarrow \text{LHS of }(*)$$

$$\geq 4\sum_{cyc}\cot^2\frac{A}{2}+27+2\sum_{cyc}\cot\frac{A}{2}\cot\frac{B}{2}$$

$$=\left(4\sum_{cyc}\cot^2\frac{A}{2}+8\sum_{cyc}\cot\frac{A}{2}\cot\frac{B}{2}\right)-6\sum_{cyc}\cot\frac{A}{2}\cot\frac{B}{2}+27$$

$$=4\left(\sum_{cyc}\cot\frac{A}{2}\right)^2-6\sum_{cyc}\cot\frac{A}{2}\cot\frac{B}{2}+27\overset{\text{via }(1)\text{ and }(2)}{=}\frac{4s^2}{r^2}-\frac{6(4R+r)}{r}+27$$

$$=\frac{4s^2-24Rr+21r^2}{r^2}\overset{?}{\geq}\left(\frac{5\sqrt{3}s}{6r}+\frac{3}{2}\right)^2=\frac{25s^2+27r^2+30\sqrt{3}rs}{12r^2}$$

$$\Leftrightarrow \boxed{23s^2-288Rr+225r^2\underset{(**)}{\overset{?}{\geq}}30\sqrt{3}rs}\text{ and }\because 23s^2-288Rr+225r^2\overset{\text{Gerretsen}}{\geq}$$

$$23(16Rr-5r^2)-288Rr+225r^2=80Rr+110r^2>0$$

$$\therefore (**)\Leftrightarrow\left(23s^2-288Rr+225r^2\right)^2\geq 2700r^2s^2$$

$$\Leftrightarrow 529s^4-(13248Rr-7650r^2)s^2+r^2(82944R^2-129600Rr+50625r^2)$$

$\boxed{\overset{(***)}{\geq}}$ 0 *and* $\because 529(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0$ $\therefore$ in order to prove $(***)$,

it suffices to prove : LHS of $(***)\geq 529(s^2-16Rr+5r^2)^2$

$$\Leftrightarrow (92R+59r)s^2\boxed{\overset{(****)}{\geq}}r(1312R^2+1124Rr-935r^2)$$

Again, $(92R+59r)s^2\overset{\text{Gerretsen}}{\geq}(92R+59r)(16Rr-5r^2)$

$$\overset{?}{\geq}r(1312R^2+1124Rr-935r^2)\Leftrightarrow 6400r^2(R-2r)^2\overset{?}{\geq}0\rightarrow\text{true}$$

$$\Rightarrow(****)\Rightarrow(***)\Rightarrow(**)\Rightarrow(*)\text{ is true}$$

$$\therefore\boxed{\sqrt{\cot^2\frac{A}{2}+\cot^2\frac{B}{2}+3}+\sqrt{\cot^2\frac{B}{2}+\cot^2\frac{C}{2}+3}+\sqrt{\cot^2\frac{C}{2}+\cot^2\frac{A}{2}+3}\geq\frac{5\sqrt{3}s}{6r}+\frac{3}{2}}\text{ and so,}$$

$$1+\frac{4R}{r}\geq\sqrt{\cot^2\frac{A}{2}+\cot^2\frac{B}{2}+3}+\sqrt{\cot^2\frac{B}{2}+\cot^2\frac{C}{2}+3}+\sqrt{\cot^2\frac{C}{2}+\cot^2\frac{A}{2}+3}$$

$$\geq\frac{5\sqrt{3}s}{6r}+\frac{3}{2}\ \forall\,\Delta\,ABC,''=''\text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$

1621. In any Δ ABC, the following relationship holds:

$$\frac{a}{9R^2-a^2}+\frac{b}{9R^2-b^2}+\frac{c}{9R^2-c^2}\le\frac{9}{2\sqrt{3(a^2+b^2+c^2)}}$$

(Inspired by a problem of *Zaza Mzhavanadze*)

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a}{9R^2-a^2}+\frac{b}{9R^2-b^2}+\frac{c}{9R^2-c^2}=$$

$$=\frac{\sum_{cyc}\left(a\left(81R^4-9R^2(b^2+c^2)+b^2c^2\right)\right)}{729R^6-81R^4\sum_{cyc}a^2+9R^2\sum_{cyc}b^2c^2-16R^2r^2s^2}$$

$$=\frac{2s.81R^4-9R^2\left(2s(s^2+4Rr+r^2)-12Rrs\right)+4Rrs(s^2+4Rr+r^2)}{729R^6-162R^4(s^2-4Rr-r^2)+9R^2((s^2+4Rr+r^2)^2-16Rrs^2)-16R^2r^2s^2}$$

$$\le\frac{9}{2\sqrt{3(a^2+b^2+c^2)}}$$

$$\Leftrightarrow\left(\frac{2s.81R^4-9R^2\left(2s(s^2+4Rr+r^2)-12Rrs\right)+4Rrs(s^2+4Rr+r^2)}{729R^6-162R^4(s^2-4Rr-r^2)+9R^2((s^2+4Rr+r^2)^2-16Rrs^2)-16R^2r^2s^2}\right)^2$$

$$\le\frac{27}{8(s^2-4Rr-r^2)}$$

$$\Leftrightarrow 14348907R^{10}+25509168R^9r+23383404R^8r^2+13541904R^7r^3+$$
$$5401890R^6r^4+1504656R^5r^5+288684R^4r^6+34992R^3r^7+2187R^2r^8$$
$$-(405R^2-1152Rr+128r^2)s^8$$
$$-(32076R^4+24624R^3r+3924R^2r^2-640Rr^3+128r^4)s^6$$
$$+\binom{852930R^6+664848R^5r+219348R^4r^2+29520R^3r^3-}{94R^2r^4-128Rr^5+2796r^6}s^4$$
$$-\binom{6377292R^8+7663248R^7r+4534380R^6r^2+1635552R^5r^3}{+283572R^4r^4+8656R^3r^5-2796R^2r^6-384Rr^7-128r^8}s^2\overset{(*)}{\ge}0$$

$$\text{Now, }-(405R^2-1152Rr+128r^2)s^8$$
$$-(32076R^4+24624R^3r+3924R^2r^2-640Rr^3+128r^4)s^6$$
$$=-(405R-342r)(R-2r)s^8+556r^2s^8$$
$$-(32076R^4+24624R^3r+3924R^2r^2-640Rr^3+128r^4)s^6\overset{\text{Gerretsen}}{\ge}$$
$$-(405R-342r)(R-2r)(4R^2+4Rr+3r^2)s^6+556r^2(16Rr-5r^2)s^6$$
$$-(32076R^4+24624R^3r+3924R^2r^2-640Rr^3+128r^4)s^6$$
$$=-(33696R^4+21636R^3r+3267R^2r^2-10256Rr^3+4960r^4)s^6$$

$\therefore$ in order to prove $(*)$, it suffices to prove:

$$14348907R^{10}+25509168R^9r+23383404R^8r^2+13541904R^7r^3$$
$$+5401890R^6r^4+1504656R^5r^5+288684R^4r^6+34992R^3r^7+2187R^2r^8$$
$$-(33696R^4+21636R^3r+3267R^2r^2-10256Rr^3+4960r^4)s^6$$
$$+\binom{852930R^6+664848R^5r+219348R^4r^2+29520R^3r^3}{-94R^2r^4-128Rr^5+2796r^6}s^4$$

$$-\begin{pmatrix}6377292R^8+7663248R^7r+4534380R^6r^2+1635552R^5r^3+\\283572R^4r^4+8656R^3r^5-2796R^2r^6-384Rr^7-128r^8\end{pmatrix}s^2\ \boxed{\overset{(**)}{\geq}}\ 0\ \textit{and}$$

$$\because -\begin{pmatrix}33696R^4+21636R^3r+3267R^2r^2\\-10256Rr^3+4960r^4\end{pmatrix}(s^2-4R^2-4Rr-3r^2)^3$$

$$\overset{\text{Gerretsen}}{\geq}\ 0\ \therefore \text{in order to prove } (**), \text{it suffices to prove : LHS of } (**) \geq$$

$$-\begin{pmatrix}33696R^4+21636R^3r+3267R^2r^2\\-10256Rr^3+4960r^4\end{pmatrix}(s^2-4R^2-4Rr-3r^2)^3 \Leftrightarrow$$

$$12192363R^{10}+17654832R^9r+7698348R^8r^2-5559664R^7r^3-10151374R^6r^4$$
$$-6243072R^5r^5-1837528R^4r^6-63424R^3r^7-228294R^2r^8$$
$$-258768Rr^9-133920r^{10}$$
$$+\begin{pmatrix}448578R^6+864R^5r-382752R^4r^2-81336R^3r^3+34055R^2r^4\\+32656Rr^5-44512r^6\end{pmatrix}s^4$$
$$-\begin{pmatrix}4759884R^8+3389904R^7r-1743012R^6r^2-3208224R^5r^3\\-1829556R^4r^4-56180R^3r^5+52227R^2r^6-80592Rr^7-134048r^8\end{pmatrix}s^2\ \boxed{\overset{(***)}{\geq}}\ 0\ \textit{and}$$

$$\because 448578R^6+864R^5r-382752R^4r^2-81336R^3r^3+$$
$$34055R^2r^4+32656Rr^5-44512r^6$$
$$=(R-2r)\begin{pmatrix}448578R^5+898020R^4r+1413288R^3r^2+\\2745240R^2r^3+5524535Rr^4+11081726r^5\end{pmatrix}+22118940r^6$$
$$\overset{\text{Euler}}{\geq}\ 22118940r^6>0\ \therefore$$
$$\begin{pmatrix}448578R^6+864R^5r-382752R^4r^2-81336R^3r^3\\+34055R^2r^4+32656Rr^5-44512r^6\end{pmatrix}(s^2-4R^2-4Rr-3r^2)^2$$
$$\overset{\text{Gerretsen}}{\geq}\ 0\ \therefore \text{in order to prove } (***), \text{it suffices to prove : LHS of } (***) \geq$$
$$\begin{pmatrix}448578R^6+864R^5r-382752R^4r^2-81336R^3r^3+\\34055R^2r^4+32656Rr^5-44512r^6\end{pmatrix}(s^2-4R^2-4Rr-3r^2)^2$$
$$\Leftrightarrow 5015115R^{10}+3286512R^9r-4148388R^8r^2-2810656R^7r^3+3158640R^6r^4$$
$$+4576384R^5r^5+1864304R^4r^6-30576R^3r^7+461947R^2r^8$$
$$+515616Rr^9+266688r^{10}\ \boxed{\overset{(****)}{\geq}}$$
$$\begin{pmatrix}1171260R^8-205632R^7r-1379376R^6r^2+499296R^5r^3+\\845204R^4r^4-101852R^3r^5-57255R^2r^6+79568Rr^7+133024r^8\end{pmatrix}s^2$$

$$\text{Now, } 1171260R^8-205632R^7r-1379376R^6r^2+499296R^5r^3+845204R^4r^4$$
$$-101852R^3r^5-57255R^2r^6+79568Rr^7+133024r^8$$
$$=(R-2r)\begin{pmatrix}1171260R^7+2136888R^6r+2894400R^5r^2+6288096R^4r^3+\\13421396R^3r^4+26740940R^2r^5+53424625Rr^6+106928818r^7\end{pmatrix}$$
$$+213990660r^8\ \overset{\text{Euler}}{\geq}\ 213990660r^8>0\ \therefore \text{RHS of } (****)\ \overset{\text{Blundon-Gerretsen}}{\leq}$$
$$\begin{pmatrix}1171260R^8-205632R^7r-1379376R^6r^2+499296R^5r^3+845204R^4r^4\\-101852R^3r^5-57255R^2r^6+79568Rr^7+133024r^8\end{pmatrix}\cdot\frac{R(4R+r)^2}{4R-2r}$$
$$\overset{?}{\leq} \text{LHS of } (****)$$
$$\Leftrightarrow 1320300t^{11}-2964150t^{10}-622764t^9+306056t^8+2117616t^7$$
$$+6356960t^6-809860t^5-4564108t^4-798733t^3-5190t^2$$
$$-97504t-533376\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)\left(\begin{matrix}1320300t^9+2317050t^8+3364236t^7+4494800t^6+\\6639872t^5+14937248t^4+312379644t^3+65205476t^2\\+130504595t+261191286\end{matrix}\right)\begin{matrix}\\ \\ \\ +522649260\end{matrix}\right)$$

$$\overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (****) \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{a}{9R^2-a^2}+\frac{b}{9R^2-b^2}+\frac{c}{9R^2-c^2} \leq \frac{9}{2\sqrt{3(a^2+b^2+c^2)}}$$

$$\forall \Delta ABC, '' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1622.

In any Δ ABC, the following relationship holds :

$$\sum_{cyc} \sqrt[2023]{\frac{m_a^2}{m_b^2+m_c^2}}+\frac{R^3}{r^3} \geq 8+max\left\{\sum_{cyc}\sqrt[2023]{\frac{a}{b+c}}, \sum_{cyc}\sqrt[2023]{\frac{w_a^2}{w_b^2+w_c^2}}\right\}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\sqrt[2023]{\frac{m_a^2}{m_b^2+m_c^2}} \overset{\text{Panaitopol}}{\geq} \sum_{cyc}\sqrt[2023]{\frac{4r^2}{R^2}\cdot\frac{h_a^2}{h_b^2+h_c^2}} = \sum_{cyc}\sqrt[2023]{\frac{4r^2}{R^2}\cdot\frac{b^2c^2}{a^2(b^2+c^2)}}$$

$$\overset{\text{Bandila}}{\geq} \sum_{cyc}\sqrt[2023]{\frac{4r^2}{R^2}\cdot\frac{b^2c^2}{a^2.\frac{R}{r}.bc}} = \sqrt[2023]{\frac{4r^3}{R^3}}\cdot\sum_{cyc}\sqrt[2023]{\frac{bc}{a^2}} \overset{A-G}{\geq} 3.\sqrt[2023]{\frac{4r^3}{R^3}}\cdot\sqrt[6069]{\prod_{cyc}\frac{bc}{a^2}}$$

$$\therefore \sum_{cyc}\sqrt[2023]{\frac{m_a^2}{m_b^2+m_c^2}} \geq 3.\sqrt[2023]{\frac{4r^3}{R^3}} \to (1)$$

$$\text{Now, via Power}-\text{Mean Inequality}, \left(\frac{\sum_{cyc} x^{\frac{1}{2023}}}{3}\right)^{2023} \leq \frac{\sum_{cyc} x}{3}$$

$$\Rightarrow \sum_{cyc} x^{\frac{1}{2023}} \leq 3.\sqrt[2023]{\frac{\sum_{cyc} x}{3}} \to (i)$$

$$r_b+r_c = s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right) = \frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)} = 4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c \overset{(i)}{=} 4R\cos^2\frac{A}{2}$$

$$\text{Now, } (b+c)^2 \overset{?}{\geq} 32Rr\cos^2\frac{A}{2} \overset{\text{via (i)}}{=} 8r(r_b+r_c) = 8r^2s\left(\frac{1}{s-b}+\frac{1}{s-c}\right)$$

$$= 8(s-a)(s-b)(s-c)\frac{a}{(s-b)(s-c)} = 4a(b+c-a)$$

$$\Leftrightarrow (b+c)^2+4a^2-4a(b+c) \overset{?}{\geq} 0$$

$$\Leftrightarrow (b+c-2a)^2 \overset{?}{\geq} 0 \to \text{true} \therefore b+c \geq \sqrt{32Rr}.\cos\frac{A}{2} \textit{ and analogs} \to \text{(ii)}$$

$$\text{We have}: \frac{w_a}{h_a} = \frac{\frac{2bc}{b+c}\cos\frac{A}{2}.a}{2rs} \overset{\text{via (ii)}}{\leq} \frac{2.4Rrs.\cos\frac{A}{2}}{2rs.4.\sqrt{2Rr}.\cos\frac{A}{2}} \Rightarrow w_a^2 \leq \frac{R}{2r}.h_a^2 \textit{ and analogs}$$

$$\Rightarrow \sum_{cyc}\sqrt[2023]{\frac{w_a^2}{w_b^2+w_c^2}} \overset{\text{via (i)}}{\leq} 3.\sqrt[2023]{\frac{\sum_{cyc}\frac{w_a^2}{w_b^2+w_c^2}}{3}} \overset{\text{Reverse Bergstrom}}{\leq} 3.\sqrt[2023]{\frac{1}{12}\sum_{cyc}\left(\frac{w_a^2}{w_b^2}+\frac{w_a^2}{w_c^2}\right)}$$

$$\leq 3.\sqrt[2023]{\frac{1}{12}.\frac{R}{2r}.\sum_{cyc}\left(\frac{h_a^2}{h_b^2}+\frac{h_a^2}{h_c^2}\right)} = 3.\sqrt[2023]{\frac{R}{24r}\sum_{cyc}\left(\frac{b^2}{c^2}+\frac{c^2}{b^2}\right)}$$

$$\overset{\text{Bandila}}{\leq} 3.\sqrt[2023]{\frac{R}{24r}\sum_{cyc}\left(\frac{R^2}{r^2}-2\right)} \therefore \sum_{cyc}\sqrt[2023]{\frac{w_a^2}{w_b^2+w_c^2}} \leq 3.\sqrt[2023]{\frac{R}{8r}\left(\frac{R^2}{r^2}-2\right)} \to (2)$$

$$\text{Also,} \sum_{cyc}\sqrt[2023]{\frac{a}{b+c}} \overset{\text{via (ii)}}{\leq} \sum_{cyc}\sqrt[2023]{\frac{4R\cos\frac{A}{2}\sin\frac{A}{2}}{4.\sqrt{2Rr}.\cos\frac{A}{2}}} = \sqrt[4046]{\frac{R}{2r}}.\sum_{cyc}\sqrt[2023]{\sin\frac{A}{2}}$$

$$\overset{\text{Jensen}}{\leq} 3.\sqrt[4046]{\frac{R}{2r}}.\sqrt[2023]{\frac{1}{2}}$$

$$\left(\therefore f(x) = \sqrt[2023]{\sin\frac{x}{2}}\ \forall x \in (0,\pi) \Rightarrow f''(x) = \frac{2022+\sin^2\frac{x}{2}}{16370116\left(\sin\frac{x}{2}\right)^{\frac{4045}{2023}}} < 0\right)$$

$$\therefore \sum_{cyc}\sqrt[2023]{\frac{a}{b+c}} \leq 3.\sqrt[2023]{\sqrt{\frac{R}{2r}}.\frac{1}{2}} \to (3) \therefore (1) \textit{ and } (2) \Rightarrow \text{in order to prove}:$$

$$\sum_{cyc}\sqrt[2023]{\frac{m_a^2}{m_b^2+m_c^2}} + \frac{R^3}{r^3} - 8 \geq \sum_{cyc}\sqrt[2023]{\frac{w_a^2}{w_b^2+w_c^2}}, \text{it suffices to prove}:$$

$$3.\sqrt[2023]{\frac{4r^3}{R^3}} + \frac{R^3}{r^3} - 8 \geq 3.\sqrt[2023]{\frac{R}{8r}\left(\frac{R^2}{r^2}-2\right)}$$

$$\Leftrightarrow \boxed{t^3 - 8 + 3.\sqrt[2023]{\frac{4}{t^3}} - 3.\sqrt[2023]{\frac{t}{8}(t^2-2)} \overset{(*)}{\geq} 0 \left(t = \frac{R}{r}\right)}$$

$$\text{Let } F(t) = t^3 - 8 + 3.\sqrt[2023]{\frac{4}{t^3}} - 3.\sqrt[2023]{\frac{t}{8}(t^2-2)}\ \forall t \geq 2 \textit{ and then}:$$

$$F'(t) = 3t^2 - \frac{6t^{\frac{2024}{2023}}}{2023.\sqrt[2023]{8}.(t^2-2)^{\frac{2022}{2023}}} - \frac{3.\sqrt[2023]{t^2-2}}{2023.\sqrt[2023]{8}.t^{\frac{2022}{2023}}} - \frac{9.\sqrt[2023]{4}}{2023t^{\frac{2026}{2023}}} \to (\blacklozenge)$$

$$\text{Now}, t^2(t^2-2)^{\frac{2022}{2023}}.2023.\sqrt[2023]{8} \overset{\text{Euler}}{\geq} t^2.2^{\frac{2022}{2023}}.2023.2^{\frac{3}{2023}}$$

$$= t^2.\left(2023.2^{\frac{2025}{2023}}\right) \overset{t\geq 2}{>} t^{\frac{2024}{2023}}.(6) \Rightarrow t^2 > \frac{6t^{\frac{2024}{2023}}}{2023.\sqrt[2023]{8}.(t^2-2)^{\frac{2022}{2023}}} \to (\bullet)$$

$$\text{Again}, t^2.2023.\sqrt[2023]{8}.t^{\frac{2022}{2023}} = t^{\frac{6068}{2023}}.\left(2023.\sqrt[2023]{8}\right) > \sqrt[2023]{t^2-2}.(3)$$

$$\left(\because t^{\frac{6068}{2023}} \overset{t\geq 2}{>} t^{\frac{2}{2023}} \Rightarrow t^{\frac{6068}{2023}} > \sqrt[2023]{t^2-2} \text{ and } 2023^{2023}.8 > 3^{2023} \Rightarrow 2023.\sqrt[2023]{8} > 3\right)$$

$$\Rightarrow t^2 > \frac{3.\sqrt[2023]{t^2-2}}{2023.\sqrt[2023]{8}.t^{\frac{2022}{2023}}} \to (\bullet\bullet)$$

$$\text{Also}, t^2.2023t^{\frac{2026}{2023}} \overset{t\geq 2}{>} 2023 > 9.\sqrt[2023]{4} \therefore t^2 > \frac{9.\sqrt[2023]{4}}{2023t^{\frac{2026}{2023}}} \to (\bullet\bullet\bullet)$$

$\therefore (\bullet)+(\bullet\bullet)+(\bullet\bullet\bullet) \Rightarrow$ **LHS of** $(\blacklozenge)$ **is true** $\Rightarrow F'(t) > 0\ \forall\, t \geq 2 \Rightarrow F(t)$ **is** $\uparrow$ **on** $[2,\infty)$

$\Rightarrow F(t) \geq F(2) = 0 \Rightarrow (*)$ **is true**

$$\therefore \boxed{\sum_{cyc}\sqrt[2023]{\frac{m_a^2}{m_b^2+m_c^2}} + \frac{R^3}{r^3} - 8 \overset{(\blacksquare)}{\geq} \sum_{cyc}\sqrt[2023]{\frac{w_a^2}{w_b^2+w_c^2}}}$$

Again, (1) **and** $(3) \Rightarrow$ **in order to prove** : $\displaystyle\sum_{cyc}\sqrt[2023]{\frac{m_a^2}{m_b^2+m_c^2}} + \frac{R^3}{r^3} - 8$

$\displaystyle\geq \sum_{cyc}\sqrt[2023]{\frac{a}{b+c}}$, **it suffices to prove** : $\displaystyle 3.\sqrt[2023]{\frac{4r^3}{R^3}} + \frac{R^3}{r^3} - 8 \geq 3.\sqrt[2023]{\sqrt{\frac{R}{2r}}.\frac{1}{2}}$

$$\Leftrightarrow \boxed{t^3 - 8 + 3.\sqrt[2023]{\frac{4}{t^3}} - 3.\sqrt[2023]{\sqrt{t}.\frac{1}{2}} \overset{(**)}{\geq} 0}$$

Let $\displaystyle P(t) = t^3 - 8 + 3.\sqrt[2023]{\frac{4}{t^3}} - 3.\sqrt[2023]{\sqrt{t}.\frac{1}{2}}\ \forall\, t \geq 2$ **and then** :

$$P'(t) = 3t^2 - \frac{3}{2023.2^{\frac{4049}{4046}}.t^{\frac{4045}{4046}}} - \frac{9.\sqrt[2023]{4}}{2023.t^{\frac{2026}{2023}}} \to (\blacklozenge\blacklozenge)$$

$$\text{Now}, t^2.2023.2^{\frac{4049}{4046}}.t^{\frac{4045}{4046}} = t^{2+\frac{4045}{4046}}.\left(2023.2^{\frac{4049}{4046}}\right) \overset{t\geq 2}{>} 2023.2^{\frac{4049}{4046}} > 3$$

$$\Rightarrow t^2 > \frac{3}{2023.2^{\frac{4049}{4046}}.t^{\frac{4045}{4046}}} \to (\bullet\bullet\bullet\bullet)$$

$$\text{Also}, t^2.2023.t^{\frac{2026}{2023}} = t^{2+\frac{2026}{2023}}.(2023) \overset{t\geq 2}{>} 2023 > 9.\sqrt[2023]{4} \Rightarrow t^2 > \frac{9.\sqrt[2023]{4}}{2023.t^{\frac{2026}{2023}}}$$

$\to (\bullet\bullet\bullet\bullet\bullet) \therefore (\bullet\bullet\bullet\bullet)+(\bullet\bullet\bullet\bullet\bullet) \Rightarrow$ **LHS of** $(\blacklozenge\blacklozenge)$ **is true** $\Rightarrow P'(t) > 0\ \forall\, t \geq 2$

$\Rightarrow P(t)$ **is** $\uparrow$ **on** $[2,\infty) \Rightarrow P(t) \geq F(2) = 0 \Rightarrow (**)$ **is true**

$$\therefore \boxed{\sum_{cyc}\sqrt[2023]{\frac{m_a^2}{m_b^2+m_c^2}+\frac{R^3}{r^3}-8}\overset{(\blacksquare\blacksquare)}{\geq}\sum_{cyc}\sqrt[2023]{\frac{a}{b+c}}}\ \therefore (\blacksquare)(\blacksquare\blacksquare)\Rightarrow$$

$$\sum_{cyc}\sqrt[2023]{\frac{m_a^2}{m_b^2+m_c^2}+\frac{R^3}{r^3}-8}\geq max\left\{\sum_{cyc}\sqrt[2023]{\frac{a}{b+c}},\sum_{cyc}\sqrt[2023]{\frac{w_a^2}{w_b^2+w_c^2}}\right\}$$

$$\Rightarrow\sum_{cyc}\sqrt[2023]{\frac{m_a^2}{m_b^2+m_c^2}+\frac{R^3}{r^3}}\geq 8+max\left\{\sum_{cyc}\sqrt[2023]{\frac{a}{b+c}},\sum_{cyc}\sqrt[2023]{\frac{w_a^2}{w_b^2+w_c^2}}\right\}$$

$$\forall\,\Delta\,ABC,''=''\ \text{iff}\ \Delta\,ABC\ \text{is equilateral (QED)}$$

1623.

In any acute triangle ABC, the following relationship holds :

$$\frac{1}{a}\sqrt{\cot A}+\frac{1}{b}\sqrt{\cot B}+\frac{1}{c}\sqrt{\cot C}>\frac{3}{p}$$

Proposed by Vasile Mircea Popa-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{a}\sqrt{\cot A}+\frac{1}{b}\sqrt{\cot B}+\frac{1}{c}\sqrt{\cot C}>\frac{3}{p}\Leftrightarrow\sum_{cyc}\sqrt{\cot A\,(1+\cot^2 A)}>\frac{6R}{s}$$

$$(\text{For own convenience}, p\equiv s)\Leftrightarrow\sum_{cyc}\cot A+\sum_{cyc}\cot^3 A$$

$$+2\sum_{cyc}\left(\sqrt{\cot A\,(1+\cot^2 A)}.\sqrt{\cot B\,(1+\cot^2 B)}\right)\boxed{\boxed{\overset{(*)}{>}}}\frac{36R^2}{s^2}$$

$$\text{Now, }2\sum_{cyc}\left(\sqrt{\cot A\,(1+\cot^2 A)}.\sqrt{\cot B\,(1+\cot^2 B)}\right)\overset{A-G}{>}2\sum_{cyc}\left(\sqrt{\cot A\,.\,2\cot A}\,.\,\sqrt{\cot B\,.\,2\cot B}\right)$$

$$=4\sum_{cyc}\cot A\cot B\Rightarrow 2\sum_{cyc}\left(\sqrt{\cot A\,(1+\cot^2 A)}.\sqrt{\cot B\,(1+\cot^2 B)}\right)\boxed{\overset{(\bullet)}{>}}4$$

$$\text{Again, }\sum_{cyc}\cot A+\sum_{cyc}\cot^3 A\overset{\text{Chebyshev}}{\geq}\sum_{cyc}\cot A+\frac{1}{3}\left(\sum_{cyc}\cot A\right)\left(\sum_{cyc}\cot^2 A\right)\geq$$

$$\sum_{cyc}\cot A+\frac{1}{3}\left(\sum_{cyc}\cot A\right)\left(\sum_{cyc}\cot A\cot B\right)\Rightarrow\sum_{cyc}\cot A+\sum_{cyc}\cot^3 A\boxed{\overset{(\bullet\bullet)}{\geq}}\frac{4}{3}\sum_{cyc}\cot A$$

$$\therefore(\bullet),(\bullet\bullet)\Rightarrow\text{LHS of }(*)\geq\frac{4}{3}\sum_{cyc}\cot A+4=\frac{4}{3}.\frac{s^2-4Rr-r^2}{2rs}+4\overset{?}{>}\frac{36R^2}{s^2}$$

$$\Leftrightarrow\frac{s^2-4Rr-r^2}{6rs}\overset{?}{>}\frac{9R^2-s^2}{s^2}$$

$$\Leftrightarrow\frac{(s^2-4Rr-r^2)^2}{36r^2s^2}\overset{?}{>}\frac{(9R^2-s^2)^2}{s^4}\left(\because 9R^2\overset{\text{Mitrinovic}}{\geq}\frac{4s^2}{3}>s^2\right)$$

$$\Leftrightarrow s^6-(8Rr+38r^2)s^4+r^2s^2(664R^2+8Rr+r^2)-2916R^4r^2 \underset{(**)}{\overset{?}{>}} 0 \text{ and}$$

$$\because (s^2-4R^2-4Rr-r^2)^3>0\left(\because \prod_{cyc}\cos A=\frac{s^2-4R^2-4Rr-r^2}{4R^2}>0\right)$$

$\therefore$ in order to prove $(**)$, it suffices to prove : LHS of $(*) > (s^2-4R^2-4Rr-r^2)^3$

$$\Leftrightarrow (12R^2+4Rr-35r^2)s^4-(48R^4+96R^3r-592R^2r^2+16Rr^3+2r^4)s^2$$

$$+64R^6+192R^5r-267R^4r^2+160R^3r^3+60R^2r^4+12Rr^5+r^6 \overset{(***)}{>} 0$$

and $\because (12R^2+4Rr-35r^2)(s^2-4R^2-4Rr-r^2)^2>0 \therefore$ in order to prove $(***)$, it suffices to prove : LHS of $(***) > (12R^2+4Rr-35r^2)(s^2-4R^2-4Rr-r^2)^2$

$$\Leftrightarrow (48R^4+32R^3r+368R^2r^2-288Rr^3-72r^4)s^2 \overset{(****)}{>}$$
$$128R^6+256R^5r+2532R^4r^2-1088R^3r^3-856R^2r^4-288Rr^5-36r^6$$

Once *again*, LHS of $(****) >$
$$(48R^4+32R^3r+368R^2r^2-288Rr^3-72r^4)(4R^2+4Rr+r^2)$$
$$\overset{?}{>} 128R^6+256R^5r+2532R^4r^2-1088R^3r^3-856R^2r^4-288Rr^5-36r^6$$
$$\Leftrightarrow 16t^6+16t^5-221t^4+360t^3-54t^2-72t-9 \overset{?}{>} 0 \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)\Big((t-2)(16t^4+80t^3+35t^2+180t+526)+1312\Big)+511$$
$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (****) \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true}$$
$$\therefore \frac{1}{a}\sqrt{\cot A}+\frac{1}{b}\sqrt{\cot B}+\frac{1}{c}\sqrt{\cot C}>\frac{3}{p}\ \forall \text{ acute } \Delta ABC \text{ (QED)}$$

1624. In *any* Δ ABC, the following relationship holds :

$$\frac{48r^3}{R^3} \le \frac{h_a+h_b}{h_c}+\frac{w_b+w_c}{w_a}+\frac{m_c+m_a}{m_b} \le \frac{3}{4}\cdot\left(9.\left(\frac{R}{r}\right)^3-64\right)$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{h_a+h_b}{h_c}+\frac{w_b+w_c}{w_a}+\frac{m_c+m_a}{m_b} \le \frac{m_a+m_b}{h_c}+\frac{m_b+m_c}{h_a}+\frac{m_c+m_a}{h_b}$$
$$=\frac{m_a}{h_c}+\frac{m_b}{h_c}+\frac{m_b}{h_a}+\frac{m_c}{h_a}+\frac{m_c}{h_b}+\frac{m_a}{h_b} \overset{\text{CBS}}{\le} \sqrt{2\sum_{cyc} m_a^2}\cdot\sqrt{2\sum_{cyc}\frac{1}{h_a^2}}$$
$$=\sqrt{4.\frac{3}{4}.\sum_{cyc} a^2.\frac{1}{4r^2s^2}.\sum_{cyc} a^2} \overset{\substack{\text{Leibnitz}\\ \text{and}\\ \text{Mitrinovic}}}{\le} \sqrt{\frac{3.81R^4}{4r^2.27r^2}}=\frac{3R^2}{2r^2} \overset{?}{\le} \frac{3}{4}.\left(9.\left(\frac{R}{r}\right)^3-64\right)$$

$$=\frac{3(9R^3-64r^3)}{4r^3}\Leftrightarrow 9t^3-64\overset{?}{\ge}2t^2\left(t=\frac{R}{r}\right)\Leftrightarrow 9t^3-2t^2-64\overset{?}{\ge}0$$

$$\Leftrightarrow (t-2)(9t^2+16t+32)\to \text{true} \because t\overset{\text{Euler}}{\ge}2$$

$$\therefore \frac{h_a+h_b}{h_c}+\frac{w_b+w_c}{w_a}+\frac{m_c+m_a}{m_b}\le\frac{3}{4}.\left(9.\left(\frac{R}{r}\right)^3-64\right)$$

$$\text{Again, }\frac{h_a+h_b}{h_c}+\frac{w_b+w_c}{w_a}+\frac{m_c+m_a}{m_b}\ge\frac{h_a+h_b}{m_c}+\frac{h_b+h_c}{m_a}+\frac{h_c+h_a}{m_b}\overset{\text{Chebyshev}}{\ge}$$

$$\frac{1}{3}\left(\sum_{cyc}(h_b+h_c)\right)\left(\sum_{cyc}\frac{1}{m_a}\right)\left(\begin{array}{c}\because \text{WLOG assuming } a\ge b\ge c\Rightarrow\\ h_b+h_c\ge h_c+h_a\ge h_a+h_b \text{ and } \frac{1}{m_a}\ge\frac{1}{m_b}\ge\frac{1}{m_c}\end{array}\right)$$

$$=\frac{2}{3}\left(2rs\sum_{cyc}\frac{1}{a}\right)\left(\sum_{cyc}\frac{1}{m_a}\right)\overset{\text{Bergstrom}}{\ge}\frac{2}{3}.2rs.\frac{9}{2s}.\frac{9}{\sum_{cyc}m_a}\overset{\text{Leuenberger + Euler}}{\ge}6r.\frac{9}{\frac{9R}{2}}$$

$$=\frac{12r}{R}=\frac{12R^2r}{R^3}\overset{\text{Euler}}{\ge}\frac{12r.4r^2}{R^3}\therefore\frac{h_a+h_b}{h_c}+\frac{w_b+w_c}{w_a}+\frac{m_c+m_a}{m_b}\ge\frac{48r^3}{R^3}\text{ and so,}$$

$$\frac{48r^3}{R^3}\le\frac{h_a+h_b}{h_c}+\frac{w_b+w_c}{w_a}+\frac{m_c+m_a}{m_b}\le\frac{3}{4}.\left(9.\left(\frac{R}{r}\right)^3-64\right)$$

$\forall\,\Delta\,ABC,''=''$ iff $\Delta\,ABC$ is equilateral (QED)

1625. In any Δ ABC, the following relationship holds :

$$\frac{96r^3}{28R^3-192r^3}\le\frac{m_a+m_b}{m_b+m_c}+\frac{w_b+w_c}{w_c+w_a}+\frac{h_c+h_a}{h_a+h_b}\le 168.\left(\frac{R}{4r}\right)^3-18$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{m_a+m_b}{m_b+m_c}+\frac{w_b+w_c}{w_c+w_a}+\frac{h_c+h_a}{h_a+h_b}\le\frac{m_a+m_b}{h_b+h_c}+\frac{m_b+m_c}{h_c+h_a}+\frac{m_c+m_a}{h_a+h_b}$$

$$\overset{\text{Reverse Bergstrom}}{\le}\frac{1}{4}\left(\frac{m_a+m_b}{h_b}+\frac{m_a+m_b}{h_c}\right)+\frac{1}{4}\left(\frac{m_b+m_c}{h_c}+\frac{m_b+m_c}{h_a}\right)$$

$$+\frac{1}{4}\left(\frac{m_c+m_a}{h_a}+\frac{m_c+m_a}{h_b}\right)=\frac{1}{4}\left(\frac{\sum_{cyc}m_a+m_c}{h_a}+\frac{\sum_{cyc}m_a+m_a}{h_b}+\frac{\sum_{cyc}m_a+m_b}{h_c}\right)$$

$$=\frac{1}{4}\left(\sum_{cyc}m_a\right)\left(\sum_{cyc}\frac{1}{h_a}\right)+\frac{1}{4}\left(\frac{m_a}{h_b}+\frac{m_b}{h_c}+\frac{m_c}{h_a}\right)\overset{\substack{\text{Leuenberger}\\ \text{and}\\ \text{CBS}}}{\le}$$

$$\frac{4R+r}{4r}+\frac{1}{4}.\sqrt{\sum_{cyc}m_a^2.\sum_{cyc}\frac{1}{h_a^2}}\overset{\substack{\text{Leibnitz}\\ \text{and}\\ \text{Mitrinovic}}}{\le}\frac{4R+r}{4r}+\frac{1}{4}.\sqrt{\frac{3}{4}.\frac{9R^2.9R^2}{4r^2.27r^2}}$$

$$=\frac{3R^2+4r(4R+r)}{16r^2}\overset{?}{\le}168.\left(\frac{R}{4r}\right)^3-18=\frac{21R^3-144r^3}{8r^3}$$

$$\Leftrightarrow 42t^3 - 3t^2 - 16t - 292 \overset{?}{\geq} 0\left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)(42t^2 + 81t + 146) \to \text{true}$$

$$\because t \overset{\text{Euler}}{\geq} 2 \therefore \frac{m_a + m_b}{m_b + m_c} + \frac{w_b + w_c}{w_c + w_a} + \frac{h_c + h_a}{h_a + h_b} \leq 168.\left(\frac{R}{4r}\right)^3 - 18$$

$$\text{Again,} \frac{m_a + m_b}{m_b + m_c} + \frac{w_b + w_c}{w_c + w_a} + \frac{h_c + h_a}{h_a + h_b} \geq \frac{h_a + h_b}{m_b + m_c} + \frac{h_b + h_c}{m_c + m_a} + \frac{h_c + h_a}{m_a + m_b}$$

$$= 2rs\left(\frac{1}{am_b + am_c} + \frac{1}{bm_b + bm_c} + \frac{1}{bm_c + bm_a} + \frac{1}{cm_c + cm_a} + \frac{1}{cm_a + cm_b} + \frac{1}{am_a + am_b}\right) \overset{\text{Bergstrom}}{\geq}$$

$$\frac{2rs.36}{2(am_b + bm_c + cm_a) + (am_a + bm_b + cm_c) + (bm_a + cm_b + am_c)}$$

$$\overset{\text{CBS}}{\geq} \frac{72rs}{2.\sqrt{\sum_{cyc} m_a^2 . \sum_{cyc} a^2} + \sqrt{\sum_{cyc} m_a^2 . \sum_{cyc} a^2} + \sqrt{\sum_{cyc} m_a^2 . \sum_{cyc} a^2}} \overset{\substack{\text{Leibnitz} \\ \text{and} \\ \text{Mitrinovic}}}{\geq} \frac{18r.3\sqrt{3}r}{\sqrt{\frac{3}{4}.9R^2.9R^2}}$$

$$= \frac{12r^2}{R^2} \overset{?}{\geq} \frac{96r^3}{28R^3 - 192r^3} \Leftrightarrow 7t^3 - 2t^2 - 48 \overset{?}{\geq} 0 \Leftrightarrow (t-2)(7t^2 + 12t + 24) \to \text{true}$$

$$\because t \overset{\text{Euler}}{\geq} 2 \therefore \frac{m_a + m_b}{m_b + m_c} + \frac{w_b + w_c}{w_c + w_a} + \frac{h_c + h_a}{h_a + h_b} \geq \frac{96r^3}{28R^3 - 192r^3} \text{ and so,}$$

$$\frac{96r^3}{28R^3 - 192r^3} \leq \frac{m_a + m_b}{m_b + m_c} + \frac{w_b + w_c}{w_c + w_a} + \frac{h_c + h_a}{h_a + h_b} \leq 168.\left(\frac{R}{4r}\right)^3 - 18$$

$$\forall \Delta ABC, "=" \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1626.

In $\Delta ABC, I$ –incenter, IM, IN, IP –medians in $\Delta IBC, \Delta ICA, \Delta IAB$

$M \in (BC), N \in (CA), P \in (AB), IM = v_a, IN = v_b, IP = v_c$. Prove that:

$$2Rr - r^2 \leq v_a^2 + v_b^2 + v_c^2 \leq 2R^2 - 4Rr + 3r^2$$

Proposed by Mehmet Şahin-Turkiye

Solution by Daniel Sitaru-Romania

$$\sum_{cyc} v_a^2 = \sum_{cyc} \frac{1}{2}(IB^2 + IC^2) - \frac{1}{4}\sum_{cyc} a^2 =$$

$$= \sum_{cyc} IA^2 - \frac{1}{4} \cdot 2(s^2 - r^2 - 4Rr) = \sum_{cyc} \frac{r^2}{\sin^2 \frac{A}{2}} - \frac{1}{2}(s^2 - r^2 - 4Rr) =$$

$$= r^2\sum_{cyc}\frac{1}{\sin^2\frac{A}{2}} - \frac{1}{2}(s^2 - r^2 - 4Rr) = r^2\cdot\frac{s^2 + r^2 - 8Rr}{r^2} - \frac{s^2}{2} + \frac{r^2}{2} + 2Rr$$

$$= s^2 + r^2 - 8Rr - \frac{s^2}{2} + \frac{r^2}{2} + 2Rr = \frac{s^2}{2} + \frac{3r^2}{2} - 6Rr$$

$$\sum_{cyc} v_a^2 = \frac{s^2}{2} + \frac{3r^2}{2} - 6Rr \overset{GERRETSEN}{\leq} \frac{1}{2}(4R^2 + 4Rr + 3r^2) + \frac{3r^2}{2} - 6Rr =$$

$$= 2R^2 - 4Rr + 3r^2$$

$$\sum_{cyc} v_a^2 = \frac{s^2}{2} + \frac{3r^2}{2} - 6Rr \overset{GERRETSEN}{\geq} \frac{1}{2}(16Rr - 5r^2) + \frac{3r^2}{2} - 6Rr =$$

$$= 2Rr - r^2$$

Equality holds for $a = b = c$.

1627. In acute $\Delta ABC, AD, BE, CF$ are altitudes and r is inradii. If r_1, r_2, r_3 are inradies of $\Delta AFE, \Delta BDF, \Delta CDE$ then:

$$r_1 + r_2 + r_3 \leq \frac{3r}{2}$$

Proposed by Ertan Yildirim-Turkiye

Solution by Daniel Sitaru-Romania

$$\sum_{cyc} r_1 = \sum_{cyc}\frac{[AFE]}{\frac{AF + FE + EA}{2}} = \sum_{cyc}\frac{\frac{1}{2}\cdot AE\cdot AF\cdot \sin A}{\frac{1}{2}(a\cos A + b\cos A + c\cos A)} =$$

$$= \sum_{cyc}\frac{b\cos A\cdot c\cos A\cdot \sin A}{(a + b + c)\cdot \cos A} = \sum_{cyc}\frac{bc\sin A\cos A}{2s} =$$

$$= \sum_{cyc}\frac{2F\cos A}{2s} = \sum_{cyc}\frac{rs\cos A}{s} = r\sum_{cyc}\cos A \overset{JENSEN}{\leq}$$

$$\leq 3r\cos\left(\frac{A + B + C}{3}\right) = 3r\cos\frac{\pi}{3} = \frac{3r}{2}$$

Equality holds for $a = b = c$.

1628. In any Δ ABC, the following relationship holds :

$$3.\left(\frac{2r}{R}\right)^6 \le \frac{m_a m_b}{w_a w_b}+\frac{w_b w_c}{h_b h_c}+\frac{h_c h_a}{m_c m_a}\le 3.\left(\frac{R}{2r}\right)^6$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{m_a m_b}{w_a w_b}+\frac{w_b w_c}{h_b h_c}+\frac{h_c h_a}{m_c m_a}\overset{A-G}{\ge} 3.\sqrt[2]{\frac{m_a m_b}{w_a w_b}\cdot\frac{w_b w_c}{h_b h_c}\cdot\frac{h_c h_a}{m_c m_a}} = 3.\sqrt[3]{\frac{m_b w_c h_a}{w_a h_b m_c}}$$

$$\ge 3.\sqrt[3]{\frac{h_b h_c h_a}{m_a m_b m_c}}\overset{m_a m_b m_c\le\frac{Rs^2}{2}}{\ge} 3.\sqrt[3]{\frac{\frac{2r^2s^2}{R}}{\frac{Rs^2}{2}}} = 3.\sqrt[3]{\frac{4r^2.2r}{R^2.2r}}\overset{Euler}{\ge} 3.\sqrt[3]{\frac{8r^3}{R^3}} = 3.\left(\frac{2r}{R}\right)$$

$$=\frac{3.\left(\frac{2r}{R}\right)^6}{\left(\frac{2r}{R}\right)^5}\overset{Euler}{\ge} 3.\left(\frac{2r}{R}\right)^6 \therefore \frac{m_a m_b}{w_a w_b}+\frac{w_b w_c}{h_b h_c}+\frac{h_c h_a}{m_c m_a}\ge 3.\left(\frac{2r}{R}\right)^6$$

Again, $\frac{m_a m_b}{w_a w_b}+\frac{w_b w_c}{h_b h_c}+\frac{h_c h_a}{m_c m_a}\le\frac{m_a m_b}{h_a h_b}+\frac{m_b m_c}{h_b h_c}+\frac{m_c m_a}{h_c h_a}$

$$\overset{Panaitopol}{\le}\left(\frac{R}{2r}\right)^2.\sum_{cyc}\frac{h_b h_c}{h_b h_c}=\frac{3.\left(\frac{R}{2r}\right)^6}{\left(\frac{R}{2r}\right)^4}\overset{Euler}{\le} 3.\left(\frac{R}{2r}\right)^6 \therefore \frac{m_a m_b}{w_a w_b}+\frac{w_b w_c}{h_b h_c}+\frac{h_c h_a}{m_c m_a}$$

$$\le 3.\left(\frac{R}{2r}\right)^6 \text{ and so, } 3.\left(\frac{2r}{R}\right)^6\le\frac{m_a m_b}{w_a w_b}+\frac{w_b w_c}{h_b h_c}+\frac{h_c h_a}{m_c m_a}\le 3.\left(\frac{R}{2r}\right)^6$$

$\forall\,\Delta$ ABC," =" iff Δ ABC is equilateral (QED)

Proof of $m_a m_b m_c\le\frac{Rs^2}{2}$

$$m_a^2 m_b^2 m_c^2=\frac{1}{64}(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)$$

$$\overset{(1)}{=}\frac{1}{64}\left\{-4\sum_{cyc}a^6+6\left(\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4\right)+3a^2b^2c^2\right\}$$

$$\text{Now, }\sum_{cyc}a^6=\left(\sum_{cyc}a^2\right)^3-3(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

$$= \left(\sum_{cyc} a^2\right)^3 - 3\left(2a^2b^2c^2 + \sum_{cyc}\left(a^2b^2\left(\sum_{cyc} a^2 - c^2\right)\right)\right)$$

$$= \left(\sum_{cyc} a^2\right)^3 + 3a^2b^2c^2 - 3\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right)$$

$$\therefore \sum_{cyc} a^6 \overset{(2)}{=} \left(\sum_{cyc} a^2\right)^3 + 3a^2b^2c^2 - 3\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right)$$

$$\sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4 = \sum_{cyc}\left(a^2b^2\left(\sum_{cyc} a^2 - c^2\right)\right) \overset{(3)}{=}$$

$$\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) - 3a^2b^2c^2 \therefore (1),(2),(3) \Rightarrow m_a^2m_b^2m_c^2$$

$$= \frac{1}{64}\left(\begin{matrix} -4\left(\sum_{cyc} a^2\right)^3 - 12a^2b^2c^2 + 12\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) \\ +6\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) - 18a^2b^2c^2 + 3a^2b^2c^2 \end{matrix}\right)$$

$$= \frac{1}{64}\left(-4\left(\sum_{cyc} a^2\right)^3 + 18\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) - 27a^2b^2c^2\right)$$

$$= \frac{1}{64}\left(-4\left(\sum_{cyc} a^2\right)^3 + 18\left(\left(\sum_{cyc} ab\right)^2 - 16Rrs^2\right)\left(\sum_{cyc} a^2\right) - 27a^2b^2c^2\right)$$

$$= \frac{1}{64}\left\{\begin{matrix} -32(s^2-4Rr-r^2)^3 + 36(s^2-4Rr-r^2)(s^2+4Rr+r^2)^2 \\ -576Rrs^2(s^2-4Rr-r^2) - 432R^2r^2s^2 \end{matrix}\right\}$$

$$= \frac{1}{16}\{s^6 - s^4(12Rr - 33r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4) - r^3(4R+r)^3\}$$

$$\le \frac{R^2s^4}{4} \Leftrightarrow$$

$$s^6 - s^4(4R^2 + 12Rr - 33r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4) - r^3(4R+r)^3 \overset{(\bullet)}{\le} 0$$

Now, LHS of $(\bullet) \overset{\text{Gerretsen}}{\le} -s^4(8Rr - 36r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4)$

$$-r^3(4R+r)^3 \overset{?}{\le} 0$$

$$\Leftrightarrow s^4(8R-16r)+s^2(60R^2r+120Rr^2+33r^3)+r^2(4R+r)^3 \underbrace{\overset{?}{\ge}}_{(\bullet\bullet)} 20rs^4$$

$$\text{Now, LHS of } (\bullet\bullet) \underbrace{\overset{\text{Gerretsen}}{\ge}}_{(a)} s^2(16Rr-5r^2)(8R-16r)$$

$$+s^2(60R^2r+120Rr^2+33r^3)+r^2(4R+r)^3 \text{ and}$$

$$\text{RHS of } (\bullet\bullet) \underbrace{\overset{\text{Geretsen}}{\le}}_{(b)} 20rs^2(4R^2+4Rr+3r^2)$$

$$(a),(b) \Rightarrow \text{in order to prove } (\bullet\bullet), \text{it suffices to prove :}$$

$$s^2(16Rr-5r^2)(8R-16r)+s^2(60R^2r+120Rr^2+33r^3)+r^2(4R+r)^3$$
$$\ge 20rs^2(4R^2+4Rr+3r^2)$$

$$\Leftrightarrow s^2(108R^2-256Rr+53r^2)+r(4R+r)^3 \ge 0$$

$$\Leftrightarrow s^2(108R^2-256Rr+80r^2)+r(4R+r)^3 \overset{(\bullet\bullet\bullet)}{\ge} 27r^2s^2$$

$$\text{Now, LHS of } (\bullet\bullet\bullet) \underbrace{\overset{\text{Gerretsen}}{\ge}}_{(c)} (108R^2-256Rr+80r^2)(16Rr-5r^2)+r(4R+r)^3$$

$$\text{and RHS of } (\bullet\bullet\bullet) \underbrace{\overset{\text{Geretsen}}{\le}}_{(d)} 27r^2(4R^2+4Rr+3r^2)$$

$$(c),(d) \Rightarrow \text{in order to prove } (\bullet\bullet\bullet), \text{it suffices to prove :}$$

$$(108R^2-256Rr+80r^2)(16Rr-5r^2)+r(4R+r)^3 \ge 27r^2(4R^2+4Rr+3r^2)$$

$$\Leftrightarrow 224t^3-587t^2+308t-60 \ge 0 \left(\text{where } t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\big((t-2)(224t+309)+648\big) \ge 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (\bullet\bullet\bullet) \Rightarrow (\bullet\bullet)$$

$$\Rightarrow (\bullet) \text{ is true} \Rightarrow m_a^2m_b^2m_c^2 \le \frac{R^2s^4}{4} \Rightarrow m_am_bm_c \le \frac{Rs^2}{2} \text{ (QED)}$$

1629. In any Δ ABC, the following relationship holds :

$$\frac{192r^4}{3R^4-32r^4} \le \left(\frac{m_a}{m_b}+\frac{w_b}{w_a}\right)^2+\left(\frac{w_b}{w_c}+\frac{h_c}{h_b}\right)^2+\left(\frac{h_c}{h_a}+\frac{m_a}{m_c}\right)^2 \le 9.\left(\frac{243}{32}.\left(\frac{R}{r}\right)^6-484\right)^2-24$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\frac{m_a}{m_b}+\frac{w_b}{w_a}\right)^2+\left(\frac{w_b}{w_c}+\frac{h_c}{h_b}\right)^2+\left(\frac{h_c}{h_a}+\frac{m_a}{m_c}\right)^2 \le$$

$$2\left(\frac{m_a^2}{m_b^2}+\frac{w_b^2}{w_a^2}+\frac{w_b^2}{w_c^2}+\frac{h_c^2}{h_b^2}+\frac{h_c^2}{h_a^2}+\frac{m_a^2}{m_c^2}\right) \le 2\left(\frac{m_a^2}{h_b^2}+\frac{m_b^2}{h_a^2}+\frac{m_b^2}{h_c^2}+\frac{m_c^2}{h_b^2}+\frac{m_c^2}{h_a^2}+\frac{m_a^2}{h_c^2}\right)$$

$$=2\sum_{cyc}\frac{m_b^2+m_c^2}{h_a^2}=2\sum_{cyc}\frac{\sum_{cyc}m_a^2-m_a^2}{h_a^2}=\frac{2}{4r^2s^2}\left(\sum_{cyc}m_a^2\right)\left(\sum_{cyc}a^2\right)-2\sum_{cyc}\frac{m_a^2}{h_a^2}$$

$$\overset{\substack{\text{Leibnitz}\\ \text{and}\\ \text{Mitrinovic}}}{\leq}\frac{\frac{3}{4}.81R^4}{2r^2.27r^2}-6\therefore\left(\frac{m_a}{m_b}+\frac{w_b}{w_a}\right)^2+\left(\frac{w_b}{w_c}+\frac{h_c}{h_b}\right)^2+\left(\frac{h_c}{h_a}+\frac{m_a}{m_c}\right)^2\overset{(1)}{\leq}\frac{9R^4}{8r^4}-6$$

$$\text{Again, }9.\left(\frac{243}{32}.\left(\frac{R}{r}\right)^6-484\right)^2-24\overset{\text{Euler}}{\geq}9.\left(\frac{243}{32}.16.\left(\frac{R}{r}\right)^2-484\right)^2-24$$

$$\overset{?}{\geq}\frac{9R^4}{8r^4}-6\Leftrightarrow\frac{(243t^2-968)^2}{4}\overset{?}{\geq}\frac{t^4}{8}+2=\frac{t^4+16}{8}$$

$$\Leftrightarrow 118097t^4-940896t^2+1874032\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)(t+2)\big((t-2)(118097t+236194)+3880\big)\overset{?}{\geq}0\to\text{true}\because t\overset{\text{Euler}}{\geq}2$$

$$\therefore 9.\left(\frac{243}{32}.\left(\frac{R}{r}\right)^6-484\right)^2-24\geq\frac{9R^4}{8r^4}-6$$

$$\overset{\text{via (1)}}{\geq}\left(\frac{m_a}{m_b}+\frac{w_b}{w_a}\right)^2+\left(\frac{w_b}{w_c}+\frac{h_c}{h_b}\right)^2+\left(\frac{h_c}{h_a}+\frac{m_a}{m_c}\right)^2$$

$$\therefore\boxed{\left(\frac{m_a}{m_b}+\frac{w_b}{w_a}\right)^2+\left(\frac{w_b}{w_c}+\frac{h_c}{h_b}\right)^2+\left(\frac{h_c}{h_a}+\frac{m_a}{m_c}\right)^2\leq 9.\left(\frac{243}{32}.\left(\frac{R}{r}\right)^6-484\right)^2-24}$$

$$\text{Also, }\left(\frac{m_a}{m_b}+\frac{w_b}{w_a}\right)^2+\left(\frac{w_b}{w_c}+\frac{h_c}{h_b}\right)^2+\left(\frac{h_c}{h_a}+\frac{m_a}{m_c}\right)^2\overset{A-G}{\geq}4\left(\frac{m_a}{m_b}.\frac{w_b}{w_a}+\frac{w_b}{w_c}.\frac{h_c}{h_b}+\frac{h_c}{h_a}.\frac{m_a}{m_c}\right)$$

$$\overset{A-G}{\geq}12\sqrt[3]{\frac{m_a^2w_b^2h_c^2}{(m_bm_c)(w_cw_a)(h_ah_b)}}=12\sqrt[3]{\frac{m_a^3w_b^3h_c^3}{(\prod_{cyc}m_a)(\prod_{cyc}w_a)(\prod_{cyc}h_a)}}$$

$$\geq 12\left(\prod_{cyc}h_a\right)\sqrt[3]{\frac{1}{(\prod_{cyc}m_a)^3}}\geq 12\frac{(\prod_{cyc}h_a)}{(\prod_{cyc}m_a)}\overset{m_am_bm_c\leq\frac{Rs^2}{2}}{\geq}12.\frac{\frac{2r^2s^2}{R}}{\frac{Rs^2}{2}}=\frac{48r^2}{R^2}$$

$$\overset{?}{\geq}\frac{192r^4}{3R^4-32r^4}\Leftrightarrow 3t^4-4t^2-32\overset{?}{\geq}0\Leftrightarrow(t-2)(3t^3+6t^2+8t+16)\overset{?}{\geq}0$$

$$\to\text{true}\because t\overset{\text{Euler}}{\geq}2\therefore\boxed{\left(\frac{m_a}{m_b}+\frac{w_b}{w_a}\right)^2+\left(\frac{w_b}{w_c}+\frac{h_c}{h_b}\right)^2+\left(\frac{h_c}{h_a}+\frac{m_a}{m_c}\right)^2\geq\frac{192r^4}{3R^4-32r^4}}$$

$$\text{and so, }\frac{192r^4}{3R^4-32r^4}\leq\left(\frac{m_a}{m_b}+\frac{w_b}{w_a}\right)^2+\left(\frac{w_b}{w_c}+\frac{h_c}{h_b}\right)^2+\left(\frac{h_c}{h_a}+\frac{m_a}{m_c}\right)^2$$

$$\leq 9.\left(\frac{243}{32}.\left(\frac{R}{r}\right)^6-484\right)^2-24\ \forall\,\Delta\,ABC,''=''\text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$

1630. In any Δ ABC, the following relationship holds :

$$\frac{64r^5}{R^6} \le \left(\frac{\sqrt{m_a}}{m_b}\right)^2 + \left(\frac{\sqrt{w_b}}{w_c}\right)^2 + \left(\frac{\sqrt{h_c}}{h_a}\right)^2 \le \frac{1}{r}\cdot\left(\frac{81}{32}\left(\frac{R}{r}\right)^5 - 80\right)$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\frac{\sqrt{m_a}}{m_b}\right)^2 + \left(\frac{\sqrt{w_b}}{w_c}\right)^2 + \left(\frac{\sqrt{h_c}}{h_a}\right)^2 \le \frac{m_a}{h_b^2} + \frac{m_b}{h_c^2} + \frac{m_c}{h_a^2}$$

$$= \frac{1}{4r^2s^2}\cdot(b^2m_a + c^2m_b + a^2m_c) \overset{\text{CBS}}{\le} \frac{1}{4r^2s^2}\cdot\sqrt{\sum_{\text{cyc}} a^4}\cdot\sqrt{\sum_{\text{cyc}} m_a^2} \overset{\text{Mitrinovic}}{\le}$$

$$\frac{1}{4r^2.27r^2}\cdot\sqrt{2\sum_{\text{cyc}} a^2b^2 - 16r^2s^2}.\sqrt{\frac{3}{4}\sum_{\text{cyc}} a^2} \overset{\substack{\text{Goldstone}\\ \text{and}\\ \text{Leibnitz}}}{\le} \frac{1}{4r^2.27r^2}\cdot\sqrt{(8R^2 - 16r^2)s^2}.\sqrt{\frac{3}{4}.9R^2}$$

$$\overset{\text{Mitrinovic}}{\le} \frac{1}{4r^2.27r^2}\cdot\sqrt{(8R^2 - 16r^2).\frac{27R^2}{4}}\cdot\sqrt{\frac{3}{4}.9R^2} = \frac{R^2.\sqrt{\frac{R^2 - 2r^2}{2}}}{4r^4}$$

$$\overset{?}{\le} \frac{1}{r}\cdot\left(\frac{81}{32}\left(\frac{R}{r}\right)^5 - 80\right) = \frac{81R^5 - 2560r^5}{32r^6}$$

$$\Leftrightarrow (81R^5 - 2560r^5)^2 \overset{?}{\ge} 64R^4r^4\left(\frac{R^2 - 2r^2}{2}\right)$$

$$\Leftrightarrow 6561t^{10} - 32t^6 - 414720t^5 + 64t^4 + 6553600 \overset{?}{\ge} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)\begin{pmatrix}6561t^8 + 26244t^7 + 78732t^6 + 209952t^5 + 524848t^4\\ +844864t^3 + 1280128t^2 + 1741056t + 1843712\end{pmatrix} +410624\right) \overset{?}{\ge} 0$$

$$\to \text{true} \because t \overset{\text{Euler}}{\ge} 2$$

$$\therefore \left(\frac{\sqrt{m_a}}{m_b}\right)^2 + \left(\frac{\sqrt{w_b}}{w_c}\right)^2 + \left(\frac{\sqrt{h_c}}{h_a}\right)^2 \le \frac{1}{r}\cdot\left(\frac{81}{32}\left(\frac{R}{r}\right)^5 - 80\right)$$

Again, $\left(\frac{\sqrt{m_a}}{m_b}\right)^2 + \left(\frac{\sqrt{w_b}}{w_c}\right)^2 + \left(\frac{\sqrt{h_c}}{h_a}\right)^2 \ge \frac{h_a}{m_b^2} + \frac{h_b}{m_c^2} + \frac{h_c}{m_a^2} \overset{\text{Bergstrom}}{\ge} \frac{\left(\sum_{\text{cyc}}\frac{1}{m_a}\right)^2}{\sum_{\text{cyc}}\frac{1}{h_a}}$

$$\overset{\text{Bergstrom}}{\ge} r\left(\frac{9}{\sum_{\text{cyc}} m_a}\right)^2 \overset{\text{Leuenberger + Euler}}{\ge} r\left(\frac{9}{\frac{9R}{2}}\right)^2 = \frac{4r}{R^2} = \frac{4r^5}{R^2r^4} \overset{\text{Euler}}{\ge} \frac{4r^5}{R^2\left(\frac{R}{2}\right)^4} = \frac{64r^5}{R^6}$$

$$\therefore \left(\frac{\sqrt{m_a}}{m_b}\right)^2 + \left(\frac{\sqrt{w_b}}{w_c}\right)^2 + \left(\frac{\sqrt{h_c}}{h_a}\right)^2 \ge \frac{64r^5}{R^6} \text{ and so,}$$

$$\frac{64r^5}{R^6} \le \left(\frac{\sqrt{m_a}}{m_b}\right)^2 + \left(\frac{\sqrt{w_b}}{w_c}\right)^2 + \left(\frac{\sqrt{h_c}}{h_a}\right)^2 \le \frac{1}{r}\cdot\left(\frac{81}{32}\left(\frac{R}{r}\right)^5 - 80\right)$$

$\forall\,\Delta\,ABC, '' ='' \text{ iff } \Delta\,ABC$ is equilateral (QED)

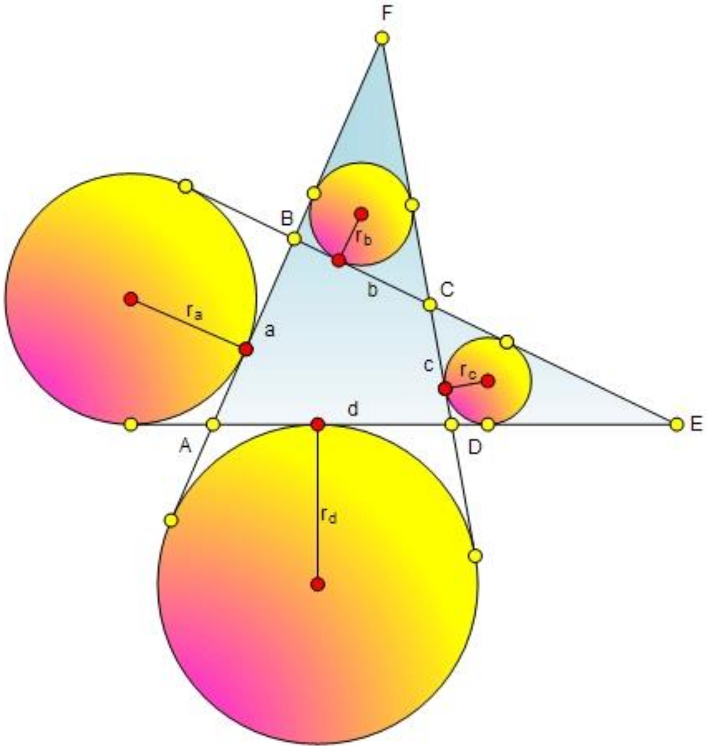
1631.

In any cyclic quadrilateral, the following relationship holds:

$$\frac{r_a^4}{a^7} + \frac{r_b^4}{b^7} + \frac{r_c^4}{c^7} + \frac{r_d^4}{d^7} \ge \frac{2}{s^3}$$

Proposed by Daniel Sitaru-Romania

Solution by Soumava Chakraborty-Kolkata-India



Via Ptolemy's theorem 1 and Ptolemy's theorem 2, we have: $pq = ac + bd$ and

$\frac{p}{q} = \frac{ad+bc}{ab+cd}$ and consequently, $AC = p = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}$

Now, $\cos B = \frac{a^2+b^2-\frac{(ac+bd)(ad+bc)}{ab+cd}}{2ab} \Rightarrow 2\cos^2\frac{B}{2} = \frac{a^2+b^2-\frac{(ac+bd)(ad+bc)}{ab+cd}}{2ab} + 1$

$$= \frac{(a^2+b^2)(ab+cd)-(ac+bd)(ad+bc)+2ab(ab+cd)}{2ab(ab+cd)}$$

$$= \frac{(a+b+c-d)(a+b+d-c)}{2ab(ab+cd)} \Rightarrow 1+\tan^2\frac{B}{2} = \frac{4ab(ab+cd)}{(a+b+c-d)(a+b+d-c)}$$

$$\Rightarrow \tan^2\frac{B}{2}=\frac{4ab(ab+cd)-(a+b+c-d)(a+b+d-c)}{(a+b+c-d)(a+b+d-c)}$$

$$=\frac{(c+d+a-b)(c+d+b-a)}{(a+b+c-d)(a+b+d-c)}\Rightarrow \tan^2\frac{B}{2}=\frac{(s-b)(s-a)}{(s-c)(s-d)}$$

$$\text{and analogously, } \tan^2\frac{A}{2}=\frac{(s-a)(s-d)}{(s-b)(s-c)},\ \tan^2\frac{C}{2}=\frac{(s-c)(s-b)}{(s-d)(s-a)}$$

$$\text{and } \tan^2\frac{D}{2}=\frac{(s-d)(s-c)}{(s-a)(s-b)}\to(1)$$

Let a perpendicular (r_a) be dropped from the center of the circle with radius r_a onto AB intersecting AB at X.

$$\text{Now, } \tan\left(90^\circ-\frac{A}{2}\right)=\frac{r_a}{AX}\Rightarrow AX=r_a\tan\frac{A}{2} \text{ and } \tan\left(90^\circ-\frac{B}{2}\right)=\frac{r_a}{BX}$$

$$\Rightarrow BX=r_a\tan\frac{B}{2}\Rightarrow AX+BX=a=r_a\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\overset{\text{via }(1)}{=}$$

$$r_a\left(\sqrt{\frac{(s-a)(s-d)}{(s-b)(s-c)}}+\sqrt{\frac{(s-b)(s-a)}{(s-c)(s-d)}}\right)$$

$$=\frac{r_a(s-a)(s-d+s-b)}{\sqrt{(s-a)(s-b)(s-c)(s-d)}}=\frac{r_a(s-a)(c+a)}{F}$$

$$\Rightarrow r_a=\frac{aF}{(s-a)(c+a)} \text{ and analogously, } r_b=\frac{bF}{(s-b)(b+d)},$$

$$r_c=\frac{cF}{(s-c)(c+a)} \text{ and } r_d=\frac{dF}{(s-d)(b+d)}\to(2)$$

$$\text{Via }(2), \frac{r_a^4}{a^7}+\frac{r_b^4}{b^7}+\frac{r_c^4}{c^7}+\frac{r_d^4}{d^7}$$

$$=F^4\left(\frac{\left(\frac{1}{(s-a)(c+a)}\right)^4}{a^3}+\frac{\left(\frac{1}{(s-b)(b+d)}\right)^4}{b^3}+\frac{\left(\frac{1}{(s-c)(c+a)}\right)^4}{c^3}+\frac{\left(\frac{1}{(s-d)(b+d)}\right)^4}{d^3}\right)$$

$$\overset{\text{Radon}}{\geq}\frac{F^4}{(a+b+c+d)^3}\left(\begin{array}{c}\left(\frac{1}{(s-a)(c+a)}+\frac{1}{(s-c)(c+a)}\right)\\+\left(\frac{1}{(s-b)(b+d)}+\frac{1}{(s-d)(b+d)}\right)\end{array}\right)^4$$

$$=\frac{F^4}{8s^3}\left(\frac{b+d}{(c+a)(s-a)(s-c)}+\frac{c+a}{(b+d)(s-b)(s-d)}\right)^4$$

$$\overset{A-G}{\geq}\frac{F^4}{8s^3}\left(2\sqrt{\frac{b+d}{(c+a)(s-a)(s-c)}\cdot\frac{c+a}{(b+d)(s-b)(s-d)}}\right)^4=\frac{F^4}{8s^3}\cdot\left(\frac{2}{F}\right)^4=\frac{2}{s^3}$$

$$\therefore \frac{r_a^4}{a^7}+\frac{r_b^4}{b^7}+\frac{r_c^4}{c^7}+\frac{r_d^4}{d^7}\geq\frac{2}{s^3}, ''='' \text{ iff ABCD is a square (QED)}$$

1632. If in Δ ABC, $a \le b \le c$ then :

$$\frac{a+b}{m_a+m_b} \le \frac{b+c}{m_b+m_c}$$

Proposed by Daniel Sitaru-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$m_b+m_c \overset{?}{\le} m_a+m_b \Leftrightarrow 2a^2+2b^2-c^2 \overset{?}{\le} 2b^2+2c^2-a^2$$
$$\Leftrightarrow 3(c^2-a^2) \overset{?}{\ge} 0 \to \text{true} \because c \ge a \therefore \frac{1}{m_a+m_b} \le \frac{1}{m_b+m_c}$$
$$\Rightarrow \frac{a+b}{m_a+m_b} \le \frac{a+b}{m_b+m_c} \overset{a\le c}{\le} \frac{b+c}{m_b+m_c}$$
$$\therefore \frac{a+b}{m_a+m_b} \le \frac{b+c}{m_b+m_c} \text{ in } \Delta \text{ ABC such that} : a \le b \le c, '' ='' \text{ iff } a=c \text{ (QED)}$$

1633. In any Δ ABC with

K $\to$ Lemoine point, the following relationship holds :

$$\frac{[BKC]}{ar_a}\cdot\sqrt{1+\frac{2[BKC]}{ar_a}}+\frac{[CKA]}{br_b}\cdot\sqrt{1+\frac{2[CKA]}{br_b}}+\frac{[AKB]}{cr_c}\cdot\sqrt{1+\frac{[AKB]}{cr_c}} \le \frac{\sqrt{3}}{3}$$

Proposed by Daniel Sitaru-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } AK_A = s_a \therefore \frac{KK_A}{AK} \overset{\text{Honsberger}}{=} \frac{a^2}{b^2+c^2} \Rightarrow \frac{KK_A}{AK}+1 = \frac{a^2}{b^2+c^2}+1$$
$$\Rightarrow \frac{s_a}{AK} = \frac{a^2+b^2+c^2}{b^2+c^2} \Rightarrow AK = \frac{b^2+c^2}{a^2+b^2+c^2}\cdot s_a = \frac{b^2+c^2}{a^2+b^2+c^2}\cdot\frac{2bc}{b^2+c^2}\cdot m_a$$
$$\Rightarrow AK = \frac{2bc}{\sum_{cyc} a^2}\cdot m_a \text{ and analogs} \to (1)$$
$$\text{Now, } \Delta \text{ BKC has sides BK, CK and } 'a' \therefore 16[BKC]^2$$
$$= 2a^2BK^2+2a^2CK^2+2BK^2CK^2-a^4-BK^4-CK^4$$
$$= 2a^2.\frac{4c^2a^2}{\left(\sum_{cyc} a^2\right)^2}.m_b^2+2a^2.\frac{4a^2b^2}{\left(\sum_{cyc} a^2\right)^2}.m_c^2+2.\frac{4c^2a^2}{\left(\sum_{cyc} a^2\right)^2}.m_b^2.\frac{4a^2b^2}{\left(\sum_{cyc} a^2\right)^2}.m_c^2$$
$$-a^4-\frac{16c^4a^4}{\left(\sum_{cyc} a^2\right)^4}.m_b^4-\frac{16a^4b^4}{\left(\sum_{cyc} a^2\right)^4}.m_c^4 =$$

$$\frac{1}{\left(\sum_{cyc}a^2\right)^4}\begin{pmatrix}2a^2.\left(\sum_{cyc}a^2\right)^2.c^2a^2(2c^2+2a^2-b^2)+2a^2.\left(\sum_{cyc}a^2\right)^2.a^2b^2(2a^2+2b^2-c^2)\\+2c^2a^2.a^2b^2(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)-a^4\left(\sum_{cyc}a^2\right)^4\\-c^4a^4(2c^2+2a^2-b^2)^2-a^4b^4(2a^2+2b^2-c^2)^2\end{pmatrix}$$

$\rightarrow$ (2)

$$\text{Now, }2c^2a^2.a^2b^2(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)-c^4a^4(2c^2+2a^2-b^2)^2$$
$$-a^4b^4(2a^2+2b^2-c^2)^2=-\left(c^2a^2(2c^2+2a^2-b^2)-a^2b^2(2a^2+2b^2-c^2)\right)^2$$
$$=-a^4\left(2(c^2+b^2)(c^2-b^2)+2a^2(c^2-b^2)\right)^2=-4a^4(c^2-b^2)^2\left(\sum_{cyc}a^2\right)^2\rightarrow\text{(i)}$$

$$\text{Again, }2a^2.\left(\sum_{cyc}a^2\right)^2.c^2a^2(2c^2+2a^2-b^2)+$$
$$2a^2.\left(\sum_{cyc}a^2\right)^2.a^2b^2(2a^2+2b^2-c^2)-a^4\left(\sum_{cyc}a^2\right)^4$$
$$=a^4\left(\sum_{cyc}a^2\right)^2\left(2c^2(2c^2+2a^2-b^2)+2b^2(2a^2+2b^2-c^2)-\left(\sum_{cyc}a^2\right)^2\right)$$

$\rightarrow$ (ii) $\therefore$ (2), (i), (ii) $\Rightarrow 16[\mathrm{BKC}]^2$

$$=\frac{a^4}{\left(\sum_{cyc}a^2\right)^2}.\begin{pmatrix}2c^2(2c^2+2a^2-b^2)+2b^2(2a^2+2b^2-c^2)\\-\left(\sum_{cyc}a^2\right)^2-4(c^2-b^2)^2\end{pmatrix}$$

$$=\frac{a^4}{\left(\sum_{cyc}a^2\right)^2}.\left(2\sum_{cyc}a^2b^2-\sum_{cyc}a^4\right)=\frac{a^4}{\left(\sum_{cyc}a^2\right)^2}.16F^2\Rightarrow[\mathrm{BKC}]=\frac{a^2.F}{\sum_{cyc}a^2}$$

$$\Rightarrow\frac{[\mathrm{BKC}]}{ar_a}=\frac{a^2F(s-a)}{\left(\sum_{cyc}a^2\right)aF}\Rightarrow\boxed{\frac{[\mathrm{BKC}]}{ar_a}=\frac{a(s-a)}{\sum_{cyc}a^2}}\text{ and analogs}\rightarrow(*)$$

$$\text{Now, }\frac{[\mathrm{BKC}]}{ar_a}.\sqrt{1+\frac{2[\mathrm{BKC}]}{ar_a}}+\frac{[\mathrm{CKA}]}{br_b}.\sqrt{1+\frac{2[\mathrm{CKA}]}{br_b}}+\frac{[\mathrm{AKB}]}{cr_c}.\sqrt{1+\frac{[\mathrm{AKB}]}{cr_c}}$$
$$=\sum_{cyc}x.\sqrt{1+2x}\left(x=\frac{[\mathrm{BKC}]}{ar_a}\text{ and analogs}\right)$$
$$=\sum_{cyc}\left(\sqrt{x}.\sqrt{x+2x^2}\right)\underbrace{\overset{\text{CBS}}{\leq}}_{(**)}\sqrt{\sum_{cyc}x}.\sqrt{\sum_{cyc}x+2\sum_{cyc}x^2}$$

$$\sum_{cyc} x^2 \overset{\text{via } (*)}{=} \frac{\sum_{cyc} a^2(s-a)^2}{\left(\sum_{cyc} a^2\right)^2} = \frac{1}{\left(\sum_{cyc} a^2\right)^2} \cdot \sum_{cyc} a^2(s^2 - 2sa + a^2)$$

$$= \frac{1}{\left(\sum_{cyc} a^2\right)^2} \cdot \left(\begin{matrix} 2s^2(s^2-4Rr-r^2) - 4s^2(s^2-6Rr-3r^2) + 2(s^2+4Rr+r^2)^2 \\ -32Rrs^2 - 16r^2s^2 \end{matrix} \right)$$

$$= \frac{2r^2\left((4R+r)^2 - s^2\right)}{\left(\sum_{cyc} a^2\right)^2} \Rightarrow \sum_{cyc} x + 2\sum_{cyc} x^2 \overset{\text{via } (*)}{=}$$

$$\frac{\sum_{cyc} a(s-a)}{\sum_{cyc} a^2} + \frac{4r^2\left((4R+r)^2 - s^2\right)}{\left(\sum_{cyc} a^2\right)^2} = \frac{4Rr + r^2}{s^2 - 4Rr - r^2} + \frac{r^2\left((4R+r)^2 - s^2\right)}{(s^2 - 4Rr - r^2)^2}$$

$$= \frac{4Rrs^2}{(s^2-4Rr-r^2)^2} \therefore \text{via } (**),$$

$$\frac{[BKC]}{ar_a} \cdot \sqrt{1 + \frac{2[BKC]}{ar_a}} + \frac{[CKA]}{br_b} \cdot \sqrt{1 + \frac{2[CKA]}{br_b}} + \frac{[AKB]}{cr_c} \cdot \sqrt{1 + \frac{[AKB]}{cr_c}}$$

$$\le \sqrt{\sum_{cyc} \frac{[BKC]}{ar_a}} \cdot \sqrt{\frac{4Rrs^2}{(s^2-4Rr-r^2)^2}} \overset{\text{via } (*)}{=} \sqrt{\frac{a(s-a)}{\sum_{cyc} a^2}} \cdot \sqrt{\frac{4Rrs^2}{(s^2-4Rr-r^2)^2}}$$

$$= \sqrt{\frac{4Rr + r^2}{s^2 - 4Rr - r^2}} \cdot \sqrt{\frac{4Rrs^2}{(s^2-4Rr-r^2)^2}} \overset{?}{\le} \frac{\sqrt{3}}{3}$$

$$\Leftrightarrow (s^2 - 4Rr - r^2)^3 \underset{(***)}{\overset{?}{\ge}} 12Rrs^2(4Rr + r^2)$$

$$\text{Firstly, } \left(\sum_{cyc} ab\right)^2 \ge 3abc \sum_{cyc} a = 12Rrs.\,2s \Rightarrow \left(\sum_{cyc} ab\right)^2 \ge 24Rrs^2$$

$$\Rightarrow \left(\sum_{cyc} a^2\right)^2 \ge 24Rrs^2 \Rightarrow (s^2 - 4Rr - r^2)^2 \overset{(\bullet)}{\ge} 6Rrs^2$$

$$\text{Also, } s^2 - 12Rr - 3r^2 = s^2 - 16Rr + 5r^2 + 4r(R - 2r) \overset{\substack{\text{Gerretsen} \\ \text{and} \\ \text{Euler}}}{\ge} 0$$

$$\Rightarrow s^2 - 4Rr - r^2 \overset{(\bullet\bullet)}{\ge} 8Rr + 2r^2 \therefore (\bullet).(\bullet\bullet) \Rightarrow (***) \text{ is true}$$

$$\therefore \frac{[BKC]}{ar_a} \cdot \sqrt{1 + \frac{2[BKC]}{ar_a}} + \frac{[CKA]}{br_b} \cdot \sqrt{1 + \frac{2[CKA]}{br_b}} + \frac{[AKB]}{cr_c} \cdot \sqrt{1 + \frac{[AKB]}{cr_c}}$$

$$\le \frac{\sqrt{3}}{3} \ \forall \Delta ABC, '' = '' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1634. In any Δ ABC, the following relationship holds :

$$\frac{64r^5}{R^6} \le \left(\frac{\sqrt{m_a}}{w_b}\right)^2 + \left(\frac{\sqrt{w_b}}{h_c}\right)^2 + \left(\frac{\sqrt{h_c}}{m_a}\right)^2 \le \frac{1}{r}\cdot\left(\frac{81}{32}\left(\frac{R}{r}\right)^5 - 80\right)$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\frac{\sqrt{m_a}}{w_b}\right)^2 + \left(\frac{\sqrt{w_b}}{h_c}\right)^2 + \left(\frac{\sqrt{h_c}}{m_a}\right)^2 \le \frac{m_a}{h_b^2} + \frac{m_b}{h_c^2} + \frac{m_c}{h_a^2}$$

$$= \frac{1}{4r^2s^2}\cdot\left(b^2m_a + c^2m_b + a^2m_c\right) \overset{CBS}{\le} \frac{1}{4r^2s^2}\cdot\sqrt{\sum_{cyc} a^4}\cdot\sqrt{\sum_{cyc} m_a^2} \overset{Mitrinovic}{\le}$$

$$\frac{1}{4r^2.27r^2}\cdot\sqrt{2\sum_{cyc} a^2b^2 - 16r^2s^2}\cdot\sqrt{\frac{3}{4}\sum_{cyc} a^2} \overset{\substack{Goldstone\\ and\\ Leibnitz}}{\le} \frac{1}{4r^2.27r^2}\cdot\sqrt{(8R^2 - 16r^2)s^2}\cdot\sqrt{\frac{3}{4}.9R^2}$$

$$\overset{Mitrinovic}{\le} \frac{1}{4r^2.27r^2}\cdot\sqrt{(8R^2 - 16r^2).\frac{27R^2}{4}}\cdot\sqrt{\frac{3}{4}.9R^2} = \frac{R^2.\sqrt{\frac{R^2 - 2r^2}{2}}}{4r^4}$$

$$\overset{?}{\le} \frac{1}{r}\cdot\left(\frac{81}{32}\left(\frac{R}{r}\right)^5 - 80\right) = \frac{81R^5 - 2560r^5}{32r^6}$$

$$\Leftrightarrow \left(81R^5 - 2560r^5\right)^2 \overset{?}{\ge} 64R^4r^4\left(\frac{R^2 - 2r^2}{2}\right)$$

$$\Leftrightarrow 6561t^{10} - 32t^6 - 414720t^5 + 64t^4 + 6553600 \overset{?}{\ge} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)\begin{pmatrix} 6561t^8 + 26244t^7 + 78732t^6 + 209952t^5 + 524848t^4 \\ +844864t^3 + 1280128t^2 + 1741056t + 1843712 \end{pmatrix} +410624\right) \overset{?}{\ge} 0$$

$$\to \text{true} \because t \overset{Euler}{\ge} 2$$

$$\therefore \left(\frac{\sqrt{m_a}}{w_b}\right)^2 + \left(\frac{\sqrt{w_b}}{h_c}\right)^2 + \left(\frac{\sqrt{h_c}}{m_a}\right)^2 \le \frac{1}{r}\cdot\left(\frac{81}{32}\left(\frac{R}{r}\right)^5 - 80\right)$$

Again, $\left(\frac{\sqrt{m_a}}{w_b}\right)^2 + \left(\frac{\sqrt{w_b}}{h_c}\right)^2 + \left(\frac{\sqrt{h_c}}{m_a}\right)^2 \ge \frac{h_a}{m_b^2} + \frac{h_b}{m_c^2} + \frac{h_c}{m_a^2} \overset{Bergstrom}{\ge} \frac{\left(\sum_{cyc}\frac{1}{m_a}\right)^2}{\sum_{cyc}\frac{1}{h_a}}$

$$\overset{Bergstrom}{\ge} r\left(\frac{9}{\sum_{cyc} m_a}\right)^2 \overset{Leuenberger + Euler}{\ge} r\left(\frac{9}{\frac{9R}{2}}\right)^2 = \frac{4r}{R^2} = \frac{4r^5}{R^2r^4} \overset{Euler}{\ge} \frac{4r^5}{R^2\left(\frac{R}{2}\right)^4} = \frac{64r^5}{R^6}$$

$$\therefore \left(\sqrt{\frac{m_a}{w_b}}\right)^2+\left(\sqrt{\frac{w_b}{h_c}}\right)^2+\left(\sqrt{\frac{h_c}{m_a}}\right)^2 \ge \frac{64r^5}{R^6} \text{ and so,}$$

$$\frac{64r^5}{R^6} \le \left(\sqrt{\frac{m_a}{w_b}}\right)^2+\left(\sqrt{\frac{w_b}{h_c}}\right)^2+\left(\sqrt{\frac{h_c}{m_a}}\right)^2 \le \frac{1}{r}\cdot\left(\frac{81}{32}\left(\frac{R}{r}\right)^5-80\right)$$

$\forall\,\Delta\,ABC,''=''$ iff $\Delta\,ABC$ is equilateral (QED)

1635. In any $\Delta\,ABC$, the following relationship holds :

$$\frac{4}{3R^2}\left(\frac{2r}{R}\right)^{10} \le \frac{m_a m_b}{w_a^2 w_b^2}+\frac{w_b w_c}{h_b^2 h_c^2}+\frac{h_c h_a}{m_c^2 m_a^2} \le \frac{1}{3r^2}\cdot\left(\frac{19683}{1024}\left(\frac{R}{r}\right)^{10}-19682\right)$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{m_a m_b}{w_a^2 w_b^2}+\frac{w_b w_c}{h_b^2 h_c^2}+\frac{h_c h_a}{m_c^2 m_a^2} \le \frac{m_a m_b}{h_a^2 h_b^2}+\frac{m_b m_c}{h_b^2 h_c^2}+\frac{m_c m_a}{h_c^2 h_a^2}$$

$$\overset{\text{Panaitopol}}{\le} \left(\frac{R}{2r}\right)^2\cdot\sum_{\text{cyc}}\frac{1}{h_b h_c} = \left(\frac{R}{2r}\right)^2\cdot\sum_{\text{cyc}}\frac{4R^2.bc}{ca.ab.bc} \le \left(\frac{R}{2r}\right)^2\cdot\frac{4R^2}{16R^2r^2s^2}\cdot\sum_{\text{cyc}}a^2$$

$$\overset{\substack{\text{Leibnitz}\\ \text{and}\\ \text{Mitrinovic}}}{\le} \frac{R^2.9R^2}{16r^4.27r^2} \therefore \frac{m_a m_b}{w_a^2 w_b^2}+\frac{w_b w_c}{h_b^2 h_c^2}+\frac{h_c h_a}{m_c^2 m_a^2} \le \frac{R^4}{48r^6} \to (1)$$

$$\text{Again, } \frac{1}{3r^2}\cdot\left(\frac{19683}{1024}\left(\frac{R}{r}\right)^{10}-19682\right) \overset{\text{Euler}}{\ge} \frac{1}{3r^2}\cdot\left(\frac{19683}{1024}.64.\left(\frac{R}{r}\right)^4-19682\right)$$

$$= \frac{19683R^4-19682.16r^4}{48r^6} \overset{?}{\ge} \frac{R^4}{48r^6} \Leftrightarrow 19682\left(R^4-16r^4\right) \overset{?}{\ge} 0 \to \text{true} \because R \overset{\text{Euler}}{\ge} 2r$$

$$\therefore \frac{1}{3r^2}\cdot\left(\frac{19683}{1024}\left(\frac{R}{r}\right)^{10}-19682\right) \ge \frac{R^4}{48r^6} \overset{\text{via }(1)}{\ge} \frac{m_a m_b}{w_a^2 w_b^2}+\frac{w_b w_c}{h_b^2 h_c^2}+\frac{h_c h_a}{m_c^2 m_a^2}$$

$$\Rightarrow \frac{m_a m_b}{w_a^2 w_b^2}+\frac{w_b w_c}{h_b^2 h_c^2}+\frac{h_c h_a}{m_c^2 m_a^2} \le \frac{1}{3r^2}\cdot\left(\frac{19683}{1024}\left(\frac{R}{r}\right)^{10}-19682\right)$$

$$\text{Also, } \frac{m_a m_b}{w_a^2 w_b^2}+\frac{w_b w_c}{h_b^2 h_c^2}+\frac{h_c h_a}{m_c^2 m_a^2} \ge \frac{h_a h_b}{m_a^2 m_b^2}+\frac{h_b h_c}{m_b^2 m_c^2}+\frac{h_c h_a}{m_c^2 m_a^2} \overset{\text{Panaitopol}}{\ge}$$

$$\frac{4r^2}{R^2}\cdot\sum_{\text{cyc}}\frac{1}{m_b m_c} \overset{\text{Bergstrom}}{\ge} \frac{4r^2}{R^2}\cdot\frac{9}{\sum_{\text{cyc}} m_b m_c} \ge \frac{4r^2}{R^2}\cdot\frac{9}{\sum_{\text{cyc}} m_a^2} = \frac{4r^2}{R^2}\cdot\frac{9}{\frac{3}{4}\cdot\sum_{\text{cyc}} a^2}$$

$$\overset{\text{Euler}}{\ge} \frac{4r^2}{R^2}\cdot\frac{9}{\frac{3}{4}\cdot 9R^2} = \frac{4}{3R^2}\cdot\left(\frac{2r}{R}\right)^2 \overset{\text{Euler}}{\ge} \frac{4}{3R^2}\cdot\left(\frac{2r}{R}\right)^2\cdot\left(\frac{2r}{R}\right)^8$$

$$\therefore \frac{m_a m_b}{w_a^2 w_b^2}+\frac{w_b w_c}{h_b^2 h_c^2}+\frac{h_c h_a}{m_c^2 m_a^2} \ge \frac{4}{3R^2}\left(\frac{2r}{R}\right)^{10} \text{ and so,}$$

$$\frac{4}{3R^2}\left(\frac{2r}{R}\right)^{10} \le \frac{m_a m_b}{w_a^2 w_b^2} + \frac{w_b w_c}{h_b^2 h_c^2} + \frac{h_c h_a}{m_c^2 m_a^2} \le \frac{1}{3r^2}\cdot\left(\frac{19683}{1024}\left(\frac{R}{r}\right)^{10} - 19682\right)$$

$\forall \Delta ABC, '' ='' \text{ iff } \Delta ABC$ is equilateral (QED)

1636. In any ΔABC, the following relationship holds :

$$27r^2.\left(\frac{2r}{R}\right)^6 \le \frac{m_a^2 m_b^2}{w_a w_b} + \frac{w_b^2 w_c^2}{h_b h_c} + \frac{h_c^2 h_a^2}{m_c m_a} \le \frac{27}{4}.\left(27R^2.\left(\frac{R}{2r}\right)^6 - 104r^2\right)$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{m_a^2 m_b^2}{w_a w_b} + \frac{w_b^2 w_c^2}{h_b h_c} + \frac{h_c^2 h_a^2}{m_c m_a} \le \frac{m_a^2 m_b^2}{h_a h_b} + \frac{m_b^2 m_c^2}{h_b h_c} + \frac{m_c^2 m_a^2}{h_c h_a} \overset{\text{Panaitopol}}{\le}$$

$$\left(\frac{R}{2r}\right)^2 \sum_{\text{cyc}} m_b m_c \le \frac{R^2}{4r^2}.\sum_{\text{cyc}} m_a^2 \overset{\text{Panaitopol}}{\le} \frac{R^2}{4r^2}.\frac{3}{4}.9R^2$$

$$\therefore \frac{m_a^2 m_b^2}{w_a w_b} + \frac{w_b^2 w_c^2}{h_b h_c} + \frac{h_c^2 h_a^2}{m_c m_a} \le \frac{27R^4}{16r^2} \to (1)$$

$$\text{Again, } \frac{27}{4}.\left(27R^2.\left(\frac{R}{2r}\right)^6 - 104r^2\right) \overset{\text{Euler}}{\ge} \frac{27}{4}.\left(27R^2.\left(\frac{R}{2r}\right)^2 - 104r^2\right)$$

$$= \frac{27(27R^4 - 416r^4)}{16r^2} \overset{?}{\ge} \frac{27R^4}{16r^2} \Leftrightarrow 26(R^4 - 16r^4) \overset{?}{\ge} 0 \to \text{true} \because R \overset{\text{Euler}}{\ge} 2r$$

$$\therefore \frac{27}{4}.\left(27R^2.\left(\frac{R}{2r}\right)^6 - 104r^2\right) \ge \frac{27R^4}{16r^2} \overset{\text{via }(1)}{\ge} \frac{m_a^2 m_b^2}{w_a w_b} + \frac{w_b^2 w_c^2}{h_b h_c} + \frac{h_c^2 h_a^2}{m_c m_a}$$

$$\therefore \frac{m_a^2 m_b^2}{w_a w_b} + \frac{w_b^2 w_c^2}{h_b h_c} + \frac{h_c^2 h_a^2}{m_c m_a} \le \frac{27}{4}.\left(27R^2.\left(\frac{R}{2r}\right)^6 - 104r^2\right)$$

$$\text{Also, } \frac{m_a^2 m_b^2}{w_a w_b} + \frac{w_b^2 w_c^2}{h_b h_c} + \frac{h_c^2 h_a^2}{m_c m_a} \ge \frac{h_a^2 h_b^2}{m_a m_b} + \frac{h_b^2 h_c^2}{m_b m_c} + \frac{h_c^2 h_a^2}{m_c m_a} \overset{\text{Panaitopol}}{\ge} \frac{4r^2}{R^2}.\sum_{\text{cyc}} h_b h_c$$

$$= \frac{4r^2}{R^2}.\sum_{\text{cyc}} \frac{ca.ab}{4R^2} = \frac{4r^2.4Rrs.2s}{R^2.4R^2} = \frac{8r^3}{R^3}.s^2 \overset{\text{Mitrinovic}}{\ge} \left(\frac{2r}{R}\right)^3.27r^2 \overset{\text{Euler}}{\ge}$$

$$\left(\frac{2r}{R}\right)^3.27r^2.\left(\frac{2r}{R}\right)^3 \Rightarrow \frac{m_a^2 m_b^2}{w_a w_b} + \frac{w_b^2 w_c^2}{h_b h_c} + \frac{h_c^2 h_a^2}{m_c m_a} \ge 27r^2.\left(\frac{2r}{R}\right)^6 \text{ and so,}$$

$$27r^2.\left(\frac{2r}{R}\right)^6 \le \frac{m_a^2 m_b^2}{w_a w_b} + \frac{w_b^2 w_c^2}{h_b h_c} + \frac{h_c^2 h_a^2}{m_c m_a} \le \frac{27}{4}.\left(27R^2.\left(\frac{R}{2r}\right)^6 - 104r^2\right)$$

$\forall \Delta ABC, '' ='' \text{ iff } \Delta ABC$ is equilateral (QED)

1637. Let ABC be a triangle and J denote the midpoint of GH.

Prove that the center of the Euler's circle lies on the incircle if and only if

$$IJ = \frac{\sqrt{2}}{3}IH$$

Proposed by Mihaly Bencze, Neculai Stanciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$HI^2 = 4R^2 + 4Rr + 3r^2 - s^2 \to (1), GH^2 = \frac{4}{9}\left(9R^2 - \sum_{cyc} a^2\right) \to (2),$$

$$GI^2 = \frac{1}{9}(s^2 - 16Rr + 5r^2) \to (3)$$

$$\because \text{center of the Euler circle lies on the incircle}, \therefore NI = r$$

$$(N \to \text{center of Euler circle}) \Rightarrow \frac{2F.OI^2}{abc} = r \Rightarrow \frac{2rs.R(R-2r)}{4Rrs} = r \Rightarrow R = 4r \to (4)$$

$$\text{Now, in } \Delta\, HIG, IJ \text{ is a median} \Rightarrow 4IJ^2 = 2HI^2 + 2GI^2 - GH^2 \overset{?}{=} \frac{8}{9}HI^2 \overset{\text{via (1),(2),(3)}}{\Leftrightarrow}$$

$$\frac{10}{9}(4R^2 + 4Rr + 3r^2 - s^2) \overset{?}{=} \frac{4}{9}\left(9R^2 - \sum_{cyc} a^2\right) - \frac{2}{9}(s^2 - 16Rr + 5r^2)$$

$$\Leftrightarrow 10(4R^2 + 4Rr + 3r^2 - s^2) \overset{?}{=} 4\left(9R^2 - 2(s^2 - 4Rr - r^2)\right) - 2(s^2 - 16Rr + 5r^2)$$

$$\Leftrightarrow R^2 - 6Rr + 8r^2 \overset{?}{=} 0 \Leftrightarrow (R-2r)(R-4r) \overset{?}{=} 0 \overset{\text{via (4)}}{\Leftrightarrow} 2r(R-4r) \overset{?}{=} 0$$

$$\to \text{true via (4)} \therefore 4IJ^2 = \frac{8}{9}HI^2 \Rightarrow IJ = \frac{\sqrt{2}}{3}IH \text{ (QED)}$$

1638. In any Δ ABC, the following relationship holds :

$$\min\left\{\sum_{cyc}\frac{\sin\frac{A}{2}}{\sin\frac{B}{2} + \sin\frac{C}{2}}, \sum_{cyc}\frac{\cos A}{\cos B + \cos C}\right\} + \frac{R^n}{r^n} \geq 2^n + \sum_{cyc}\frac{a^2}{b^2 + c^2}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{\cos A}{\cos B + \cos C} = \sum_{cyc}\frac{\sum_{cyc}\cos A - (\cos B + \cos C)}{\cos B + \cos C}$$

$$=\left(\sum_{cyc}\cos A\right)\sum_{cyc}\frac{1}{2\sin\frac{A}{2}\cos\frac{B-C}{2}}-3\overset{0<\cos\frac{B-C}{2}\le 1 \text{ and analogs}}{\ge}\frac{1+\frac{r}{R}}{2}.\sum_{cyc}\frac{1}{\sin\frac{A}{2}}-3$$

$$\overset{A-G}{\ge}\frac{3\left(1+\frac{r}{R}\right)}{2}.\sqrt[3]{\prod_{cyc}\frac{1}{\sin\frac{A}{2}}}-3=\frac{3\left(1+\frac{r}{R}\right)}{2}.\sqrt[3]{\frac{4R}{r}}-3\overset{Euler}{\ge}\frac{3\left(1+\frac{r}{R}\right)}{2}.\sqrt[3]{8}-3$$

$$\Rightarrow\sum_{cyc}\frac{\cos A}{\cos B+\cos C}\ge\frac{3r}{R}\to(1)$$

$$\text{Now},\sum_{cyc}\frac{a^2}{b^2+c^2}\overset{\text{Reverse Bergstrom}}{\le}\frac{1}{4}\sum_{cyc}\left(\frac{a^2}{b^2}+\frac{a^2}{c^2}\right)=\frac{1}{4}\sum_{cyc}\left(\left(\frac{b}{c}+\frac{c}{b}\right)^2-2\right)$$

$$\overset{Bandila}{\le}\frac{1}{4}\sum_{cyc}\left(\left(\frac{R}{r}\right)^2-2\right)\Rightarrow\sum_{cyc}\frac{a^2}{b^2+c^2}\le\frac{3R^2}{4r^2}-\frac{3}{2}\to(2)$$

$$\therefore \text{via (1) and (2), in order to prove}:\sum_{cyc}\frac{\cos A}{\cos B+\cos C}+\frac{R^2}{r^2}-4\ge\sum_{cyc}\frac{a^2}{b^2+c^2},$$

$$\text{it suffices to prove}:\frac{3r}{R}+\frac{R^2}{r^2}-4-\frac{3R^2}{4r^2}+\frac{3}{2}\ge 0\Leftrightarrow t^3-10t+12\ge 0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)\left((t^2-4)+2(t-2)+2\right)\ge 0\to\text{true}\because t\overset{Euler}{\ge}2$$

$$\therefore\sum_{cyc}\frac{\cos A}{\cos B+\cos C}+\frac{R^2}{r^2}-4\ge\sum_{cyc}\frac{a^2}{b^2+c^2}\to(i)$$

$$\text{Again, via Nesbitt and (2), in order to prove}:\sum_{cyc}\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}+\sin\frac{C}{2}}+\frac{R^2}{r^2}-4$$

$$\ge\sum_{cyc}\frac{a^2}{b^2+c^2},\text{it suffices to prove}:\frac{3}{2}+\frac{R^2}{r^2}-4-\frac{3R^2}{4r^2}+\frac{3}{2}\ge 0\Leftrightarrow R^2\ge 4r^2$$

$$\to\text{true via Euler}\therefore\sum_{cyc}\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}+\sin\frac{C}{2}}+\frac{R^2}{r^2}-4\ge\sum_{cyc}\frac{a^2}{b^2+c^2}\to(ii)$$

$$\therefore(i),(ii)\Rightarrow\min\left\{\sum_{cyc}\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}+\sin\frac{C}{2}},\sum_{cyc}\frac{\cos A}{\cos B+\cos C}\right\}+\frac{R^2}{r^2}-4\ge\sum_{cyc}\frac{a^2}{b^2+c^2}$$

$$\Rightarrow\frac{R^2}{r^2}-4\ge\sum_{cyc}\frac{a^2}{b^2+c^2}-\min\left\{\sum_{cyc}\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}+\sin\frac{C}{2}},\sum_{cyc}\frac{\cos A}{\cos B+\cos C}\right\}\to(iii)$$

$$\text{Let }F(n)=t^n-2^n\ \forall\, t=\frac{R}{r}\ge 2\ (t\to\text{fixed})\text{ and }\forall\, n\ge 2\text{ and then}:$$

$$F'(n)=t^n.(\ln t)-2^n.(\ln 2)\ge 0$$

$$(\because t^n\ge 2^n\text{ and }\ln t\ge\ln 2\Rightarrow t^n.(\ln t)-2^n.(\ln 2)\ge 0)$$

$$\therefore F(n)\text{ is}\uparrow\forall\, n\ge 2\Rightarrow F(n)\ge F(2)\Rightarrow\text{when both }t=\frac{R}{r}\text{ and n vary,}$$

$$\left(\frac{R}{r}\right)^n-2^n\ge\frac{R^2}{r^2}-4\overset{\text{via (iii)}}{\ge}\sum_{cyc}\frac{a^2}{b^2+c^2}-\min\left\{\sum_{cyc}\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}+\sin\frac{C}{2}},\sum_{cyc}\frac{\cos A}{\cos B+\cos C}\right\}$$

$$\Rightarrow\min\left\{\sum_{cyc}\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}+\sin\frac{C}{2}},\sum_{cyc}\frac{\cos A}{\cos B+\cos C}\right\}+\frac{R^n}{r^n}\ge 2^n+\sum_{cyc}\frac{a^2}{b^2+c^2}\ \forall\Delta ABC,$$

$''=''$ iff Δ ABC is equilateral (QED)

1639. In any Δ ABC, the following relationship holds :

$$\frac{m_a+m_b}{(w_a+w_b)^2}+\frac{w_b+w_c}{(h_b+h_c)^2}+\frac{h_c+h_a}{(m_c+m_a)^2}\le\frac{1}{2r}\cdot\left(\frac{81}{32}\cdot\left(\frac{R}{r}\right)^5-80\right)$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{m_a+m_b}{(w_a+w_b)^2}+\frac{w_b+w_c}{(h_b+h_c)^2}+\frac{h_c+h_a}{(m_c+m_a)^2}\le$$

$$\frac{m_a+m_b}{(h_a+h_b)^2}+\frac{m_b+m_c}{(h_b+h_c)^2}+\frac{m_c+m_a}{(h_c+h_a)^2}\overset{\text{Panaitopol}}{\le}\frac{R}{2r}\cdot\sum_{cyc}\frac{h_b+h_c}{(h_b+h_c)^2}$$

$$=\frac{R}{2r}\cdot 2R\sum_{cyc}\frac{1}{ca+ab}\overset{A-G}{\le}\frac{R^2}{2r}\cdot\sum_{cyc}\frac{1}{a\sqrt{bc}}=\frac{R^2}{2r}\cdot\sum_{cyc}\left(\sqrt{\frac{1}{ab}}\cdot\sqrt{\frac{1}{ac}}\right)$$

$$\overset{CBS}{\le}\frac{R^2}{2r}\cdot\sqrt{\sum_{cyc}\frac{1}{ab}}\cdot\sqrt{\sum_{cyc}\frac{1}{ab}}=\frac{R^2}{2r}\cdot\frac{2s}{4Rrs}$$

$$\therefore\frac{m_a+m_b}{(w_a+w_b)^2}+\frac{w_b+w_c}{(h_b+h_c)^2}+\frac{h_c+h_a}{(m_c+m_a)^2}\le\frac{R}{4r^2}\to(1)$$

$$\text{Again, }\frac{1}{2r}\cdot\left(\frac{81}{32}\cdot\left(\frac{R}{r}\right)^5-80\right)\overset{\text{Euler}}{\ge}\frac{1}{2r}\cdot\left(\frac{81}{32}\cdot 16\left(\frac{R}{r}\right)-80\right)$$

$$\Rightarrow\frac{1}{2r}\cdot\left(\frac{81}{32}\cdot\left(\frac{R}{r}\right)^5-80\right)\ge\frac{81R-160r}{4r^2}=\frac{80(R-2r)}{4r^2}+\frac{R}{4r^2}\overset{\text{Euler}}{\ge}\frac{R}{4r^2}$$

$$\overset{\text{via (1)}}{\ge}\frac{m_a+m_b}{(w_a+w_b)^2}+\frac{w_b+w_c}{(h_b+h_c)^2}+\frac{h_c+h_a}{(m_c+m_a)^2}$$

$$\therefore\frac{m_a+m_b}{(w_a+w_b)^2}+\frac{w_b+w_c}{(h_b+h_c)^2}+\frac{h_c+h_a}{(m_c+m_a)^2}$$

$$\le\frac{1}{2r}\cdot\left(\frac{81}{32}\cdot\left(\frac{R}{r}\right)^5-80\right)\ \forall\Delta ABC,''=''\text{ iff }\Delta ABC\text{ is equilateral (QED)}$$

1640. In any Δ ABC, the following relationship holds :

$$\frac{8r^2}{R^3} \le \frac{1}{m_a} + \frac{1}{w_b} + \frac{1}{h_c} \le \frac{R^2}{4r^3} \text{ and}$$

$$\frac{4}{3R^2} \cdot \left(\frac{2r}{R}\right)^4 \le \frac{1}{m_a^2} + \frac{1}{w_b^2} + \frac{1}{h_c^2} \le \frac{1}{r^2} \cdot \left(\left(\frac{R}{2r}\right)^4 - \frac{2}{3}\right)$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{m_a} + \frac{1}{w_b} + \frac{1}{h_c} \le \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r} = \frac{4r^2}{4r^3} \overset{\text{Euler}}{\le} \frac{R^2}{4r^3} \text{ and}$$

$$\frac{1}{m_a} + \frac{1}{w_b} + \frac{1}{h_c} \ge \frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} \overset{\substack{\text{Bergstrom} \\ \text{and} \\ \text{Leuenberger + Euler}}}{\ge} \frac{9}{\frac{9R}{2}} = \frac{2.4r^2}{R.4r^2} \overset{\text{Euler}}{\ge} \frac{8r^2}{R^3}$$

$$\frac{1}{m_a^2} + \frac{1}{w_b^2} + \frac{1}{h_c^2} \le \sum_{cyc} \frac{1}{h_a^2} = \frac{\sum_{cyc} a^2}{4r^2s^2} \overset{\substack{\text{Leibnitz} \\ \text{and} \\ \text{Mitrinovic}}}{\le} \frac{9R^2}{4r^2.27r^2}$$

$$\Rightarrow \frac{1}{m_a^2} + \frac{1}{w_b^2} + \frac{1}{h_c^2} \le \frac{R^2}{12r^4} \to (1) \text{ and } \frac{1}{r^2} \cdot \left(\left(\frac{R}{2r}\right)^4 - \frac{2}{3}\right) \overset{\text{Euler}}{\ge} \frac{3R^2 - 8r^2}{12r^4}$$

$$\overset{\text{Euler}}{\ge} \frac{3R^2 - 2R^2}{12r^4} \overset{\text{via (1)}}{\ge} \frac{1}{m_a^2} + \frac{1}{w_b^2} + \frac{1}{h_c^2} \text{ and, } \frac{1}{m_a^2} + \frac{1}{w_b^2} + \frac{1}{h_c^2} \ge \sum_{cyc} \frac{1}{m_a^2} \overset{\substack{\text{Bergstrom} \\ \text{and} \\ \text{Leibnitz}}}{\ge}$$

$$\frac{9}{\frac{3}{4}.9R^2} = \frac{4}{3R^2} \overset{\text{Euler}}{\ge} \frac{4}{3R^2} \cdot \left(\frac{2r}{R}\right)^4 \therefore \frac{8r^2}{R^3} \le \frac{1}{m_a} + \frac{1}{w_b} + \frac{1}{h_c} \le \frac{R^2}{4r^3} \text{ and}$$

$$\frac{4}{3R^2} \cdot \left(\frac{2r}{R}\right)^4 \le \frac{1}{m_a^2} + \frac{1}{w_b^2} + \frac{1}{h_c^2} \le \frac{1}{r^2} \cdot \left(\left(\frac{R}{2r}\right)^4 - \frac{2}{3}\right)$$

$\forall \Delta$ ABC, " = " iff Δ ABC is equilateral (QED)

1641. In any Δ ABC, the following relationship holds :

$$\frac{3a + b}{2a + c} + \frac{3b + c}{2b + a} + \frac{3c + a}{2c + b} \ge 4$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{3a+b}{2a+c}+\frac{3b+c}{2b+a}+\frac{3c+a}{2c+b}\geq 4$$

$$\Leftrightarrow (3a+b)(2b+a)(2c+b)+(3b+c)(2c+b)(2a+c)$$
$$+(3c+a)(2a+c)(2b+a)\geq 4(2a+c)(2b+a)(2c+b)$$

$$\overset{\text{expanding and re-arranging}}{\Leftrightarrow} 2\sum_{cyc}a^3+6abc\geq\sum_{cyc}a^2b+3\sum_{cyc}ab^2$$

$$\Leftrightarrow 2\sum_{cyc}(y+z)^3+6(x+y)(y+z)(z+x)$$

$$\geq\sum_{cyc}\left((y+z)^2(z+x)\right)+3\sum_{cyc}\left((y+z)(z+x)^2\right)$$

$$(a=y+z, b=z+x, c=x+y)\overset{\text{expanding and re-arranging}}{\Leftrightarrow}$$

$$\sum_{cyc}x^2y+3\sum_{cyc}xy^2\geq 12xyz\to\text{true}\because\sum_{cyc}x^2y\overset{A-G}{\geq}3xyz\text{ and }\sum_{cyc}xy^2\overset{A-G}{\geq}3xyz$$

$$\therefore\frac{3a+b}{2a+c}+\frac{3b+c}{2b+a}+\frac{3c+a}{2c+b}\geq 4\forall\Delta ABC,''='' \text{ iff }\Delta ABC\text{ is equilateral (QED)}$$

1642. In any Δ ABC, the following relationship holds :

$$1+\frac{r}{R}\leq\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\leq\sqrt{2+\frac{r}{2R}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\operatorname{cosec}\frac{A}{2}=\sqrt{\frac{bc(s-a)}{(s-b)(s-c)(s-a)}}=\frac{1}{r.\sqrt{s}}.\sum_{cyc}\sqrt{bc(s-a)}\overset{CBS}{\leq}$$

$$\frac{1}{r.\sqrt{s}}.\sqrt{s^2+4Rr+r^2}.\sqrt{\sum_{cyc}(s-a)}\overset{Gerretsen}{\leq}\frac{\sqrt{s}}{r.\sqrt{s}}.\sqrt{4R^2+8Rr+4r^2}=\frac{\sqrt{4(R+r)^2}}{r}$$

$$\therefore\sum_{cyc}\operatorname{cosec}\frac{A}{2}\leq\frac{2R+2r}{r}\to(1)$$

$$\text{Now,}\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\right)^2=\sum_{cyc}\sin^2\frac{A}{2}+2\sum_{cyc}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$=\frac{2R-r}{2R}+\frac{2r}{4R}.\sum_{cyc}\operatorname{cosec}\frac{A}{2}\overset{via\,(1)}{\leq}\frac{2R-r}{2R}+\frac{2r}{4R}.\frac{2R+2r}{r}=\frac{4R+r}{2R}=2+\frac{r}{2R}$$

$$\Rightarrow\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\leq\sqrt{2+\frac{r}{2R}}$$

$$Again, \sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}=\sum_{cyc}\frac{2\cos\frac{B+C}{2}.\cos\frac{B-C}{2}}{2\cos\frac{B-C}{2}}\overset{\because 0<\cos\frac{B-C}{2}\le 1 \text{ and analogs}}{\ge}$$

$$\sum_{cyc}\frac{2\cos\frac{B+C}{2}.\cos\frac{B-C}{2}}{2}=\sum_{cyc}\frac{\cos B+\cos C}{2}=\sum_{cyc}\cos A=1+\frac{r}{R}\therefore\sum_{cyc}\sin\frac{A}{2}\ge 1+\frac{r}{R}$$

$$and\ so,\ 1+\frac{r}{R}\le\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\le\sqrt{2+\frac{r}{2R}}$$

$$\forall\Delta ABC, ''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1643.

In any Δ ABC with I → incenter, the following relationship holds :

$$2\sqrt{\frac{AI+BI+CI}{2(AI+BI+CI+r)-4R}}\ge\sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}+\sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$2\sqrt{\frac{AI+BI+CI}{2(AI+BI+CI+r)-4R}}=\sqrt{\frac{4\sum_{cyc}\frac{r}{\sin\frac{A}{2}}}{2\sum_{cyc}\frac{r}{\sin\frac{A}{2}}-(4R-2r)}}=$$

$$\sqrt{\frac{\frac{2}{2R}\cdot\sum_{cyc}\frac{r}{\sin\frac{A}{2}}}{\frac{1}{2R}\sum_{cyc}\frac{r}{\sin\frac{A}{2}}-\frac{2R-r}{2R}}}=\sqrt{\frac{4\left(\prod_{cyc}\sin\frac{A}{2}\right).\frac{\sum_{cyc}\sin\frac{B}{2}\sin\frac{C}{2}}{\prod_{cyc}\sin\frac{A}{2}}}{2\left(\prod_{cyc}\sin\frac{A}{2}\right).\left(\frac{\sum_{cyc}\sin\frac{B}{2}\sin\frac{C}{2}}{\prod_{cyc}\sin\frac{A}{2}}\right)-\sum_{cyc}\sin^2\frac{A}{2}}}$$

$$=\sqrt{\frac{4\sum_{cyc}\sin\frac{B}{2}\sin\frac{C}{2}}{2\sum_{cyc}\sin\frac{B}{2}\sin\frac{C}{2}-\sum_{cyc}\sin^2\frac{A}{2}}}\overset{?}{\ge}\sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}+\sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}}$$

$$\Leftrightarrow\frac{4\sum_{cyc}\sin\frac{B}{2}\sin\frac{C}{2}}{2\sum_{cyc}\sin\frac{B}{2}\sin\frac{C}{2}-\sum_{cyc}\sin^2\frac{A}{2}}-2\overset{?}{\ge}\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}+\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}$$

$$\Leftrightarrow\frac{2\sum_{cyc}\sin^2\frac{A}{2}}{2\sum_{cyc}\sin\frac{B}{2}\sin\frac{C}{2}-\sum_{cyc}\sin^2\frac{A}{2}}\overset{?}{\ge}\frac{\sin^2\frac{B}{2}+\sin^2\frac{C}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$\Leftrightarrow 2yz\sum_{cyc} x^2 \overset{?}{\geq} (y^2+z^2)\left(2\sum_{cyc} xy - \sum_{cyc} x^2\right) \left(x = \sin\frac{A}{2}, y = \sin\frac{B}{2}, z = \sin\frac{C}{2}\right)$$

$$\overset{\text{expanding and re-arranging}}{\Leftrightarrow} x^2(y+z)^2 + (y^2+z^2)^2 \overset{?}{\geq} 2xyz(y+z) + 2x(y^3+z^3)$$

$$\Leftrightarrow x^2(y+z)^2 + (y^2+z^2)^2 \overset{?}{\geq} 2xyz(y+z) + 2x(y+z)(y^2+z^2-yz)$$

$$= 2x(y+z)(yz+y^2+z^2-yz) \Leftrightarrow x^2(y+z)^2 + (y^2+z^2)^2 \overset{?}{\geq} 2x(y+z)(y^2+z^2)$$

$$\Leftrightarrow \left(x(y+z) - (y^2+z^2)\right)^2 \overset{?}{\geq} 0 \to \text{true}$$

$$\therefore 2\sqrt{\frac{AI+BI+CI}{2(AI+BI+CI+r)-4R}} \geq \sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}} + \sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}} \ \forall \Delta ABC \ (QED)$$

1644. In any Δ ABC, the following relationship holds :

$$\frac{R}{2r}\sum_{cyc}\frac{a^{2024}}{b^{2024}+c^{2024}} + \frac{R^{2025}}{r^{2025}} \geq 2^{2025} + \sum_{cyc}\frac{a^2}{b^2+c^2}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } F(n) = t^n - 2^n \ \forall\, t = \frac{R}{r} \geq 2 \ (t \to \text{fixed}) \text{ and } \forall\, n \geq 2 \text{ and then :}$$

$$F'(n) = t^n.(\ln t) - 2^n.(\ln 2) \geq 0 \begin{pmatrix} \because t^n \geq 2^n \text{ and } \ln t \geq \ln 2 \\ \Rightarrow t^n.(\ln t) - 2^n.(\ln 2) \geq 0 \end{pmatrix}$$

$$\therefore F(n) \text{ is } \uparrow \forall\, n \geq 2 \Rightarrow F(n) \geq F(2) \Rightarrow \text{when both } t = \frac{R}{r} \text{ and n vary}, \left(\frac{R}{r}\right)^n - 2^n \geq$$

$$\frac{R^2}{r^2} - 4 \Rightarrow \frac{R}{2r}\sum_{cyc}\frac{a^{2024}}{b^{2024}+c^{2024}} + \frac{R^{2025}}{r^{2025}} - 2^{2025} \overset{\text{Nesbitt}}{\geq} \frac{R}{2r}.\frac{3}{2} + \frac{R^2}{r^2} - 4 \to (1)$$

$$\text{Also}, \sum_{cyc}\frac{a^2}{b^2+c^2} = \frac{1}{4}\sum_{cyc}\frac{4a^2}{b^2+c^2} \overset{\text{Reverse Bergstrom}}{\leq} \frac{1}{4}\sum_{cyc}\left(\frac{a^2}{b^2}+\frac{a^2}{c^2}\right)$$

$$= \frac{1}{4}\sum_{cyc}\left(\left(\frac{b}{c}+\frac{c}{b}\right)^2 - 2\right) \overset{\text{Bandila}}{\leq} \frac{1}{4}\sum_{cyc}\left(\left(\frac{R}{r}\right)^2 - 2\right) \Rightarrow \sum_{cyc}\frac{a^2}{b^2+c^2} \leq \frac{3R^2}{4r^2} - \frac{3}{2} \to (2)$$

$$\therefore \text{via (1) and (2), in order to prove : } \frac{R}{2r}\sum_{cyc}\frac{a^{2024}}{b^{2024}+c^{2024}} + \frac{R^{2025}}{r^{2025}} - 2^{2025} \geq$$

$$\sum_{cyc}\frac{a^2}{b^2+c^2}, \text{ it suffices to prove : } \frac{R}{2r}.\frac{3}{2} + \frac{R^2}{r^2} - 4 \geq \frac{3R^2}{4r^2} - \frac{3}{2} \Leftrightarrow$$

$$\frac{4R^2 - 16r^2 + 3Rr}{4r^2} \geq \frac{3R^2 - 6r^2}{4r^2} \Leftrightarrow R^2 + 3Rr - 10r^2 \geq 0 \Leftrightarrow (R-2r)(R+5r) \geq 0$$

$$\to \text{true via Euler} \therefore \frac{R}{2r}\sum_{cyc}\frac{a^{2024}}{b^{2024}+c^{2024}}+\frac{R^{2025}}{r^{2025}}-2^{2025}\ge\sum_{cyc}\frac{a^2}{b^2+c^2}$$

$\forall\,\Delta\,ABC$, " = " iff $\Delta\,ABC$ is equilateral (QED)

1645. In any Δ ABC, the following relationship holds :

$$\frac{m_a^2+m_b^2}{(w_a+w_b)^2}+\frac{w_b^2+w_c^2}{(h_b+h_c)^2}+\frac{h_c^2+h_a^2}{(m_c+m_a)^2}\le\frac{3}{16}\cdot\left(\frac{1823}{64}\cdot\left(\frac{R}{r}\right)^6-1815\right)$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{m_a^2+m_b^2}{(w_a+w_b)^2}+\frac{w_b^2+w_c^2}{(h_b+h_c)^2}+\frac{h_c^2+h_a^2}{(m_c+m_a)}\le\sum_{cyc}\frac{m_b^2+m_c^2}{(h_b+h_c)^2}\overset{\substack{\text{Panaitopol}\\ \text{and}\\ A-G}}{\le}$$

$$\frac{R^2}{4r^2}\cdot\sum_{cyc}\frac{h_b^2+h_c^2}{4h_bh_c}=\frac{R^2}{16r^2}\sum_{cyc}\left(\frac{c}{b}+\frac{b}{c}\right)\overset{\text{Bandila}}{\le}\frac{3R^2}{16r^3}\text{ and }\frac{3}{16}\cdot\left(\frac{1823}{64}\cdot\left(\frac{R}{r}\right)^6-1815\right)$$

$$\overset{\text{Euler}}{\ge}\frac{3}{16}\cdot\left(\frac{1823}{8}\cdot\left(\frac{R}{r}\right)^3-1815\right)\therefore\text{ it suffices to prove :}$$

$$\frac{3}{16}\cdot\left(\frac{1823R^3-8.1815r^3}{8r^3}\right)\ge\frac{3R^2}{16r^3}\Leftrightarrow 1815(R^3-8r^3)\ge 0\to\text{true via Euler}$$

$$\therefore\frac{m_a^2+m_b^2}{(w_a+w_b)^2}+\frac{w_b^2+w_c^2}{(h_b+h_c)^2}+\frac{h_c^2+h_a^2}{(m_c+m_a)^2}\le\frac{3}{16}\cdot\left(\frac{1823}{64}\cdot\left(\frac{R}{r}\right)^6-1815\right)$$

$\forall\,\Delta\,ABC$, " = " iff $\Delta\,ABC$ is equilateral (QED)

1646. In any Δ ABC, the following relationship holds :

$$\frac{m_a^2+m_b^2}{(w_a+w_b)^3}+\frac{w_b^2+w_c^2}{(h_b+h_c)^3}+\frac{h_c^2+h_a^2}{(m_c+m_a)^3}\le\frac{1}{4r}\cdot\left(27.\left(\frac{R}{2r}\right)^8-26\right)$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{m_a^2+m_b^2}{(w_a+w_b)^3}+\frac{w_b^2+w_c^2}{(h_b+h_c)^3}+\frac{h_c^2+h_a^2}{(m_c+m_a)^3}\le\sum_{cyc}\frac{m_b^2+m_c^2}{(h_b+h_c)^3}$$

$$\le\sum_{cyc}\frac{(h_b+h_c)(m_b^2+m_c^2)}{8h_bh_c(h_b^2+h_c^2)}$$

$$\left(\begin{array}{c}\because \forall x,y>0,(x+y)^4=\left((x^2+y^2)+2xy\right)^2\overset{A-G}{\geq}8xy(x^2+y^2)\\ \Rightarrow (x+y)^3\geq\dfrac{8xy(x^2+y^2)}{x+y}\end{array}\right)$$

$$\overset{Panaitopol}{\leq}\sum_{cyc}\frac{(h_b+h_c).\frac{R^2}{4r^2}(h_b^2+h_c^2)}{8h_bh_c(h_b^2+h_c^2)}=\frac{R^2}{32r^2}.\sum_{cyc}\left(\frac{1}{h_b}+\frac{1}{h_c}\right)=\frac{R^2}{32r^2}.\frac{2}{r}=\frac{R^2}{16r^3}$$

$$and\ \frac{1}{4r}.\left(27.\left(\frac{R}{2r}\right)^8-26\right)\overset{Euler}{\geq}\frac{1}{4r}.\left(27.\left(\frac{R}{2r}\right)^2-26\right)=\frac{27R^2-26.4r^2}{16r^3}$$

$$\therefore \text{it suffices to prove}:\frac{27R^2-26.4r^2}{16r^3}\geq\frac{R^2}{16r^3}\Leftrightarrow 26(R^2-4r^2)\geq 0$$

$$\rightarrow \text{true via Euler}\ \therefore\frac{m_a^2+m_b^2}{(w_a+w_b)^3}+\frac{w_b^2+w_c^2}{(h_b+h_c)^3}+\frac{h_c^2+h_a^2}{(m_c+m_a)^3}$$

$$\leq\frac{1}{4r}.\left(27.\left(\frac{R}{2r}\right)^8-26\right)\ \forall\Delta ABC,''=''\ \text{iff}\ \Delta ABC\ \text{is equilateral (QED)}$$

1647.

In any ΔABC and $\forall n,m\geq 0$, the following relationships holds :

$$\frac{r_a^n}{m_a^m}+\frac{r_b^n}{w_b^m}+\frac{r_c^n}{h_c^m}\geq\frac{3^{n-m+1}.2^m.r^n}{R^m}\quad and\quad \frac{m_a^n}{r_a^m}+\frac{w_b^n}{r_b^m}+\frac{h_c^n}{r_c^m}\geq\frac{3^{n-m+1}.2^m.r^n}{R^m}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{r_a^n}{m_a^m}+\frac{r_b^n}{w_b^m}+\frac{r_c^n}{h_c^m}\geq\frac{r_a^n}{m_a^m}+\frac{r_b^n}{m_b^m}+\frac{r_c^n}{m_c^m}\overset{A-G}{\geq}3.\sqrt[3]{\frac{\left(\prod_{cyc}r_a\right)^n}{\left(\prod_{cyc}m_a\right)^m}}\overset{m_am_bm_c\leq\frac{Rs^2}{2}}{\geq}$$

$$3.\sqrt[3]{\frac{(rs^2)^n}{\left(\frac{Rs^2}{2}\right)^m}}\overset{Mitrinovic}{\geq}3.\sqrt[3]{\frac{(r.27r^2)^n}{\left(\frac{R}{2}.\frac{27R^2}{4}\right)^m}}=\frac{3^{n-m+1}.2^m.r^n}{R^m}$$

$$\forall\Delta ABC,''=''\ \text{iff}\ \Delta ABC\ \text{is equilateral}$$

$$\frac{m_a^n}{r_a^m}+\frac{w_b^n}{r_b^m}+\frac{h_c^n}{r_c^m}\geq\frac{h_a^n}{r_a^m}+\frac{h_b^n}{r_b^m}+\frac{h_c^n}{r_c^m}\overset{A-G}{\geq}3.\sqrt[3]{\frac{\left(\prod_{cyc}h_a\right)^n}{\left(\prod_{cyc}r_a\right)^m}}=3.\sqrt[3]{\frac{\left(\frac{2r^2s^2}{R}\right)^n}{(rs^2)^m}}$$

$$\overset{\substack{Gerretsen+Euler\\ and\\ Mitrinovic+Euler}}{\geq}3.\sqrt[3]{\frac{\left(\frac{r^2.27Rr}{R}\right)^n}{\left(\frac{R}{2}.\frac{27R^2}{4}\right)^m}}=\frac{3^{n-m+1}.2^m.r^n}{R^m}$$

$\forall\,\Delta\,ABC,'' ='' \text{ iff } \Delta\,ABC$ is equilateral

$$\boxed{\text{Proof of } m_a m_b m_c \le \frac{Rs^2}{2}}$$

$$m_a^2 m_b^2 m_c^2 = \frac{1}{64}(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)$$

$$= \frac{1}{64}\left(-4\sum_{cyc} a^6 + 6\left(\sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4\right) + 3a^2b^2c^2\right) \rightarrow (1)$$

$$\text{Now, } \sum_{cyc} a^6 = \left(\sum_{cyc} a^2\right)^3 - 3(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

$$= \left(\sum_{cyc} a^2\right)^3 - 3\left(2a^2b^2c^2 + \sum_{cyc}\left(a^2b^2\left(\sum_{cyc} a^2 - c^2\right)\right)\right)$$

$$= \left(\sum_{cyc} a^2\right)^3 + 3a^2b^2c^2 - 3\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right)$$

$$\therefore \sum_{cyc} a^6 = \left(\sum_{cyc} a^2\right)^3 + 3a^2b^2c^2 - 3\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) \rightarrow (2)$$

$$\text{Also, } \sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4 = \sum_{cyc}\left(a^2b^2\left(\sum_{cyc} a^2 - c^2\right)\right) =$$

$$\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) - 3a^2b^2c^2 \rightarrow (3) \therefore (1),(2),(3) \Rightarrow m_a^2 m_b^2 m_c^2$$

$$= \frac{1}{64}\left(\begin{matrix} -4\left(\sum_{cyc} a^2\right)^3 - 12a^2b^2c^2 + 12\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) \\ +6\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) - 18a^2b^2c^2 + 3a^2b^2c^2 \end{matrix}\right)$$

$$= \frac{1}{64}\left(-4\left(\sum_{cyc} a^2\right)^3 + 18\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) - 27a^2b^2c^2\right)$$

$$=\frac{1}{64}\left(-4\left(\sum_{cyc}a^2\right)^3+18\left(\left(\sum_{cyc}ab\right)^2-16Rrs^2\right)\left(\sum_{cyc}a^2\right)-27a^2b^2c^2\right)$$

$$=\frac{1}{64}\left(\begin{matrix}-32(s^2-4Rr-r^2)^3+36(s^2-4Rr-r^2)(s^2+4Rr+r^2)^2\\-576Rrs^2(s^2-4Rr-r^2)-432R^2r^2s^2\end{matrix}\right)$$

$$=\frac{1}{16}(s^6-s^4(12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-r^3(4R+r)^3)$$

$$\le\frac{R^2s^4}{4}\Leftrightarrow$$

$$s^6-s^4(4R^2+12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-r^3(4R+r)^3\overset{(\bullet)}{\le}0$$

Now, LHS of $(\bullet)\overset{\text{Gerretsen}}{\le}-s^4(8Rr-36r^2)-s^2(60R^2r^2+120Rr^3+33r^4)$

$-r^3(4R+r)^3\overset{?}{\le}0$

$\Leftrightarrow s^4(8R-16r)+s^2(60R^2r+120Rr^2+33r^3)+r^2(4R+r)^3\underbrace{\overset{?}{\ge}}_{(\bullet\bullet)}20rs^4$

Now, LHS of $(\bullet\bullet)\underbrace{\overset{\text{Gerretsen}}{\ge}}_{(a)}s^2(16Rr-5r^2)(8R-16r)$

$+s^2(60R^2r+120Rr^2+33r^3)+r^2(4R+r)^3$ *and*

RHS of $(\bullet\bullet)\underbrace{\overset{\text{Geretsen}}{\le}}_{(b)}20rs^2(4R^2+4Rr+3r^2)$

$(a),(b)\Rightarrow$ in order to prove $(\bullet\bullet)$, it suffices to prove :

$s^2(16Rr-5r^2)(8R-16r)+s^2(60R^2r+120Rr^2+33r^3)+r^2(4R+r)^3$

$\ge 20rs^2(4R^2+4Rr+3r^2)$

$\Leftrightarrow s^2(108R^2-256Rr+53r^2)+r(4R+r)^3\ge 0$

$\Leftrightarrow s^2(108R^2-256Rr+80r^2)+r(4R+r)^3\overset{(\bullet\bullet\bullet)}{\ge}27r^2s^2$

Now, LHS of $(\bullet\bullet\bullet)\underbrace{\overset{\text{Gerretsen}}{\ge}}_{(c)}(108R^2-256Rr+80r^2)(16Rr-5r^2)+r(4R+r)^3$

and RHS of $(\bullet\bullet\bullet)\underbrace{\overset{\text{Geretsen}}{\le}}_{(d)}27r^2(4R^2+4Rr+3r^2)$

$(c),(d)\Rightarrow$ in order to prove $(\bullet\bullet\bullet)$, it suffices to prove :

$(108R^2-256Rr+80r^2)(16Rr-5r^2)+r(4R+r)^3\ \ge 27r^2(4R^2+4Rr+3r^2)$

$\Leftrightarrow 224t^3-587t^2+308t-60\ge 0\ \left(\text{where } t=\frac{R}{r}\right)$

$\Leftrightarrow (t-2)\big((t-2)(224t+309)+648\big)\ge 0\to$ true $\because t\overset{\text{Euler}}{\ge}2\Rightarrow(\bullet\bullet\bullet)\Rightarrow(\bullet\bullet)$

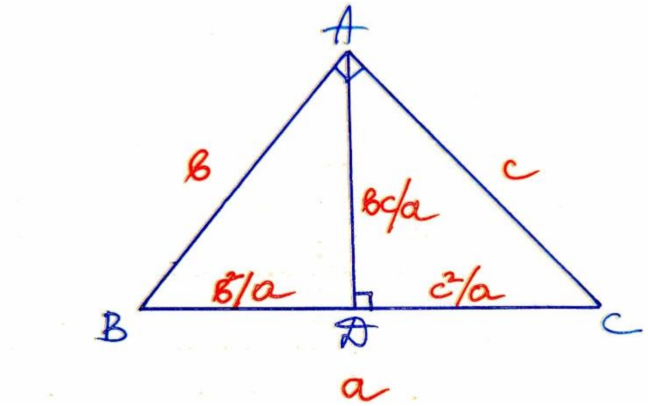
$\Rightarrow(\bullet)$ is true $\Rightarrow m_a^2m_b^2m_c^2\le\frac{R^2s^4}{4}\Rightarrow m_am_bm_c\le\frac{Rs^2}{2}$ (QED)

1648. In $\Delta ABC, m(BAC) = 90°$, $AD \perp BC, D \in (BC), R, R_1, R_2$ – circumradii and r, r_1, r_2 – inradii of $\Delta ABC, \Delta ABD, \Delta ACD$. Prove that

$$\frac{R-R_1}{R+R_1}+\frac{R-R_2}{R+R_2}=\frac{r-r_1}{r+r_1}+\frac{r-r_2}{r+r_2}$$

Proposed by Marin Chirciu-Romania

Solution by Togrul Ehmedov-Azerbaijan



$$|AB| = b, |AC| = c, |BC| = a, |BD| = \frac{b^2}{a}, |CD| = \frac{c^2}{a}, |AD| = \frac{bc}{a}$$

We know that $R = \frac{a}{2}, R_1 = \frac{b}{2}, R_2 = \frac{c}{2}$ and

$$r = \frac{b+c-a}{2}, r_1 = \frac{b}{a}r, r_2 = \frac{c}{a}r$$

$$\frac{R-R_1}{R+R_1}+\frac{R-R_2}{R+R_2}=\frac{r-r_1}{r+r_1}+\frac{r-r_2}{r+r_2} \Rightarrow \frac{\frac{a}{2}-\frac{b}{2}}{\frac{a}{2}+\frac{b}{2}}+\frac{\frac{a}{2}-\frac{c}{2}}{\frac{a}{2}+\frac{c}{2}}=\frac{r-\frac{b}{a}r}{r+\frac{b}{a}r}+\frac{r-\frac{c}{a}r}{r+\frac{c}{a}r} \Rightarrow$$

$$\Rightarrow \frac{a-b}{a+b}+\frac{a-c}{a+c}=\frac{a-b}{a+b}+\frac{a-c}{a+c}$$

1649. In $\Delta ABC, m(BAC) = 90°, AD \perp BC. R_1, R_2 \to$ circumradii and $r_1, r_2 \to$ inradii of $\Delta ABD, \Delta ACD$. Prove that :

$$\frac{r_a+r_b+r_c}{r+r_1+r_2} \geq \left(\sqrt{2}+1\right)\frac{r_a+r_b+r_c}{R+R_1+R_2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$2R\sin A = a, 2R_1 \sin\widehat{ADB} = c, 2R_2 \sin\widehat{ADC} = b$$

$$\therefore R=\frac{a}{2}, R_1=\frac{c}{2}, R_2=\frac{b}{2}\to(1)$$

$$\text{Now, } r_1=4R_1\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=c.\frac{1}{\sqrt{2}}.\left(\cos\frac{B-C}{2}-\sin\frac{A}{2}\right)$$

$$=c.\frac{1}{\sqrt{2}}.\left(\frac{b+c}{a}.\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right)=\frac{c}{2}.\frac{2(s-a)}{a}\overset{\text{via }(1)}{=}\frac{c}{a}.(2R+r-2R)$$

$$\Rightarrow r_1=\frac{c}{a}.r \text{ and analogously}, r_2=\frac{b}{a}.r\to(2)$$

$$\text{Again}, \frac{R}{r}=\frac{R}{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}=\frac{1}{\frac{2}{\sqrt{2}}.\left(\cos\frac{B-C}{2}-\sin\frac{A}{2}\right)}\overset{0<\cos\frac{B-C}{2}\le1}{\ge}\frac{1}{\sqrt{2}\left(1-\frac{1}{\sqrt{2}}\right)}$$

$$=\frac{1}{\sqrt{2}-1}\Rightarrow\frac{R}{r}\ge\sqrt{2}+1\to(\text{i})$$

$$\text{Also}, r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{2R}{\sqrt{2}}.\left(\cos\frac{B-C}{2}-\sin\frac{A}{2}\right)\overset{\text{via }(1)}{=}\frac{a}{\sqrt{2}}.\left(\frac{b+c}{a}.\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right)$$

$$=\frac{a}{2}.\frac{2(s-a)}{a}\overset{\text{via }(1)}{=}2R+r-2R\Rightarrow r=s\to(3)$$

$$\therefore\frac{\left(\frac{r_a+r_b+r_c}{r+r_1+r_2}\right)}{\left(\frac{r_a+r_b+r_c}{R+R_1+R_2}\right)}\overset{\text{via }(1),(2)\text{ and }(3)}{=}\frac{\frac{a}{2}+\frac{c}{2}+\frac{b}{2}}{s+\frac{c}{a}.s+\frac{b}{a}.s}=\frac{s}{s\left(1+\frac{b}{a}+\frac{c}{a}\right)}=\frac{a}{a+b+c}$$

$$\overset{\text{via }(1)}{=}\frac{2R}{2s}\overset{\text{via }(3)}{=}\frac{R}{r}\overset{\text{via }(1)}{\ge}\sqrt{2}+1, ''='' \text{ iff } \Delta\, ABC \text{ is isosceles right}-\text{angled (QED)}$$

1650. In any $\Delta\, ABC$, the following relationship holds :

$$\sum_{cyc}\frac{r_a}{r_a+\lambda r_b}\le\frac{3}{\lambda+1}.\frac{4(\lambda+1)R+(1-8\lambda)r}{4R+r}\ \text{for}\ \frac{1}{2}\le\lambda\le\frac{7}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{r_a}{r_a+\lambda r_b}\le\frac{3}{\lambda+1}.\frac{4(\lambda+1)R+(1-8\lambda)r}{4R+r}$$

$$\Leftrightarrow\sum_{cyc}\frac{r_a+\lambda r_b-\lambda r_b}{r_a+\lambda r_b}\le\frac{3}{\lambda+1}.\frac{4R+r+\lambda(4R-8r)}{4R+r}$$

$$\Leftrightarrow 3-\frac{3}{\lambda+1}-\lambda\sum_{cyc}\frac{r_b^2}{r_ar_b+\lambda r_b^2}\le\frac{3\lambda(4R-8r)}{(\lambda+1)(4R+r)}$$

$$\Leftrightarrow\frac{3\lambda}{\lambda+1}-\lambda\sum_{cyc}\frac{r_b^2}{r_ar_b+\lambda r_b^2}-\frac{3\lambda(4R-8r)}{(\lambda+1)(4R+r)}\le 0$$

$$\Leftrightarrow\sum_{cyc}\frac{r_b^2}{r_ar_b+\lambda r_b^2}-\frac{3}{\lambda+1}+\frac{3(4R-8r)}{(\lambda+1)(4R+r)}\overset{(*)}{\ge}0\ (\because\lambda>0)$$

$$\text{Now, }\sum_{cyc}\frac{r_b^2}{r_ar_b+\lambda r_b^2}-\frac{3}{\lambda+1}\overset{\text{Bergstrom}}{\geq}\frac{(4R+r)^2}{s^2+\lambda((4R+r)^2-2s^2)}-\frac{3}{\lambda+1}$$

$$=\frac{\lambda(4R+r)^2+(4R+r)^2-3s^2-3\lambda\left((4R+r)^2-2s^2\right)}{(\lambda+1)\left(s^2+\lambda((4R+r)^2-2s^2)\right)}$$

$$=\frac{(4R+r)^2-3s^2-2\lambda\left((4R+r)^2-3s^2\right)}{(\lambda+1)\left(s^2+\lambda((4R+r)^2-2s^2)\right)}$$

$$=\frac{(1-2\lambda)\left((4R+r)^2-3s^2\right)}{(\lambda+1)\left(s^2+\lambda((4R+r)^2-2s^2)\right)}\ \therefore\ \text{in order to prove } (*),\text{ it suffices to prove}:$$

$$\frac{(2\lambda-1)\left((4R+r)^2-3s^2\right)}{(\lambda+1)\left(s^2+\lambda((4R+r)^2-2s^2)\right)}\leq\frac{3(4R-8r)}{(\lambda+1)(4R+r)}$$

$$\Leftrightarrow 3(4R-8r)\left(\lambda(4R+r)^2-(2\lambda-1)s^2\right)\geq(2\lambda-1)(4R+r)\left((4R+r)^2-3s^2\right)$$

$$\Leftrightarrow 3\lambda(4R-8r)(4R+r)^2\geq(2\lambda-1)\left((4R+r)^3-3s^2(4R+r)+3(4R-8r)s^2\right)$$

$$\Leftrightarrow 3\lambda(4R-8r)(4R+r)^2\geq(2\lambda-1)\left((4R+r)^3-27rs^2\right)$$

$$\Leftrightarrow (4R+r)^3-27rs^2+\lambda\left(3(4R-8r)(4R+r)^2-2(4R+r)^3+54rs^2\right)\boxed{\overset{(**)}{\geq}}\ 0$$

$\boxed{\textit{Case 1}}$ $3(4R-8r)(4R+r)^2-2(4R+r)^3+54rs^2\geq 0$ *and then* : LHS of $(**)\geq$

$$(4R+r)^3-27rs^2\overset{\text{Trucht}}{\geq}3s^2(4R+r-9r)=3s^2(4R-8r)\overset{\text{Euler}}{\geq}0\Rightarrow(**)\text{ is true}$$

$\boxed{\textit{Case 2}}$ $3(4R-8r)(4R+r)^2-2(4R+r)^3+54rs^2<0$ *and then* : LHS of $(**)=$

$$(4R+r)^3-27rs^2-\lambda\left(-\left(3(4R-8r)(4R+r)^2-2(4R+r)^3+54rs^2\right)\right)\overset{0<\lambda\leq\frac{7}{2}}{\geq}$$

$$(4R+r)^3-27rs^2-\frac{7}{2}\left(-\left(3(4R-8r)(4R+r)^2-2(4R+r)^3+54rs^2\right)\right)\overset{?}{\geq}0$$

$$\Leftrightarrow 2(4R+r)^3-54rs^2+21(4R-8r)(4R+r)^2-14(4R+r)^3+7.54rs^2\overset{?}{\geq}0$$

$$\Leftrightarrow (4R+r)^2(84R-168r-48R-12r)+6.54rs^2\overset{?}{\geq}0$$

$$\Leftrightarrow (R-5r)(4R+r)^2+9rs^2\boxed{\underset{(***)}{\overset{?}{\geq}}}\,0$$

$$\text{Now, LHS of }(***)\overset{\text{Gerretsen}}{\geq}(R-5r)(4R+r)^2+9r\left(16Rr-5r^2\right)\overset{?}{\geq}0\Leftrightarrow$$

$$16t^3-72t^2+105t-50\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)(4t-5)^2\overset{?}{\geq}0\to\text{true}\because t\overset{\text{Euler}}{\geq}2$$

$\Rightarrow(***)\Rightarrow(**)$ is true $\therefore$ combining both *cases*, $(**)\Rightarrow(*)$ is true $\forall\,\Delta\,ABC$ with $\frac{1}{2}\leq\lambda\leq\frac{7}{2}\therefore\sum_{cyc}\frac{r_a}{r_a+\lambda r_b}\leq\frac{3}{\lambda+1}\cdot\frac{4(\lambda+1)R+(1-8\lambda)r}{4R+r}\ \forall\,\Delta\,ABC$ with $\frac{1}{2}\leq\lambda\leq\frac{7}{2}$,

$$''=''\text{ iff }\Delta\,ABC\text{ is equilateral and }\lambda=\frac{7}{2}\ \text{(QED)}$$

1651. In *any* $\Delta\,ABC$ *and* $\forall\,n,m\geq 0$, the fo*ll*owing re*la*tionship ho*l*ds :

$$\frac{r_a^n}{(m_a+w_b)^m}+\frac{r_b^n}{(w_b+h_c)^m}+\frac{r_c^n}{(h_c+m_a)^m}\geq\frac{3^{n-m+1}.r^n}{R^m}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{r_a^n}{(m_a+w_b)^m}+\frac{r_b^n}{(w_b+h_c)^m}+\frac{r_c^n}{(h_c+m_a)^m}\geq$$

$$\frac{r_a^n}{(m_a+m_b)^m}+\frac{r_b^n}{(m_b+m_c)^m}+\frac{r_c^n}{(m_c+m_a)^m}\overset{A-G}{\geq}3.\sqrt[3]{\frac{\left(\prod_{cyc}r_a\right)^n}{\left(\prod_{cyc}(m_a+m_b)\right)^m}}$$

$$=3.\frac{\left(\sqrt[3]{rs^2}\right)^n}{\left(\sqrt[3]{(m_a+m_b)}\right)^m}\overset{\substack{\text{Mitrinovic}\\ \text{and}\\ A-G}}{\geq}3.\frac{\left(\sqrt[3]{r.27r^2}\right)^n}{\left(\frac{\sum_{cyc}(m_a+m_b)}{3}\right)^m}=3.\frac{(3r)^n}{\left(\frac{2}{3}\left(\sum_{cyc}m_a\right)\right)^m}$$

$$\overset{\text{Leuenberger + Euler}}{\geq}3.\frac{(3r)^n}{\left(\frac{2}{3}\left(\frac{9R}{2}\right)\right)^m}=\frac{3^{n-m+1}.r^n}{R^m}\ \forall\,\Delta\,ABC,$$

$'' ='' $ iff Δ ABC is equilateral (QED)

1652.In any Δ ABC, the following relationship holds :

$$2\sum_{cyc}\frac{m_a}{h_a}\geq\frac{2s}{\sqrt[3]{abc}}+\frac{1}{\sqrt[3]{h_ah_bh_c}}\sum_{cyc}h_a$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$2\sum_{cyc}\frac{m_a}{h_a}-\frac{1}{\sqrt[3]{h_ah_bh_c}}\sum_{cyc}h_a\overset{\text{Tereshin}}{\geq}2\sum_{cyc}\frac{b^2+c^2}{4R.\frac{bc}{2R}}-\frac{1}{\sqrt[3]{\frac{2r^2s^2}{R}}}.\frac{s^2+4Rr+r^2}{2R}$$

$$\overset{\text{Gerretsen}}{\geq}\frac{\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-3abc}{abc}-\frac{1}{\sqrt[3]{\frac{r^2(27Rr+5r(R-2r))}{R}}}.\frac{s^2+4Rr+r^2}{2R}$$

$$\overset{\text{Euler}}{\geq}\frac{2s(s^2+4Rr+r^2)-12Rrs}{4Rrs}-\frac{1}{\sqrt[3]{\frac{r^2(27Rr)}{R}}}.\frac{s^2+4Rr+r^2}{2R}$$

$$=\frac{s^2-2Rr+r^2}{2Rr}-\frac{s^2+4Rr+r^2}{6Rr}=\frac{s^2-5Rr+r^2}{3Rr}\overset{?}{\geq}\frac{2s}{\sqrt[3]{abc}}=\frac{2s}{\sqrt[3]{4Rrs}}$$

$$\Leftrightarrow\frac{(s^2-5Rr+r^2)^3}{27R^3r^3}\overset{?}{\geq}\frac{8s^3}{4Rrs}\Leftrightarrow(s^2-5Rr+r^2)^3-54R^2r^2s^2\ \boxed{\overset{?}{\underset{(*)}{\geq}}}\ 0$$

and $\because\ (s^2-16Rr+5r^2)^3\overset{\text{Gerretsen}}{\geq}0\ \therefore$ in order to prove $(*)$, it suffices to prove :

LHS of $(*)\geq(s^2-16Rr+5r^2)^3\Leftrightarrow(33R-12r)s^4-rs^2(747R^2-450Rr+72r^2)$

$$+r^2(3971R^3-3765R^2r+1185Rr^2-124r^3)\boxed{\overset{(**)}{\geq}}0 \text{ and}$$

$$\because (33R-12r)(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0 \therefore \text{in order to prove } (**), \text{it suffices}$$

$$\text{to prove : LHS of } (**)\geq (33R-12r)(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow (309R^2-264Rr+48r^2)s^2\boxed{\overset{(***)}{\geq}}r(4477R^3-4587R^2r+1560Rr^2-176r^3)$$

$$\text{Now, LHS of } (***)\overset{\text{Gerretsen}}{\geq}(309R^2-264Rr+48r^2)(16Rr-5r^2)\overset{?}{\geq}$$

$$r(4477R^3-4587R^2r+1560Rr^2-176r^3)\Leftrightarrow 467t^3-1182t^2+528t-64\overset{?}{\geq}0$$

$$\left(t=\frac{R}{r}\right)\Leftrightarrow (t-2)(343t^2+124t(t-2)+32)\overset{?}{\geq}0\to \text{true} \because t\overset{\text{Euler}}{\geq}2$$

$$\Rightarrow (***)\Rightarrow (**)\Rightarrow (*) \text{ is true} \therefore 2\sum_{cyc}\frac{m_a}{h_a}-\frac{1}{\sqrt[3]{h_ah_bh_c}}\sum_{cyc}h_a\geq \frac{2s}{\sqrt[3]{abc}}\Rightarrow$$

$$2\sum_{cyc}\frac{m_a}{h_a}\geq \frac{2s}{\sqrt[3]{abc}}+\frac{1}{\sqrt[3]{h_ah_bh_c}}\sum_{cyc}h_a\ \forall\ \Delta ABC,''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1653. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{(r_a+r_b)(r_a+r_c)}{h_b+h_c}\geq 9R$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$r_b+r_c=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=\frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)}=4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c\overset{(i)}{=}4R\cos^2\frac{A}{2} \text{ and analogs}$$

$$\therefore \text{via (i)}, \sum_{cyc}\frac{(r_a+r_b)(r_a+r_c)}{h_b+h_c}=\sum_{cyc}\frac{4R\cos^2\frac{C}{2}.4R\cos^2\frac{B}{2}}{\frac{ca+ab}{2R}}$$

$$=\sum_{cyc}\frac{4R\cos^2\frac{C}{2}.4R\cos^2\frac{B}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}.4R\cos\frac{A}{2}\cos\frac{B-C}{2}}\overset{0<\cos\frac{B-C}{2}\leq 1}{\geq}2R\sum_{cyc}\frac{\cos^2\frac{B}{2}\cos^2\frac{C}{2}}{\cos^2\frac{A}{2}\sin\frac{A}{2}}\overset{A-G}{\geq}6R.\sqrt[3]{\frac{\prod_{cyc}\cos^2\frac{A}{2}}{\prod_{cyc}\sin\frac{A}{2}}}$$

$$=6R.\sqrt[3]{\frac{\frac{s^2}{16R^2}}{\frac{r}{4R}}}=6R.\sqrt[3]{\frac{2s^2}{8Rr}}\overset{\text{Gerretsen}}{\geq}6R.\sqrt[3]{\frac{27Rr+5r(R-2r)}{8Rr}}\overset{\text{Euler}}{\geq}6R.\sqrt[3]{\frac{27Rr}{8Rr}}=9R$$

$$\therefore \sum_{cyc} \frac{(r_a+r_b)(r_a+r_c)}{h_b+h_c} \geq 9R\ \forall\ \Delta\ ABC, ''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

1654. In any Δ ABC with I → incenter and R_a, R_b, R_c → circumradii of Δ BIC, Δ CIA and Δ AIB, the following relationship holds :

$$\frac{2r}{R} \leq \sum_{cyc} \frac{R_a}{r_b+r_c} \leq \left(\frac{R}{2r}\right)^2$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$r_b+r_c = s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right) = \frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)} = 4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c \stackrel{(i)}{=} 4R\cos^2\frac{A}{2} \text{ and analogs}$$

$$\sphericalangle BIC = \pi - \frac{B+C}{2} = \pi - \frac{\pi-A}{2} = \frac{\pi+A}{2} \therefore R_a = \frac{a}{2\sin\frac{\pi+A}{2}} = \frac{4R\sin\frac{A}{2}\cos\frac{A}{2}}{2\cos\frac{A}{2}} \Rightarrow$$

$$R_a = 2R\sin\frac{A}{2} \text{ and analogs} \therefore \sum_{cyc}\frac{R_a}{r_b+r_c} \stackrel{\text{via (i)}}{=} \sum_{cyc}\frac{2R\sin\frac{A}{2}}{4R\cos^2\frac{A}{2}} = \frac{1}{2}\sum_{cyc}\left(\tan\frac{A}{2}\sec\frac{A}{2}\right)$$

$$\stackrel{CBS}{\leq} \frac{1}{2}\cdot\sqrt{\sum_{cyc}\tan^2\frac{A}{2}}\cdot\sqrt{\sum_{cyc}\sec^2\frac{A}{2}} = \frac{1}{2}\cdot\sqrt{\frac{1}{s^2}\sum_{cyc}r_a^2}\cdot\sqrt{3+\frac{1}{s^2}\sum_{cyc}r_a^2}$$

$$= \frac{1}{2}\cdot\sqrt{\frac{(4R+r)^2-2s^2}{s^2}}\cdot\sqrt{3+\frac{(4R+r)^2-2s^2}{s^2}}$$

$$= \frac{1}{2}\left(\sqrt{\frac{2(4R+r)^2}{2s^2}-2}\right)\left(\sqrt{\frac{2(4R+r)^2}{2s^2}+1}\right)$$

$$\stackrel{Gerretsen}{\leq} \frac{1}{2}\left(\sqrt{\frac{2(4R+r)^2}{27Rr+5r(R-2r)}-2}\right)\left(\sqrt{\frac{2(4R+r)^2}{27Rr+5r(R-2r)}+1}\right)$$

$$\stackrel{Euler}{\leq} \frac{1}{2}\left(\sqrt{\frac{2(4R+r)^2}{27Rr}-2}\right)\left(\sqrt{\frac{2(4R+r)^2}{27Rr}+1}\right)$$

$$= \frac{\sqrt{(2(4R+r)^2-54Rr)(2(4R+r)^2+27Rr)}}{2.27Rr} \stackrel{?}{\leq} \left(\frac{R}{2r}\right)^2$$

$$\Leftrightarrow \frac{(2(4R+r)^2-54Rr)(2(4R+r)^2+27Rr)}{4.729R^2r^2} \stackrel{?}{\leq} \frac{R^4}{16r^4}$$

$$\Leftrightarrow 729R^6 \overset{?}{\geq} 4r^2\left(2(4R+r)^2 - 54Rr\right)\left(2(4R+r)^2 + 27Rr\right)$$
$$\Leftrightarrow 729t^6 - 4096t^4 - 640t^3 + 6024t^2 - 40t - 16 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)\left((t-2)\left(729t^4 + 2916t^3 + 4652t^2 + 6304t + 12632\right) + 25272\right) \overset{?}{\geq} 0$$
$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \therefore \sum_{cyc} \frac{R_a}{r_b + r_c} \leq \left(\frac{R}{2r}\right)^2 \text{ and again, } \sum_{cyc} \frac{R_a}{r_b + r_c} = \frac{1}{2}\sum_{cyc} \frac{\sin\frac{A}{2}}{\cos^2\frac{A}{2}} \overset{\text{Jensen}}{\geq}$$
$$\frac{3}{2}\cdot\frac{\sin\frac{\pi}{6}}{\cos^2\frac{\pi}{6}} \left(\because f(x) = \frac{\sin\frac{x}{2}}{\cos^2\frac{x}{2}}\ \forall x \in (0,\pi) \Rightarrow f''(x) = \frac{6\sin^3\frac{x}{2} + 5\sin\frac{x}{2}\cos^2\frac{x}{2}}{4\cos^4\frac{x}{2}} > 0 \atop \Rightarrow f''(x) \text{ is convex}\right)$$
$$= \frac{3}{2}\cdot\frac{\left(\frac{1}{2}\right)}{\left(\frac{3}{4}\right)} = 1 \overset{\text{Euler}}{\geq} \frac{2r}{R} \therefore \sum_{cyc} \frac{R_a}{r_b + r_c} \geq \frac{2r}{R} \text{ and so, } \frac{2r}{R} \leq \sum_{cyc} \frac{R_a}{r_b + r_c} \leq \left(\frac{R}{2r}\right)^2$$
$$\forall \Delta ABC, '' = '' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1655. In any Δ ABC, the following relationship holds :

$$\frac{r_a^2}{r_a + r_b} + \frac{r_b^2}{r_b + r_c} \geq \frac{1}{8r}(a^2 - 3c^2 - ab + 2ac + 3bc)$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{r_a^2}{r_a + r_b} + \frac{r_b^2}{r_b + r_c} = \frac{\frac{F^2}{(s-a)^2}}{\frac{F}{s-a} + \frac{F}{s-b}} + \frac{\frac{F^2}{(s-b)^2}}{\frac{F}{s-b} + \frac{F}{s-c}} = \frac{F(s-b)}{c(s-a)} + \frac{F(s-c)}{a(s-b)}$$
$$\Rightarrow 8r\left(\frac{r_a^2}{r_a + r_b} + \frac{r_b^2}{r_b + r_c}\right) = 8\left(\frac{F^2(s-b)}{sc(s-a)} + \frac{F^2(s-c)}{sa(s-b)}\right)$$
$$= 8\left(\frac{s(s-a)(s-b)(s-c)(s-b)}{sc(s-a)} + \frac{s(s-a)(s-b)(s-c)(s-c)}{sa(s-b)}\right)$$
$$= 8\left(\frac{(s-b)^2(s-c)}{c} + \frac{(s-c)^2(s-a)}{a}\right) = \frac{8y^2z}{x+y} + \frac{8z^2x}{y+z}$$
$$(x = s-a, y = s-b, z = s-c \Rightarrow a = y+z, b = z+x, c = x+y)$$
$$= \frac{8y^2z(y+z) + 8z^2x(x+y)}{(x+y)(y+z)} \geq a^2 - 3c^2 - ab + 2ac + 3bc$$
$$\Leftrightarrow \frac{8y^2z(y+z) + 8z^2x(x+y)}{(x+y)(y+z)} \geq$$
$$(y+z)^2 - 3(x+y)^2 - (y+z)(z+x) + 2(y+z)(x+y) + 3(z+x)(x+y)$$
$$\Leftrightarrow x^2y^2 + 2z^2x^2 + xy^3 + y^3z + y^2z^2 \overset{(*)}{\geq} x^2yz + 4xy^2z + xyz^2$$
$$\text{Now, } x^2y^2 + y^2z^2 + z^2x^2 \geq x^2yz + xy^2z + xyz^2 \rightarrow (1) \text{ and}$$
$$z^2x^2 + xy^3 + y^3z \overset{A-G}{\geq} 3xy^2z \rightarrow (2) \text{ and } (1) + (2) \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{r_a^2}{r_a+r_b}+\frac{r_b^2}{r_b+r_c}\ge\frac{1}{8r}\left(a^2-3c^2-ab+2ac+3bc\right)$$
$\forall\,\Delta\,ABC,''=''$ iff $\Delta\,ABC$ is equilateral (QED)

1656. In any $\Delta\,ABC$, the following relationship holds :

$$3\cdot\sqrt[3]{\frac{r^2}{2R^2}}\le\sum_{cyc}\frac{h_a}{\sqrt{(r_a+r_b)(r_a+r_c)}}\le\frac{4R+r}{6r}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$r_b+r_c=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=\frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)}=4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c\overset{(i)}{=}4R\cos^2\frac{A}{2}\ \textit{and analogs}$$

$$\text{Now, }\sum_{cyc}\frac{h_a}{\sqrt{(r_a+r_b)(r_a+r_c)}}\overset{via\ (i)}{=}\sum_{cyc}\frac{\frac{2rs}{a}.\cos\frac{A}{2}}{4R\cos\frac{C}{2}\cos\frac{B}{2}\cos\frac{A}{2}}=\sum_{cyc}\frac{\frac{2rs\cos\frac{A}{2}}{4R\cos\frac{A}{2}\sin\frac{A}{2}}}{4R.\frac{s}{4R}}$$

$$=\frac{r}{2R}\sum_{cyc}\frac{1}{\sin\frac{A}{2}}=\frac{r}{2R}\sum_{cyc}\sqrt{\frac{bc(s-a)}{r^2s}}\overset{CBS}{\le}\frac{1}{2R.\sqrt{s}}\cdot\sqrt{\sum_{cyc}(s-a)}.\sqrt{s^2+4Rr+r^2}$$

$$\overset{Gerretsen}{\le}\frac{1}{2R}.\sqrt{4R^2+8Rr+4r^2}=\frac{R+r}{R}\overset{?}{\le}\frac{4R+r}{6r}\Leftrightarrow 4R^2-5Rr-6r^2\overset{?}{\ge}0$$

$$\Leftrightarrow(4R+3r)(R-2r)\overset{?}{\ge}0\to\text{true}\because R\overset{Euler}{\ge}2r\ \therefore\sum_{cyc}\frac{h_a}{\sqrt{(r_a+r_b)(r_a+r_c)}}\le\frac{4R+r}{6r}$$

$$\text{Again, }\sum_{cyc}\frac{h_a}{\sqrt{(r_a+r_b)(r_a+r_c)}}=\frac{r}{2R}\sum_{cyc}\frac{1}{\sin\frac{A}{2}}\overset{A-G}{\ge}\frac{3r}{2R}\cdot\sqrt[3]{\frac{4R}{r}}=\frac{3}{2R}\cdot\sqrt[3]{\frac{r^3.4R.2R^2}{r.2R^2}}$$

$$=\frac{3.2R}{2R}\cdot\sqrt[3]{\frac{r^2}{2R^2}}\ \therefore\sum_{cyc}\frac{h_a}{\sqrt{(r_a+r_b)(r_a+r_c)}}\ge3.\sqrt[3]{\frac{r^2}{2R^2}}\ \textit{and so,}$$

$$3.\sqrt[3]{\frac{r^2}{2R^2}}\le\sum_{cyc}\frac{h_a}{\sqrt{(r_a+r_b)(r_a+r_c)}}\le\frac{4R+r}{6r}$$
$\forall\,\Delta\,ABC,''=''$ iff $\Delta\,ABC$ is equilateral (QED)

1657. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{r_b r_c}{rp^2+r_a^3}\ge\frac{1}{2r}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{r_b r_c}{rp^2+r_a^3}=\sum_{cyc}\frac{r_b r_c+r_a^2-r_a^2}{r_a r_b r_c+r_a^3}=\sum_{cyc}\frac{r_b r_c+r_a^2-r_a^2}{r_a(r_b r_c+r_a^2)}$$

$$=\sum_{cyc}\frac{1}{r_a}-\sum_{cyc}\frac{r_a}{r_b r_c+r_a^2}\overset{A-G}{\ge}\frac{1}{r}-\sum_{cyc}\frac{r_a}{2r_a\cdot\sqrt{r_b r_c}}\ge\frac{1}{r}-\frac{1}{2}\sum_{cyc}\frac{1}{w_a}\ge\frac{1}{r}-\frac{1}{2}\sum_{cyc}\frac{1}{h_a}$$

$$=\frac{1}{r}-\frac{1}{2r}=\frac{1}{2r}\therefore\sum_{cyc}\frac{r_b r_c}{rp^2+r_a^3}\ge\frac{1}{2r}\ \forall\,\Delta\,ABC,''=''\text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$

1658. In any Δ ABC, the following relationship holds :

$$\frac{\cos^{2024}A}{\cos^{2022}B+\cos^{2022}C}+\frac{\cos^{2024}B}{\cos^{2022}C+\cos^{2022}A}+\frac{\cos^{2024}C}{\cos^{2022}A+\cos^{2022}B}\ge\frac{3}{8}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{\cos^{2024}A}{\cos^{2022}B+\cos^{2022}C}+\frac{\cos^{2024}B}{\cos^{2022}C+\cos^{2022}A}+\frac{\cos^{2024}C}{\cos^{2022}A+\cos^{2022}B}$$

$$=\sum_{cyc}\frac{\cos^2A\left(\sum_{cyc}\cos^{2022}A-\left(\cos^{2022}B+\cos^{2022}C\right)\right)}{\cos^{2022}B+\cos^{2022}C}$$

$$=\left(\sum_{cyc}\cos^{2022}A\right)\left(\sum_{cyc}\frac{\cos^2A}{\cos^{2022}B+\cos^{2022}C}\right)-\sum_{cyc}\cos^2A\overset{\text{Chebyshev}}{\ge}$$

$$\frac{1}{3}\left(\sum_{cyc}\cos^{2022}A\right)\left(\sum_{cyc}\cos^2A\right)\left(\sum_{cyc}\frac{1}{\cos^{2022}B+\cos^{2022}C}\right)-\sum_{cyc}\cos^2A$$

$$\left(\begin{array}{c}\because\text{ WLOG assuming }a\ge b\ge c\Rightarrow\cos^2A\le\cos^2B\le\cos^2C\text{ and}\\ \frac{1}{\cos^{2022}B+\cos^{2022}C}\le\frac{1}{\cos^{2022}C+\cos^{2022}A}\le\frac{1}{\cos^{2022}A+\cos^{2022}B}\end{array}\right)$$

$$\overset{\text{Bergstrom}}{\ge}\frac{1}{3}\left(\sum_{cyc}\cos^{2022}A\right)\left(\sum_{cyc}\cos^2A\right)\left(\frac{9}{2\sum_{cyc}\cos^{2022}A}\right)-\sum_{cyc}\cos^2A$$

$$\left(\begin{array}{c}\text{we note that}: \cos^{2022} B + \cos^{2022} C \ge \frac{1}{2^{2021}}(\cos B + \cos C)^{2022} \\ = \frac{1}{2^{2021}}\left(2\sin\frac{A}{2}\cos\frac{B-C}{2}\right)^{2022} \overset{0<\cos\frac{B-C}{2}\le 1}{>} 0\end{array}\right)$$

$$= \frac{1}{2}\sum_{cyc}\cos^2 A = \frac{1}{2}\left(3 - \frac{\sum_{cyc} a^2}{4R^2}\right) \overset{\text{Leibnitz}}{\ge} \frac{1}{2}\left(3 - \frac{9R^2}{4R^2}\right)$$

$$\Rightarrow \frac{\cos^{2024} A}{\cos^{2022} B + \cos^{2022} C} + \frac{\cos^{2024} B}{\cos^{2022} C + \cos^{2022} A} + \frac{\cos^{2024} C}{\cos^{2022} A + \cos^{2022} B} \ge \frac{3}{8}$$

$\forall \Delta ABC, '' = ''$ iff ΔABC is equilateral (QED)

1659. In any ΔABC, the following relationship holds :

$$\left(\frac{m_b h_b}{\sin A}\right)^2 + \left(\frac{m_c h_c}{\sin B}\right)^2 + \left(\frac{m_a h_a}{\sin C}\right)^2 \ge \frac{16F^3}{\sqrt{3}}\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)$$

Proposed by Yusif Abbaszade-Azerbaijan

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\left(\frac{m_b h_b}{\sin A}\right)^2 + \left(\frac{m_c h_c}{\sin B}\right)^2 + \left(\frac{m_a h_a}{\sin C}\right)^2 = \sum_{cyc}\left(\frac{m_a h_a}{\sin C}\right)^2$$

$$= \sum_{cyc}\frac{(2b^2 + 2c^2 - a^2)b^2c^2}{16R^2 . \frac{c^2}{4R^2}} = \sum_{cyc}\frac{2b^4 + 2b^2c^2 - a^2b^2}{4} = \frac{1}{4}\left(2\sum_{cyc} a^4 + \sum_{cyc} a^2b^2\right)$$

$$\Rightarrow LHS = \frac{1}{4}\left(5\sum_{cyc} a^2b^2 - 32r^2s^2\right) \to (1)$$

$$\text{Again}, \frac{16F^3}{\sqrt{3}}\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) = \frac{16r^3s^3(\sum_{cyc} a^2b^2)}{\sqrt{3}.16R^2r^2s^2} = \frac{rs(\sum_{cyc} a^2b^2)}{\sqrt{3}.R^2} \overset{\text{Mitrinovic}}{\le}$$

$$\frac{r.3\sqrt{3}.R(\sum_{cyc} a^2b^2)}{2.\sqrt{3}.R^2} = \frac{3r\sum_{cyc} a^2b^2}{2R} \overset{?}{\le} \frac{1}{4}\left(5\sum_{cyc} a^2b^2 - 32r^2s^2\right)$$

$$\Leftrightarrow R\left(5\sum_{cyc} a^2b^2 - 32r^2s^2\right) \overset{?}{\ge} 6r\sum_{cyc} a^2b^2 \Leftrightarrow (5R - 6r)\sum_{cyc} a^2b^2 \underset{(*)}{\overset{?}{\ge}} 32Rr^2s^2$$

$$\text{Now}, (5R - 6r)\sum_{cyc} a^2b^2 \ge (5R - 6r)abc\sum_{cyc} a = 8Rrs^2(5R - 6r) \overset{?}{\ge} 32Rr^2s^2$$

$$\Leftrightarrow 5R - 6r \overset{?}{\ge} 4r \Leftrightarrow 5(R - 2r) \overset{?}{\ge} 0 \to \text{true} \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{16F^3}{\sqrt{3}}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) \le \frac{1}{4}\left(5\sum_{cyc} a^2b^2 - 32r^2s^2\right) \overset{via\,(1)}{=}$$

$$\left(\frac{m_bh_b}{\sin A}\right)^2 + \left(\frac{m_ch_c}{\sin B}\right)^2 + \left(\frac{m_ah_a}{\sin C}\right)^2, \text{"} = \text{" iff } \Delta\, ABC \text{ is equilateral (QED)}$$

Solution 2 by proposer

$$\sum_{cyc}\left(\frac{m_bh_b}{\sin < A}\right)^2 = \sum_{cyc}\left(\frac{4FRm_b}{ab}\right)^2 \ge \sum_{cyc}\left(\frac{4FR}{ab}\times\frac{a^2+c^2}{4R}\right)^2 = F^2\sum_{cyc}\left(\frac{a^2+c^2}{ab}\right)^2$$

$$F^2\sum_{cyc}\left(\frac{a^2+c^2}{ab}\right)^2 \ge^{AM-GM} F^2\sum_{cyc}\left(\frac{2ac}{ab}\right)^2 = 4F^2\sum_{cyc}\frac{c^2}{b^2}$$

$$4F^2\sum_{cyc}\frac{c^2}{b^2} = 4F^2\sum_{cyc}\left(\frac{\frac{c^2}{b^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}}{3}\right) \ge^{AM-GM} 4F^2\sum_{cyc}\sqrt[3]{\frac{a^2c^2}{b^4}} = 4F^2\sum_{cyc}\sqrt[3]{\frac{(abc)^2}{b^6}}$$

$$4F^2\sqrt[3]{(abc)^2}\sum_{cyc}\frac{1}{a^2} \ge 4F^2\times\frac{4F}{\sqrt{3}}\sum_{cyc}\frac{1}{a^2} = \frac{16F^3}{\sqrt{3}}\sum_{cyc}\frac{1}{a^2}$$

$$\left(\frac{m_bh_b}{\sin < A}\right)^2 + \left(\frac{m_ch_c}{\sin < B}\right)^2 + \left(\frac{m_ah_a}{\sin < C}\right)^2 \ge \frac{16F^3}{\sqrt{3}}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$$

Note Section :

$$\sin < A = \frac{a}{2R} \qquad h_b = \frac{2F}{b}$$

$$m_a \ge \frac{b^2+c^2}{4R} \qquad \sqrt[3]{(abc)^2} \ge \frac{4F}{\sqrt{3}}$$

1660. If $x, y, z > 0$ then in ΔABC holds:

$$\sum \frac{y+z}{x}\, s_a^2 \ge \left(12\sqrt{3}\right) r\frac{F}{R}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$LHS \overset{AM-GM}{\ge} \sum_{cyc}\frac{2\sqrt{yz}}{x}s_a^2 \overset{AG-GM}{\ge} 6\sqrt[3]{(s_a\, s_b\, s_c)^2} \overset{s_a\ge h_a}{\ge} 6\sqrt[3]{(h_a\, h_b\, h_c)^2} =$$

$$= 6\sqrt[3]{\left(\frac{4F^4}{R^2}\right)} = 6F\sqrt[3]{\left(\frac{4Rrs}{R^3}\right)} \underset{Mitrinovic}{\overset{Euler}{\ge}} = 12F\sqrt{3}\frac{r}{R}$$

1661. **In ΔABC holds:**

$$\sum \cos^3\frac{B-C}{2} \geq \frac{16r}{\sqrt{3}\,R}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\prod \cos\frac{B-C}{2} = \frac{(a+b)(b+c)(c+a)}{abc}\prod \sin\frac{A}{2} = \frac{s^2+2Rr+r^2}{8R^2}$$

$$\sum \cos^3\frac{B-C}{2} \overset{AM-GM}{\geq} 3\prod \cos\frac{B-C}{2}$$

$$\frac{16r}{\sqrt{3}\,R}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{16r}{\sqrt{3}\,R}\frac{s}{4R} \overset{Mitrinovic}{\leq} \frac{48Rr}{8R^2}$$

We need to show:

$$\frac{3(s^2+2Rr+r^2)}{8R^2} \geq \frac{48Rr}{8R^2}, \qquad s^2 \geq 14Rr - r^2$$

$$s^2 \geq 16Rr - 5r^2 \geq 14Rr - r^2 (Gerretsen) or\ R \geq 2r\ (Euler)$$

1662. **In any Δ ABC. the following relationship holds :**

$$(ab+bc+ca)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 2\sqrt{3}(m_a+m_b+m_c)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$2\sqrt{3}(m_a+m_b+m_c) \overset{\text{Chu and Yang}}{\leq} 2\sqrt{3(4s^2-16Rr+5r^2)} \overset{?}{\leq}$$

$$(ab+bc+ca)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = \frac{(s^2+4Rr+r^2)^2}{4Rrs}$$

$$\Leftrightarrow (s^2+4Rr+r^2)^4 - 192R^2r^2s^2(4s^2-16Rr+5r^2) \underset{(*)}{\overset{?}{\geq}} 0$$

and $\because (s^2-16Rr+5r^2)^4 \overset{\text{Gerretsen}}{\geq} 0 \therefore$ in order to prove $(*)$, it suffices to prove :

$$\text{LHS of } (*) \geq (s^2-16Rr+5r^2)^4 \Leftrightarrow (5R-6r)s^6 - r(138R^2-63Rr+9r^2)s^4$$
$$+r^2(1232R^3-1008R^2r+303Rr^2-31r^3)s^2$$
$$-r^3(4080R^4-5136R^3r+2394R^2r^2-501Rr^3+39r^4) \overset{(**)}{\geq} 0$$

and $\because (5R-6r)(s^2-16Rr+5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore$ in order to prove $(**)$,

$$\text{it suffices to prove : LHS of } (**) \ge (5R-6r)(s^2-16Rr+5r^2)^3$$
$$\Leftrightarrow (51R^2-30Rr+3r^2)s^4 - r(1304R^3-1080R^2r+276Rr^2-22r^3)s^2$$
$$+r^2(8200R^4-9080R^3r+3723R^2r^2-662Rr^3+43r^4) \overset{(***)}{\ge} 0 \text{ and } \because$$
$$(51R^2-30Rr+3r^2)(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\ge} 0 \therefore \text{ in order to prove } (***),$$
$$\text{it suffices to prove : LHS of } (***) \ge (51R^2-30Rr+3r^2)(s^2-16Rr+5r^2)^2$$
$$\Leftrightarrow (164R^3-195R^2r+60Rr^2-4r^3)s^2 \overset{(****)}{\ge}$$
$$r(2428R^4-3380R^3r+1560R^2r^2-284Rr^3+16r^4)$$
$$\text{Now, LHS of } (****) \overset{\text{Gerretsen}}{\ge} (164R^3-195R^2r+60Rr^2-4r^3)(16Rr-5r^2)$$
$$\overset{?}{\ge} r(2428R^4-3380R^3r+1560R^2r^2-284Rr^3+16r^4)$$
$$\Leftrightarrow 196t^4-560t^3+375t^2-80t+4 \overset{?}{\ge} 0 \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)(112t^3+84t^2(t-2)+39t-2) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2$$
$$\Rightarrow (****) \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore (ab+bc+ca)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$
$$\ge 2\sqrt{3}(m_a+m_b+m_c)\ \forall\ \Delta ABC, ''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1663. In ΔABC holds:

$$\frac{a}{h_a}+\frac{b}{h_b}+\frac{c}{h_c} \ge 2\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum\frac{a}{h_a}=\frac{\sum a^2}{2F}=\frac{2(s^2-4Rr-r^2)}{2rs}\overset{Gerretsen}{\ge}\frac{12Rr-6r^2}{2rs}$$
$$2\sum\tan\frac{A}{2}=\frac{2(4R+r)}{s}$$
$$We\ need\ to\ show:$$
$$\frac{12Rr-6r^2}{2rs} \ge \frac{2(4R+r)}{s} \text{ or } 4Rr-8r^2 \ge 0 \text{ or } R \ge 2r(Euler)$$

1664. In all acute ΔABC the following relationship holds:

$$\frac{ca+cb}{a^2+b^2-c^2}+\frac{ab+ac}{b^2+c^2-a^2}+\frac{bc+ba}{c^2+a^2-b^2} \ge 6$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\frac{ca+cb}{a^2+b^2-c^2}+\frac{ab+ac}{b^2+c^2-a^2}+\frac{bc+ba}{c^2+a^2-b^2}=\sum_{cyc}\frac{ca+cb}{a^2+b^2-c^2}=$$

$$=\sum_{cyc}\frac{c(a+b)}{2abcosC}\overset{AM-GM}{\geq}\sum_{cyc}\frac{c\cdot 2\sqrt{ab}}{2abcosC}\overset{AM-GM}{\geq}$$

$$\geq 3\cdot\sqrt[3]{\frac{abc\cdot\sqrt{ab}\cdot\sqrt{bc}\cdot\sqrt{ca}}{ab\cdot bc\cdot ca\cdot cosAcosBcosC}}=3\sqrt[3]{\frac{1}{cosAcosBcosC}}\geq$$

$$\geq 3\cdot\sqrt[3]{\frac{1}{\frac{1}{8}}}=3\cdot\sqrt[3]{8}=6$$

Equality holds for $a=b=c$.

1665. In any Δ ABC, the following relationship holds :

$$\sqrt[3]{\frac{2}{3}\cdot\sum_{cyc}\frac{a^3}{b^3+c^3}}+\frac{R^3-8r^3}{r^3}\geq\frac{2}{3}\cdot\sum_{cyc}\frac{a^4}{b^4+c^4}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{2}{3}\cdot\sum_{cyc}\frac{a^4}{b^4+c^4}\overset{A-G}{\leq}\frac{2}{3}\cdot\sum_{cyc}\frac{a^4}{bc(b^2+c^2)}=\frac{2}{3}\cdot\sum_{cyc}\frac{a^2\left(\sum_{cyc}a^2-(b^2+c^2)\right)}{bc(b^2+c^2)}$$

$$=\frac{2}{3}\cdot\left(\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}\frac{a^2}{bc(b^2+c^2)}\right)-\frac{1}{4Rrs}\cdot\sum_{cyc}a^3\right)$$

$$\overset{A-G}{\leq}\frac{2}{3}\cdot\left((s^2-4Rr-r^2)\left(\sum_{cyc}\frac{a^2}{b^2c^2}\right)-\frac{2s(s^2-6Rr-3r^2)}{4Rrs}\right)$$

$$=\frac{2}{3}\cdot\left(\left(\frac{s^2-4Rr-r^2}{16R^2r^2s^2}\right)\left(2\sum_{cyc}a^2b^2-16r^2s^2\right)-\frac{s^2-6Rr-3r^2}{2Rr}\right)$$

$$\overset{Goldstone}{\leq}\frac{2}{3}\cdot\left(\left(\frac{s^2-4Rr-r^2}{16R^2r^2s^2}\right)(8R^2s^2-16r^2s^2)-\frac{s^2-6Rr-3r^2}{2Rr}\right)$$

$$=\frac{1}{3}\cdot\left(\frac{(s^2-4Rr-r^2)(R^2-2r^2)-Rr(s^2-6Rr-3r^2)}{R^2r^2}\right)$$

$$=\frac{(R^2-Rr-2r^2)s^2-(4Rr+r^2)(R^2-2r^2)+Rr(6Rr+3r^2)}{3R^2r^2}$$

$$\overset{\text{Gerretsen}}{\leq}\frac{(R^2-Rr-2r^2)(4R^2+4Rr+3r^2)-(4Rr+r^2)(R^2-2r^2)+Rr(6Rr+3r^2)}{3R^2r^2}$$

$$\left(\because R^2-Rr-2r^2=(R-2r)(R+r)\overset{\text{Euler}}{\geq}0\right)\overset{?}{\leq}\frac{R^3-7r^3}{r^3}$$

$$\Leftrightarrow 3t^5-4t^4+4t^3-17t^2+4\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left(3t^4+2t^3+7t^2+\frac{t(t-2)+t^2-4}{2}\right)\overset{?}{\geq}0\to\text{true}\because t\overset{\text{Euler}}{\geq}2$$

$$\therefore\frac{2}{3}\cdot\sum_{cyc}\frac{a^4}{b^4+c^4}\leq\frac{R^3-7r^3}{r^3}\to(1)$$

Now, $\sqrt[3]{\frac{2}{3}\cdot\sum_{cyc}\frac{a^3}{b^3+c^3}}\overset{\text{Nesbitt}}{\geq}\sqrt[3]{\frac{2}{3}\cdot\frac{3}{2}}=1\therefore\sqrt[3]{\frac{2}{3}\cdot\sum_{cyc}\frac{a^3}{b^3+c^3}}+\frac{R^3-8r^3}{r^3}\geq\frac{R^3-7r^3}{r^3}$

$$\overset{\text{via }(1)}{\geq}\frac{2}{3}\cdot\sum_{cyc}\frac{a^4}{b^4+c^4}\ \forall\Delta ABC,''=''\text{ iff }\Delta ABC\text{ is equilateral (QED)}$$

1666. In ΔABC the following relationship holds:

$$\sqrt[6]{\prod_{cyc}(c+a-b)}\cdot\sqrt{\frac{a+b+c}{3}}\leq\sqrt[3]{abc}$$

Proposed by Pavlos Trifon-Greece

Solution by Tapas Das-India

$$\sqrt[6]{\prod_{cyc}(c+a-b)}\cdot\sqrt{\frac{a+b+c}{3}}\leq\sqrt[3]{abc}$$

$$\sqrt[6]{\prod_{cyc}(2s-2b)}\cdot\sqrt{\frac{2s}{3}}\leq\sqrt[3]{4Rrs}$$

$$8(s-a)(s-b)(s-c)\cdot\frac{8s^3}{27}\leq 16R^2r^2s^2$$

$$\frac{1}{s}\cdot s(s-a)(s-b)(s-c)\cdot\frac{8s^3}{27}\leq 2R^2r^2s^2$$

$$\frac{F^2}{s}\cdot 4s\leq 27R^2r^2$$

$$4r^2s^2 \le 27R^2r^2, \quad s^2 \le \frac{27R^2}{4}, \quad s \le \frac{3\sqrt{3}}{2}\cdot R\ (MITRINOVIC)$$

Equality holds for $a = b = c$.

1667.

In any Δ ABC, the following relationship holds :

$$6.(3r)^6 \le \frac{r_a^6 + r_b^6}{r_c} + \frac{r_b^6 + r_c^6}{r_a} + \frac{r_c^6 + r_a^6}{r_b} \le 2.\left(\frac{9}{r}\right)^3 \left(2187\left(\frac{R}{2}\right)^8 - 2186r^8\right)$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} r_a^2 = (4R+r)^2 - 2s^2 \overset{\substack{\text{Euler}\\ and\\ \text{Gerretsen + Euler}}}{\le} \frac{81R^2}{4} - 27Rr \Rightarrow \sum_{cyc} r_a^2 \le \frac{27R(3R-4r)}{4} \to (1)$$

$$\sum_{cyc} r_a^4 = \left(\sum_{cyc} r_a^2\right)^2 - 2rs^2(s^2 - 8Rr - 2r^2) \overset{\substack{\text{via (1),Gerretsen}\\ and\\ \text{Gerretsen + Euler}}}{\le} \frac{729R^2(3R-4r)^2}{16}$$
$$- r.27Rr(8Rr - 7r^2) \Rightarrow \sum_{cyc} r_a^4 \le \frac{27R(243R^3 - 648R^2r + 304Rr^2 + 112r^3)}{16}$$
$$\to (2)$$

$$\text{Now, } \sum_{cyc} \frac{r_b^6 + r_c^6}{r_a} = \frac{(\sum_{cyc} r_a^2 - r_a^2)(r_b^4 + r_c^4 - r_b^2r_c^2)}{r_a}$$
$$= \left(\sum_{cyc} r_a^2\right)\sum_{cyc}\frac{\sum_{cyc} r_a^4 - r_a^4 - r_b^2r_c^2}{r_a} - \sum_{cyc} r_a(r_b^4 + r_c^4 - r_b^2r_c^2)$$
$$\overset{\substack{\text{via (2)}\\ and\\ A-G}}{\le} \left(\sum_{cyc} r_a^2\right)\left(\frac{27R(243R^3 - 648R^2r + 304Rr^2 + 112r^3)}{16r} - 2\sum_{yc}\frac{r_a^2r_br_c}{r_a}\right) - \sum_{yc} r_ar_b^2r_c^2$$
$$= \left(\sum_{cyc} r_a^2\right)\left(\frac{27R(243R^3 - 648R^2r + 304Rr^2 + 112r^3)}{16r} - 6rs^2\right)$$
$$- rs^4 \overset{\substack{\text{Gerretsen + Euler}\\ and\\ \text{Mitrinovic}}}{\le} \left(\sum_{cyc} r_a^2\right)\left(\frac{27R(243R^3 - 648R^2r + 304Rr^2 + 112r^3)}{16r}\right.$$
$$\left. - 6r.\frac{27Rr}{2}\right) - r.729r^4$$

$$=\left(\sum_{cyc} r_a^2\right).\frac{27R(243R^3-648R^2r+304Rr^2+48r^3)}{16r} \overset{via\,(1)}{\leq} \frac{27R(3R-4r)}{4}.\frac{27R(243R^3-648R^2r+304Rr^2+48r^3)}{16r}-729r^5$$

$$\left(\because 243R^3-648R^2r+304Rr^2+48r^3=(R-2r)(243R^2-162Rr-20r^2)+24r^3 \overset{Euler}{\geq} 24r^3>0\right)$$

$$=\frac{729\left(R^2(3R-4r)(243R^3-648R^2r+304Rr^2+48r^3)-64r^6\right)}{64r} \overset{?}{\leq} 2.\left(\frac{9}{r}\right)^3\left(2187\left(\frac{R}{2}\right)^8-2186r^8\right)=\frac{729\left(2187R^8-2186(256r^8)\right)}{128r^3}$$

$$\Leftrightarrow 2187t^8-1458t^6+5832t^5-7008t^4+2048t^3+512t^2-559488 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(2187t^7+4374t^6+7290t^5+20412t^4+33816t^3+69680t^2+139872t+279744) \overset{?}{\geq} 0 \to \text{true} \because t \overset{Euler}{\geq} 2$$

$$\therefore \frac{r_a^6+r_b^6}{r_c}+\frac{r_b^6+r_c^6}{r_a}+\frac{r_c^6+r_a^6}{r_b} \leq 2.\left(\frac{9}{r}\right)^3\left(2187\left(\frac{R}{2}\right)^8-2186r^8\right)$$

$$\text{Again}, \sum_{cyc}\frac{r_b^6+r_c^6}{r_a} \overset{Chebyshev}{\geq} \frac{1}{3}\left(\sum_{cyc}(r_b^6+r_c^6)\right)\left(\sum_{cyc}\frac{1}{r_a}\right) \left(\because \text{WLOG assuming } a \geq b \geq c\right.$$

$$\left.\Rightarrow r_b^6+r_c^6 \leq r_c^6+r_a^6 \leq r_a^6+r_b^6 \text{ and } \frac{1}{r_a} \leq \frac{1}{r_b} \leq \frac{1}{r_c}\right)=\frac{2}{3}\sum_{cyc} r_a^6 \overset{A-G}{\geq} 2rs^4$$

$$\overset{Mitrinovic}{\geq} 2r(729r^4) \therefore \frac{r_a^6+r_b^6}{r_c}+\frac{r_b^6+r_c^6}{r_a}+\frac{r_c^6+r_a^6}{r_b} \geq 6.(3r)^6 \text{ and so}, 6.(3r)^6$$

$$\leq \frac{r_a^6+r_b^6}{r_c}+\frac{r_b^6+r_c^6}{r_a}+\frac{r_c^6+r_a^6}{r_b} \leq 2.\left(\frac{9}{r}\right)^3\left(2187\left(\frac{R}{2}\right)^8-2186r^8\right) \text{ (QED)}$$

1668. In any Δ ABC, the following relationship holds :

$$18r \leq \frac{r_a^4+r_b^4}{r_c^3}+\frac{r_b^4+r_c^4}{r_a^3}+\frac{r_c^4+r_a^4}{r_b^3} \leq \frac{18}{r^9}\left(19683\left(\frac{R}{2}\right)^{10}-19682r^{10}\right)$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{r_a^3}=\sum_{cyc}\frac{(s-a)^3}{r^3s^3}=\frac{1}{r^3s^3}\left(\left(\sum_{cyc}(s-a)\right)^3-3\prod_{cyc}(s-b+s-c)\right)$$

$$=\frac{s^3-12Rrs}{r^3s^3} \overset{Mitrinovic}{\leq} \frac{\frac{27R^2}{4}-12Rr}{r^3.27r^2} \Rightarrow \sum_{cyc}\frac{1}{r_a^3} \leq \frac{9R^2-16Rr}{36r^5} \to (1)$$

$$\sum_{cyc} r_a^4 = ((4R+r)^2 - 2s^2)^2 - 2s^2(s^2 - 8Rr - 2r^2)$$

$$= (4R+r)^4 + 2s^4 - 4s^2(4R+r)^2 + 4rs^2(4R+r) \overset{\substack{\text{Trucht} \\ \text{and} \\ \because s^2 \geq 12Rr+3r^2}}{\leq}$$

$$(4R+r)^4 + \frac{2(4R+r)^4}{9} - 12r(4R+r)^3 + \frac{4r(4R+r)^3}{3}$$

$$= \frac{(44R-85r)(4R+r)^3}{9} \overset{\text{Euler}}{\leq} \frac{(44R-85r)\left(\frac{9R}{2}\right)^3}{9}$$

$$\therefore \sum_{cyc} \frac{r_b^4 + r_c^4}{r_a^3} = \left(\sum_{cyc} r_a^4\right)\left(\sum_{cyc} \frac{1}{r_a^3}\right) - \sum_{cyc} r_a \leq$$

$$\frac{(44R-85r)\left(\frac{9R}{2}\right)^3}{9} \cdot \left(\sum_{cyc} \frac{1}{r_a^3}\right) - (4R+r) \overset{\substack{\text{via (1)} \\ \text{and} \\ \text{via Euler}}}{\leq}$$

$$\frac{(44R-85r)(81R^3)}{8} \cdot \frac{9R^2 - 16Rr}{36r^5} - 9r$$

$$\therefore \frac{r_a^4 + r_b^4}{r_c^3} + \frac{r_b^4 + r_c^4}{r_a^3} + \frac{r_c^4 + r_a^4}{r_b^3} \leq 9 . \frac{R^3(44R-85r)(9R^2-16Rr) - 32r^6}{32r^5} \to (2)$$

$$\text{Again, } \frac{18}{r^9}\left(19683\left(\frac{R}{2}\right)^{10} - 19682r^{10}\right) \overset{\text{Euler}}{\geq} \frac{18}{r^9}\left(\frac{19683R^6 . 16r^4 - 1024 . 19682r^{10}}{1024}\right)$$

$$\Rightarrow \frac{18}{r^9}\left(19683\left(\frac{R}{2}\right)^{10} - 19682r^{10}\right) \geq 9 . \frac{19683R^6 - 64 . 19682r^6}{32r^5} \to (3) \therefore (2),(3) \Rightarrow$$

$$\text{in order to prove : } \sum_{cyc} \frac{r_b^4 + r_c^4}{r_a^3} \leq \frac{18}{r^9}\left(19683\left(\frac{R}{2}\right)^{10} - 19682r^{10}\right), \text{it suffices}$$

$$\text{to prove : } \frac{19683R^6 - 64 . 19682r^6}{32r^5} \geq \frac{R^3(44R-85r)(9R^2-16Rr) - 32r^6}{32r^5}$$

$$\Leftrightarrow 19287t^6 + 1469t^5 - 1360t^4 - 1259616 \overset{?}{\geq} 0 \; \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(19287t^5 + 40043t^4 + 78726t^3 + 157452t^2 + 314904t + 629808) \overset{?}{\geq} 0$$

$$\to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \therefore \frac{r_a^4 + r_b^4}{r_c^3} + \frac{r_b^4 + r_c^4}{r_a^3} + \frac{r_c^4 + r_a^4}{r_b^3} \leq \frac{18}{r^9}\left(19683\left(\frac{R}{2}\right)^{10} - 19682r^{10}\right)$$

$$\text{Also, } \frac{r_a^4 + r_b^4}{r_c^3} + \frac{r_b^4 + r_c^4}{r_a^3} + \frac{r_c^4 + r_a^4}{r_b^3} \overset{\text{Radon}}{\geq} \frac{\left(2\sum_{cyc} r_a\right)^4}{\left(2\sum_{cyc} r_a\right)^3} = 2(4R+r) \overset{\text{Euler}}{\geq} 18r$$

$$\text{and so, } 18r \leq \frac{r_a^4 + r_b^4}{r_c^3} + \frac{r_b^4 + r_c^4}{r_a^3} + \frac{r_c^4 + r_a^4}{r_b^3} \leq \frac{18}{r^9}\left(19683\left(\frac{R}{2}\right)^{10} - 19682r^{10}\right)$$

$\forall \Delta$ ABC,'' ='' iff Δ ABC is equilateral (QED)

1669. **In ΔABC the following relationship holds:**

$$a\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)+b\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)+c\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\geq 12r$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$a\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)+b\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)+c\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)=$$

$$=\sum_{cyc}a\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)=\sum_{cyc}a\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=$$

$$=\sum_{cyc}a\left(\frac{\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{B}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}\right)=\sum_{cyc}a\left(\frac{\sin\left(\frac{B+C}{2}\right)}{\cos\frac{B}{2}\cos\frac{C}{2}}\right)=$$

$$=\sum_{cyc}a\left(\frac{\sin\left(\frac{\pi-A}{2}\right)}{\cos\frac{B}{2}\cos\frac{C}{2}}\right)=\sum_{cyc}a\left(\frac{\cos\left(\frac{A}{2}\right)}{\cos\frac{B}{2}\cos\frac{C}{2}}\right)\overset{AM-GM}{\geq}3\sqrt[3]{\frac{abc}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}}=$$

$$=3\sqrt[3]{\frac{4Rrs}{\frac{s}{4R}}}=3\sqrt[3]{16R^2r}\overset{EULER}{\geq}3\sqrt[3]{16\cdot 4r^2\cdot r}=12r$$

Equality holds for $a=b=c$.

1670. **In ΔABC the following relationship holds:**

$$(b+c)\tan\frac{A}{2}+(c+a)\tan\frac{B}{2}+(a+b)\tan\frac{C}{2}\geq 12r$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$(b+c)\tan\frac{A}{2}+(c+a)\tan\frac{B}{2}+(a+b)\tan\frac{C}{2}=$$

$$=\sum_{cyc}(b+c)\tan\frac{A}{2}=2R\sum_{cyc}(\sin B+\sin C)\tan\frac{A}{2}=$$

$$= 4R\sum_{cyc} sin\frac{B+C}{2}cos\frac{B-C}{2}tan\frac{A}{2} = 4R\sum_{cyc} sin\frac{\pi-A}{2}cos\frac{B-C}{2}tan\frac{A}{2} =$$

$$= 4R\sum_{cyc} cos\frac{A}{2}cos\frac{B-C}{2}\cdot\frac{sin\frac{A}{2}}{cos\frac{A}{2}} = 4R\sum_{cyc} sin\frac{A}{2}cos\frac{B-C}{2} \overset{AM-GM}{\geq}$$

$$\geq 12R\cdot\sqrt[3]{\prod_{cyc} sin\frac{A}{2}\cdot\prod_{cyc} cos\frac{B-C}{2}} = 12R\sqrt[3]{\frac{r}{4R}\cdot\frac{s^2+r^2+2Rr}{8R^2}} \geq$$

$$\overset{GERRETSEN}{\geq} 12R\sqrt[3]{\frac{r}{4R}\cdot\frac{16Rr-5r^2+r^2+2Rr}{8R^2}} = 12R\sqrt[3]{\frac{r(18Rr-4r^2)}{32R^3}} \geq$$

$$\overset{EULER}{\geq} 12\sqrt[3]{\frac{r(36r^2-4r^2)}{32}} = 12r$$

Equality holds for $a = b = c$.

1671. **In all acute triangles ABC the following relationship holds:**

$$a(cotB + cotC) + b(cotC + cotA) + c(cotA + cotB) \geq 12r$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$a(cotB + cotC) + b(cotC + cotA) + c(cotA + cotB) = \sum_{cyc} a(cotB + cotC) =$$

$$= \sum_{cyc} a\left(\frac{cosB}{sinB} + \frac{cosC}{sinC}\right) = \sum_{cyc} a\cdot\frac{cosBsinC + cosCsinB}{sinBsinC} =$$

$$= \sum_{cyc} a\cdot\frac{sin(B+C)}{sinBsinC} = \sum_{cyc} a\cdot\frac{sin(\pi-A)}{sinBsinC} = \sum_{cyc} a\cdot\frac{sinA}{sinBsinC} \overset{AM-GM}{\geq}$$

$$\geq 3\cdot\sqrt[3]{abc\cdot\frac{sinAsinBsinC}{sinBsinC\cdot sinCsinA\cdot sinAsinB}} =$$

$$=3\sqrt[3]{\frac{abc}{sinAsinBsinC}}=3\sqrt[3]{\frac{abc}{\frac{a}{2R}\cdot\frac{b}{2R}\cdot\frac{c}{2R}}}=3\sqrt[3]{8R^3}=6R\overset{EULER}{\geq}6\cdot 2r=12r$$

Equality holds for $a=b=c$.

1672. In any Δ ABC, the following relationship holds :

$$\frac{a^4}{(b+c)m_b^2m_c^2}+\frac{b^4}{(c+a)m_c^2m_a^2}+\frac{c^4}{(a+b)m_a^2m_b^2}\geq\frac{8\sqrt{3}}{9R}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{m_a^2}{bc}\overset{?}{\geq}\frac{9}{4}\Leftrightarrow\frac{1}{16Rrs}\cdot\sum_{cyc}a(2b^2+2c^2-a^2)\overset{?}{\geq}\frac{9}{4}$$

$$\Leftrightarrow\frac{1}{4Rrs}\cdot\sum_{cyc}a(2b^2+2c^2+2a^2-3a^2)\overset{?}{\geq}9$$

$$\Leftrightarrow 2\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a\right)-3\sum_{cyc}a^3\overset{?}{\geq}36Rrs$$

$$\Leftrightarrow 8s(s^2-4Rr-r^2)-6s(s^2-6Rr-3r^2)\overset{?}{\geq}36Rrs\Leftrightarrow s^2\overset{?}{\geq}16Rr-5r^2$$

$\rightarrow$ true via Gerretsen $\therefore\sum_{cyc}\frac{m_a^2}{bc}\geq\frac{9}{4}$ and implementing : $\sum_{cyc}\frac{m_a^2}{bc}\geq\frac{9}{4}$ on

a triangle with sides $\frac{2m_a}{3},\frac{2m_b}{3},\frac{2m_c}{3}$ whose medians as a consequence of

Apollonius' theorem $=\frac{a}{2},\frac{b}{2},\frac{c}{2}$ respectively, we arrive at : $\sum_{cyc}\frac{\frac{1}{4}\cdot a^2}{\frac{4}{9}m_bm_c}\geq\frac{9}{4}$

$$\Rightarrow\sum_{cyc}\frac{a^2}{m_bm_c}\geq 4\rightarrow(1)$$

$$\text{Now, }\sum_{cyc}\frac{a^4}{(b+c)m_b^2m_c^2}=\sum_{cyc}\frac{\left(\frac{a^2}{m_bm_c}\right)^2}{b+c}\overset{Bergstrom}{\geq}\frac{\left(\sum_{cyc}\frac{a^2}{m_bm_c}\right)^2}{4s}\overset{\text{via (1) and via Mitrinovic}}{\geq}\frac{16}{4R.\frac{3\sqrt{3}}{2}}$$

$$=\frac{8\sqrt{3}}{9R}\ \forall\Delta\text{ ABC},''=''\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$

Solution 2 by Tapas Das-India

$$\frac{a^4}{(b+c)m_b^2m_c^2}+\frac{b^4}{(c+a)m_c^2m_a^2}+\frac{c^4}{(a+b)m_a^2m_b^2}\overset{Bergstrom}{\geq}$$

$$\geq\frac{(a^2+b^2+c^2)^2}{\sum(b+c)m_b^2m_c^2}\overset{Chebyshev}{\geq}\frac{\frac{16}{9}(\sum m_a^2)^2}{\frac{1}{3}(4s)(\sum m_b^2m_c^2)}\geq$$

$$=\frac{\frac{16}{9}3(\sum m_b^2m_c^2)}{\frac{1}{3}(4s)(\sum m_b^2m_c^2)}=\frac{12}{3s}\overset{Mitrinovic}{\geq}\frac{12}{9\sqrt{3}\frac{R}{2}}=\frac{8\sqrt{3}}{9R}$$

Equality holds for $a=b=c$.

1673. In any Δ ABC, the following relationship holds :

$$\frac{m_a^4}{(b+c)b^2c^2}+\frac{m_b^4}{(c+a)c^2a^2}+\frac{m_c^4}{(a+b)a^2b^2}\geq\frac{9\sqrt{3}}{32R}$$

Proposed by Daniel Sitaru-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{m_a^4}{(b+c)b^2c^2}=\sum_{cyc}\frac{\left(\frac{m_a^2}{bc}\right)^2}{b+c}\overset{Bergstrom}{\geq}\frac{\left(\sum_{cyc}\frac{m_a^2}{bc}\right)^2}{4s}\overset{Mitrinovic}{\geq}$$

$$\geq\frac{\left(\sum_{cyc}\frac{m_a^2}{bc}\right)^2}{4R.\frac{3\sqrt{3}}{2}}\overset{?}{\geq}\frac{9\sqrt{3}}{32R}\Leftrightarrow\sum_{cyc}\frac{m_a^2}{bc}\overset{?}{\geq}\frac{9}{4}\Leftrightarrow\frac{1}{16Rrs}.\sum_{cyc}a(2b^2+2c^2-a^2)\overset{?}{\geq}\frac{9}{4}$$

$$\Leftrightarrow\frac{1}{4Rrs}.\sum_{cyc}a(2b^2+2c^2+2a^2-3a^2)\overset{?}{\geq}9$$

$$\Leftrightarrow 2\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a\right)-3\sum_{cyc}a^3\overset{?}{\geq}36Rrs$$

$$\Leftrightarrow 8s(s^2-4Rr-r^2)-6s(s^2-6Rr-3r^2)\overset{?}{\geq}36Rrs\Leftrightarrow s^2\overset{?}{\geq}16Rr-5r^2$$

$$\to\text{true via Gerretsen}\ \therefore\sum_{cyc}\frac{m_a^4}{(b+c)b^2c^2}\geq\frac{9\sqrt{3}}{32R}$$

$\forall\,\Delta$ ABC, " = " iff Δ ABC is equilateral (QED)

1674. In any Δ ABC, the following relationship holds :

$$\frac{m_a^2 m_b^2}{(a+b)a^2b^2}+\frac{m_b^2 m_c^2}{(b+c)b^2c^2}+\frac{m_c^2 m_a^2}{(c+a)c^2a^2}\ge\frac{9\sqrt{3}}{32R}$$

Proposed by Daniel Sitaru-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{m_a^2 m_b^2}{(a+b)a^2b^2}=\frac{81}{16}\sum_{cyc}\frac{\left(\frac{2}{3}m_a\right)^2\left(\frac{2}{3}m_b\right)^2}{(a+b)a^2b^2}=\frac{81}{16}\cdot\sum_{cyc}\frac{\frac{AG^2.BG^2}{a^2b^2}}{a+b}\overset{BERGSTROM}{\ge}$$

$$\ge\frac{81}{16}\cdot\frac{\left(\sum_{cyc}\frac{AG.BG}{ab}\right)^2}{a+b+b+c+c+a}\overset{HAYASHI}{\ge}\frac{81}{16}\cdot\frac{1}{4s}\overset{MITRINOVIC}{\ge}$$

$$\ge\frac{81}{16}\cdot\frac{1}{4R.\frac{3\sqrt{3}}{2}}=\frac{9\sqrt{3}}{32R}$$

" =" iff Δ ABC is equilateral (QED)

1675. In any Δ ABC, the following relationship holds :

$$\frac{r_a r_b a^2b^2}{(r_a+r_b)m_a^2m_b^2}+\frac{r_b r_c b^2c^2}{(r_b+r_c)m_b^2m_c^2}+\frac{r_c r_a c^2a^2}{(r_c+r_a)m_c^2m_a^2}\ge 8r$$

Proposed by Daniel Sitaru-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{m_am_b}{ab}=\frac{9}{4}\sum_{cyc}\frac{AG.BG}{ab}\overset{Hayashi}{\ge}\frac{9}{4}$$

and implementing : $\sum_{cyc}\frac{m_am_b}{ab}\ge\frac{9}{4}$ on a triangle with sides $\frac{2m_a}{3},\frac{2m_b}{3},\frac{2m_c}{3}$

whose medians as a consequence of Apollonius' theorem

$=\frac{a}{2},\frac{b}{2},\frac{c}{2}$ respectively, we arrive at :

$$\sum_{cyc}\frac{\frac{1}{4}.ab}{\frac{4}{9}.m_am_b}\ge\frac{9}{4}\Rightarrow\sum_{cyc}\frac{ab}{m_am_b}\ge 4\to(1)$$

$$\text{Now, } \sum_{cyc} \frac{r_a r_b a^2 b^2}{(r_a + r_b) m_a^2 m_b^2} = \sum_{cyc} \frac{\left(\frac{ab}{m_a m_b}\right)^2}{\frac{1}{r_a} + \frac{1}{r_b}} \overset{\text{Bergstrom}}{\geq}$$

$$\geq \frac{\left(\sum_{cyc} \frac{ab}{m_a m_b}\right)^2}{2 \sum_{cyc} \frac{1}{r_a}} \overset{\text{via (1)}}{\geq} \frac{16}{\frac{2}{r}} = 8r$$

$$\therefore \sum_{cyc} \frac{r_a r_b a^2 b^2}{(r_a + r_b) m_a^2 m_b^2} \geq 8r \ \forall \ \Delta\ ABC, " = " \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

1676. In ΔABC the following relationship holds:

$$\frac{1}{(a+b)(m_a+m_b-m_c)^2} + \frac{1}{(b+c)(m_b+m_c-m_a)^2} + \frac{1}{(c+a)(m_c+m_a-m_b)^2} \geq \frac{2\sqrt{3}}{9R^3}$$

Proposed by Daniel Sitaru-Romania

Solution by Tapas Das-India

$$\frac{1}{(a+b)(m_a+m_b-m_c)^2} + \frac{1}{(b+c)(m_b+m_c-m_a)^2} + \frac{1}{(c+a)(m_c+m_a-m_b)^2} =$$

$$= \sum \frac{\left(\frac{1}{m_a+m_b-m_c}\right)^2}{a+b} \overset{BERGSTROM}{\geq} \frac{\left(\frac{1}{m_a+m_b-m_c} + \frac{1}{m_b+m_c-m_a} + \frac{1}{m_c+m_a-m_b}\right)^2}{2(a+b+c)}$$

$$\geq$$

$$\overset{BERGSTROM}{\geq} \frac{81}{4s(m_a+m_b+m_c)^2} \overset{LEUENBERGER}{\geq} \frac{81}{4s(4R+r)^2} \geq$$

$$\overset{MITRINOVIC}{\geq} \frac{81}{\frac{4.3\sqrt{3}R}{2} \cdot (4R+r)^2} \overset{EULER}{\geq} \frac{81}{\frac{4.3\sqrt{3}R}{2} \cdot \frac{81R^2}{4}} = \frac{2\sqrt{3}}{9R^3}$$

Equality holds for $a = b = c$.

1677. In any Δ ABC, the following relationship holds :

$$\sum_{cyc} \tan^2 \frac{A}{2} \geq \frac{26}{27} + 27 \left(\sum_{cyc} \cot^2 \frac{A}{2}\right)^{-3}$$

Proposed by Daniel Sitaru-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\tan^2\frac{A}{2}=\frac{1}{s^2}\sum_{cyc}r_a^2=\frac{(4R+r)^2-2s^2}{s^2}\overset{\text{Trucht}}{\geq}\frac{3s^2-2s^2}{s^2}$$

$$\Rightarrow\sum_{cyc}\tan^2\frac{A}{2}\geq 1\to(1)$$

$$\text{Again, }\sum_{cyc}\cot^2\frac{A}{2}=\sum_{cyc}\frac{s^2}{r_a^2}=\frac{s^2}{r^2s^4}\cdot\sum_{cyc}r_b^2r_c^2=\frac{1}{r^2s^2}\cdot\left(\left(\sum_{cyc}r_br_c\right)^2-2r_ar_br_c\sum_{cyc}r_a\right)$$

$$=\frac{s^4-2rs^2(4R+r)}{r^2s^2}=\frac{s^2-8Rr-2r^2}{r^2}\Rightarrow\frac{26}{27}+27\left(\sum_{cyc}\cot^2\frac{A}{2}\right)^{-3}$$

$$=\frac{26}{27}+\frac{27r^6}{(s^2-8Rr-2r^2)^3}\overset{?}{\leq}1\Leftrightarrow\frac{729r^6}{(s^2-8Rr-2r^2)^3}\overset{?}{\leq}1\Leftrightarrow 9r^2\overset{?}{\leq}s^2-8Rr-2r^2$$

$$\Leftrightarrow s^2-8Rr-11r^2\overset{?}{\geq}0\Leftrightarrow s^2-16Rr+5r^2+8r(R-2r)\overset{?}{\geq}0$$

$$\to\text{true}\because s^2-16Rr+5r^2\overset{\text{Gerretsen}}{\geq}0\text{ and }8r(R-2r)\overset{\text{Euler}}{\geq}0$$

$$\therefore\frac{26}{27}+27\left(\sum_{cyc}\cot^2\frac{A}{2}\right)^{-3}\leq 1\overset{\text{via (1)}}{\leq}\sum_{cyc}\tan^2\frac{A}{2}\Rightarrow$$

$$\sum_{cyc}\tan^2\frac{A}{2}\geq\frac{26}{27}+27\left(\sum_{cyc}\cot^2\frac{A}{2}\right)^{-3}\quad\forall\Delta ABC,''=''\text{ iff }\Delta ABC\text{ is equilateral (QED)}$$

1678. If in $\Delta ABC, a\neq b\neq c\neq a$ then:

$$\left(\frac{a^3(c-b)+b^3(a-c)+c^3(b-a)}{a^2(c-b)+b^2(a-c)+c^2(b-a)}\right)^2<25R^2+\frac{8\sqrt{3}\,F}{9}$$

Proposed by Daniel Sitaru – Romania

Solution by Tapas Das-India

$$a^3(b-c)+b^3(c-a)+c^3(a-b)=a^3(b-c)+b^3c-c^3b+b^3a+c^3a$$

$$=a^2(b-c)+bc(b^2-c^2)-a(b^3-c^3)$$

$$=a^3(b-c)+bc(b+c)(b-c)-a(b-c)(b^2+bc+c^2)$$

$$=(b-c)\{a^3+b^2c+bc^2-ab^2-abc-ac^2\}$$

$$= (b-c)\{a(a^2-b^2)+bc(b-a)+c^2(b-a)\}$$

$$= (b-c)(a-b)\{a(a+b)-bc-c^2\} = (b-c)(a-b)\{a^2+ab-bc-c^2\}$$

$$= (b-c)(a-b)\{(a-c)(a+c)+b(a-c)\} = -(a-b)(b-c)(c-a)(a+b+c)$$

$$\therefore a^3(c-b)+b^3(a-c)+c^3(b-a) = -\{-(a-b)(b-c)(c-a)(a+b+c)\}$$

$$= (a-b)(b-c)(c-a)(a+b+c) \quad \textbf{(1)}$$

$$a^2(b-c)+b^2(c-a)+c^2(a-b)$$

$$= a^2b-a^2c+b^2c-b^2a+ac^2-bc^2+abc-abc$$

$$= ab(a-b+c)-ac(a-c)+bc(-a+b-c)$$

$$= (ab-bc)(a-b+c)-ac(a-c) = (a-c)\{b(a-b)+c(b-a)\}$$

$$= (a-c)(a-b)(b-c) = -(a-b)(b-c)(c-a)$$

$$\therefore a^2(c-b)+b^2(a-c)+c^2(b-a)$$

$$= -\{-(a-b)(b-c)(c-a)\} = (a-b)(b-c)(c-a) \quad \textbf{(2)}$$

$$\therefore \left(\frac{a^3(c-b)+b^3(a-c)+c^3(b-a)}{a^2(c-b)+b^2(a-c)+c^2(b-a)}\right)^2 = (a+b+c)^2 = 4s^2 \text{ (using (1) and (2))}$$

We need to show:

$$4s^2 < 25R^2 + \frac{8\sqrt{3}F}{9}$$

$$\text{or } 4(4R^2+4Rr+3r^2) < 25R^2 + \frac{8\sqrt{3}}{9}\cdot r\cdot 3\sqrt{3}r$$

$$\text{(Gerretsen and Mitrinovic) or } 9R^2-16Rr-4r^2 > 0$$

$$\text{or } (R-2r)(9R+2r) > 0 \text{ True (Euler)}$$

Conclusion:

$$\left(\frac{a^3(c-b)+b^3(a-c)+c^3(b-a)}{a^2(c-b)+b^2(a-c)+c^2(b-a)}\right)^2 = 4s^2 < 25R^2 + \frac{8\sqrt{3}F}{9}$$

$$\text{or, } \left(\frac{a^3(c-b)+b^3(a-c)+c^3(b-a)}{a^2(c-b)+b^2(a-c)+c^2(b-a)}\right)^2 < 25R^2 + \frac{8\sqrt{3}F}{9}$$

1679. In ΔABC the following relationship holds:

$$(b^2+c^2)tan\frac{A}{2} + (c^2+a^2)tan\frac{B}{2} + (a^2+b^2)tan\frac{C}{2} \ge 24\sqrt{3}r^2$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$(b^2+c^2)tan\frac{A}{2}+(c^2+a^2)tan\frac{B}{2}+(a^2+b^2)tan\frac{C}{2}=\sum_{cyc}(b^2+c^2)tan\frac{A}{2}\geq$$

$$\overset{AM-GM}{\geq}2\sum_{cyc}bc\cdot tan\frac{A}{2}\overset{AM-GM}{\geq}2\cdot 3\sqrt[3]{(abc)^2\prod_{cyc}tan\frac{A}{2}}=$$

$$=6\cdot\sqrt[3]{(4Rrs)^2\cdot\frac{r}{s}}=6\cdot\sqrt[3]{16R^2r^3s}=6r\cdot\sqrt[3]{16R^2s}\overset{EULER}{\geq}$$

$$\geq 6r\cdot\sqrt[3]{64r^2s}=24r\cdot\sqrt[3]{r^2s}\overset{MITRINOVIC}{\geq}24r\cdot\sqrt[3]{3\sqrt{3}\cdot r^3}=$$

$$=24r^2\cdot\sqrt[3]{\left(\sqrt{3}\right)^3}=24\sqrt{3}r^2$$

Equality holds for $a=b=c$.

1680. In $\Delta ABC, n\in\mathbb{N}$ the following relationship holds:

$$min\left(\sum_{cyc}(b^n+c^n)tan\frac{A}{2},\sum_{cyc}a^n\left(tan\frac{B}{2}+tan\frac{C}{2}\right)\right)\geq 2^{n+1}\cdot 3^{\frac{n+1}{2}}\cdot r^n$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$\sum_{cyc}(b^n+c^n)tan\frac{A}{2}\overset{AM-GM}{\geq}2\sum_{cyc}\sqrt{b^nc^n}\cdot tan\frac{A}{2}\overset{AM-GM}{\geq}$$

$$\geq 2\cdot 3\cdot\sqrt[3]{\sqrt{b^nc^n}\cdot\sqrt{c^na^n}\cdot\sqrt{a^nb^n}\cdot tan\frac{A}{2}tan\frac{B}{2}tan\frac{C}{2}}=$$

$$=6\cdot\sqrt[3]{(abc)^n\cdot\frac{r}{s}}=6\cdot\sqrt[3]{4^n\cdot R^n\cdot r^n\cdot s^n\cdot\frac{r}{s}}=$$

$$=6\cdot\sqrt[3]{4^n\cdot R^n\cdot r^{n+1}\cdot s^{n-1}}\overset{EULER}{\geq}6\cdot\sqrt[3]{2^{2n}\cdot 2^n\cdot r^n\cdot r^{n+1}\cdot s^{n-1}}=$$

$$=6\cdot 2^n\cdot\sqrt[3]{r^{2n+1}\cdot s^{n-1}}\overset{MITRINOVIC}{\geq}3\cdot 2^{n+1}\cdot\sqrt[3]{r^{2n+1}\cdot(3\sqrt{3}r)^{n-1}}=$$

$$=2^{n+1}\cdot 3\cdot 3^{\frac{n}{2}}\cdot\sqrt[3]{r^{2n+1+n-1}}=2^{n+1}\cdot 3^{\frac{n+1}{2}}\cdot r^n$$

$$\sum_{cyc} a^n\left(tan\frac{B}{2}+tan\frac{C}{2}\right) \overset{AM-GM}{\geq} 2\sum_{cyc} a^n\sqrt{tan\frac{B}{2}tan\frac{C}{2}} \overset{AM-GM}{\geq}$$

$$\geq 2\cdot 3\cdot\sqrt[3]{(abc)^n\cdot\prod_{cyc}\sqrt{tan\frac{B}{2}tan\frac{C}{2}}} = 6\cdot\sqrt[3]{(4Rrs)^n\cdot\frac{r}{s}} = 6\cdot\sqrt[3]{4^n\cdot R^n\cdot r^n\cdot s^n\cdot\frac{r}{s}} =$$

$$= 6\cdot\sqrt[3]{4^n\cdot R^n\cdot r^{n+1}\cdot s^{n-1}} \overset{EULER}{\geq} 6\cdot\sqrt[3]{2^{2n}\cdot 2^n\cdot r^n\cdot r^{n+1}\cdot s^{n-1}} =$$

$$= 6\cdot 2^n\cdot\sqrt[3]{r^{2n+1}\cdot s^{n-1}} \overset{MITRINOVIC}{\geq} 3\cdot 2^{n+1}\cdot\sqrt[3]{r^{2n+1}\cdot(3\sqrt{3}r)^{n-1}} =$$

$$= 2^{n+1}\cdot 3\cdot 3^{\frac{n}{2}}\cdot\sqrt[3]{r^{2n+1+n-1}} = 2^{n+1}\cdot 3^{\frac{n+1}{2}}\cdot r^n$$

Equality holds for $a=b=c$.

1681. In all acute triangles $ABC, n\in\mathbb{N}$ the following relationship holds:

$$a^n(cotB+cotC)+b^n(cotC+cotA)+c^n(cotA+cotB)$$
$$\geq 2^{n+1}\cdot 3^{\frac{n+1}{2}}\cdot r^n$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$a^n(cotB+cotC)+b^n(cotC+cotA)+c^n(cotA+cotB) =$$

$$= \sum_{cyc} a^n\left(\frac{cosB}{sinB}+\frac{cosC}{sinC}\right) = \sum_{cyc} a^n\cdot\frac{sinCcosB+sinBcosC}{sinBsinC} =$$

$$= \sum_{cyc} a^n\cdot\frac{sin(B+C)}{sinBsinC} = \sum_{cyc} a^n\cdot\frac{sin(\pi-A)}{sinBsinC} = \sum_{cyc} a^n\cdot\frac{sinA}{sinBsinC} \overset{AM-GM}{\geq}$$

$$\geq 3\cdot\sqrt[3]{(abc)^n\cdot\frac{sinA}{sinBsinC}\cdot\frac{sinB}{sinCsinA}\cdot\frac{sinC}{sinCsinB}} =$$

$$= 3\cdot\sqrt[3]{(4RF)^n\cdot\frac{1}{sinAsinBsinC}} = 3\cdot\sqrt[3]{(4Rrs)^n\cdot\frac{2R\cdot 2R\cdot 2R}{abc}} =$$

$$= 6R\cdot\sqrt[3]{(4Rrs)^{n-1}} \overset{EULER}{\geq} 12r\sqrt[3]{(8r^2s)^{n-1}} \geq$$

$$\overset{MITRINOVIC}{\geq} 12r\sqrt[3]{\left(8r^3\cdot 3\sqrt{3}\right)^{n-1}} = 12r\left(2\sqrt{3}r\right)^{n-1} =$$

$$= 3r \cdot 2^2 \cdot 2^{n-1} \cdot 3^{\frac{n-1}{2}} \cdot r^{n-1} = 2^{n+1} \cdot 3^{\frac{n+1}{2}} \cdot r^n$$

Equality holds for $a = b = c$.

1682. In all acute triangles ABC the following relationship holds:

$$(cotB + cotC)tan\frac{A}{2} + (cotC + cotA)tan\frac{B}{2} + (cotA + cotB)tan\frac{C}{2} \geq 2$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$(cotB + cotC)tan\frac{A}{2} + (cotC + cotA)tan\frac{B}{2} + (cotA + cotB)tan\frac{C}{2} =$$

$$= \sum_{cyc}(cotB + cotC)tan\frac{A}{2} = \sum_{cyc}\left(\frac{cosB}{sinB} + \frac{cosC}{sinC}\right)tan\frac{A}{2} =$$

$$= \sum_{cyc}\frac{cosBsinC + cosCsinB}{sinBsinC} \cdot tan\frac{A}{2} = \sum_{cyc}\frac{sin(B+C)}{sinBsinC} \cdot tan\frac{A}{2} =$$

$$= \sum_{cyc}\frac{sin(\pi - A)}{sinBsinC} \cdot tan\frac{A}{2} = \sum_{cyc}\frac{sinA}{sinBsinC} \cdot tan\frac{A}{2} \overset{AM-GM}{\geq}$$

$$\geq 3\sqrt[3]{\prod_{cyc}\frac{1}{sinA} \cdot \prod_{cyc}tan\frac{A}{2}} = 3\sqrt[3]{\frac{2R^2}{sr} \cdot \frac{r}{s}} = 3\sqrt[3]{2 \cdot \frac{R^2}{s^2}} \geq$$

$$\overset{MITRINOVIC}{\geq} 3\sqrt[3]{\frac{2R^2}{\frac{27R^2}{4}}} = 3\sqrt[3]{\frac{8}{27}} = 2$$

Equality holds for $a = b = c$.

1683. In ΔABC the following relationship holds:

$$h_a\left(tan\frac{B}{2} + tan\frac{C}{2}\right) + h_b\left(tan\frac{C}{2} + tan\frac{A}{2}\right) + h_c\left(tan\frac{A}{2} + tan\frac{B}{2}\right) \geq 6\sqrt{3}r$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$h_a\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)+h_b\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)+h_c\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)=$$

$$=\sum_{cyc}h_a\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)=\sum_{cyc}h_a\cdot\frac{\sin\left(\frac{B+C}{2}\right)}{\cos\frac{B}{2}\cos\frac{C}{2}}=$$

$$=2F\sum_{cyc}\frac{\cos\frac{A}{2}}{a\cos\frac{B}{2}\cos\frac{C}{2}}\overset{AM-GM}{\geq}2F\cdot 3\sqrt[3]{\frac{1}{abc}\cdot\prod_{cyc}\frac{1}{\cos\frac{A}{2}}}=$$

$$=6F\cdot\sqrt[3]{\frac{1}{4RF}\cdot\frac{4R}{s}}=6rs\cdot\sqrt[3]{\frac{1}{Fs}}=6r\cdot\sqrt[3]{\frac{s^3}{rs^2}}=$$

$$=6r\cdot\sqrt[3]{\frac{s}{r}}\overset{MITRINOVIC}{\geq}6r\cdot\sqrt[3]{\frac{3\sqrt{3}r}{r}}=6r\sqrt[3]{\left(\sqrt{3}\right)^3}=6\sqrt{3}r$$

Equality holds for $a=b=c$.

1684. In ΔABC the following relationship holds:

$$\min\left(\sum_{cyc}w_a\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right),\sum_{cyc}m_a\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)\right)\geq 6\sqrt{3}r$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

Lemma: In ΔABC the following relationship holds:

$$\sum_{cyc}h_a\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)\geq 6\sqrt{3}r$$

Proof:

$$\sum_{cyc}h_a\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)=\sum_{cyc}h_a\cdot\frac{\sin\left(\frac{B+C}{2}\right)}{\cos\frac{B}{2}\cos\frac{C}{2}}=$$

$$=2F\sum_{cyc}\frac{\cos\frac{A}{2}}{a\cos\frac{B}{2}\cos\frac{C}{2}}\overset{AM-GM}{\geq}2F\cdot 3\sqrt[3]{\frac{1}{abc}\cdot\prod_{cyc}\frac{1}{\cos\frac{A}{2}}}=6F\cdot\sqrt[3]{\frac{1}{4RF}\cdot\frac{4R}{s}}=6rs\cdot\sqrt[3]{\frac{1}{Fs}}=$$

$$=6r\cdot\sqrt[3]{\frac{s^3}{rs^2}}=6r\cdot\sqrt[3]{\frac{s}{r}}\overset{MITRINOVIC}{\geq}6r\cdot\sqrt[3]{\frac{3\sqrt{3}r}{r}}=6r\sqrt[3]{\left(\sqrt{3}\right)^3}=6\sqrt{3}r$$

Equality holds for $a=b=c$.

Back to the problem:

$$w_a\geq h_a\Rightarrow\sum_{cyc}w_a\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)\geq\sum_{cyc}h_a\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)\overset{Lemma}{\Rightarrow}\geq 6\sqrt{3}r$$

$$m_a\geq h_a\Rightarrow\sum_{cyc}m_a\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)\geq\sum_{cyc}h_a\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)\overset{Lemma}{\Rightarrow}\geq 6\sqrt{3}r$$

$$\min\left(\sum_{cyc}w_a\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right),\sum_{cyc}m_a\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)\right)\geq 6\sqrt{3}r$$

Equality holds for $a=b=c$.

1685. In ΔABC the following relationship holds:

$$(h_b+h_c)\cdot\tan\frac{A}{2}+(h_c+h_a)\cdot\tan\frac{B}{2}+(h_a+h_b)\cdot\tan\frac{C}{2}\geq 6\sqrt{3}r$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Ertan Yildirim-Turkiye

Lemma: $h_a+h_b+h_c\geq 9r$

Let $a\geq b\geq c$ then $h_a\leq h_b\leq h_c$ and $h_a+h_b\leq h_a+h_c\leq h_b+h_c$

$$\tan\frac{A}{2}\geq\tan\frac{B}{2}\geq\tan\frac{C}{2}$$

$$\therefore\sum(h_b+h_c)\tan\frac{A}{2}\overset{Chebyshev}{\geq}\frac{1}{3}\cdot 2(h_a+h_b+h_c)\cdot\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)$$

$$\overset{Lemma}{\geq}\frac{1}{3}\cdot 2\cdot 9r\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)=6r\cdot\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)$$

$f(x)=\tan x\Rightarrow f'(x)=\sec^2 x\Rightarrow f''(x)=2\sec^2 x\cdot\tan x>0$ where $x\in\left(0,\frac{\pi}{2}\right)$

Therefore $f(x)$ is convex

$$\frac{f\left(\frac{A}{2}\right)+f\left(\frac{B}{2}\right)+f\left(\frac{C}{2}\right)}{3}\geq f\left(\frac{A+B+C}{6}\right)$$

$$\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\geq 3\cdot f\left(\frac{\pi}{6}\right)=3\cdot\tan\frac{\pi}{6}=3\frac{\sqrt{3}}{3}=\sqrt{3}$$

$$\therefore\sum(h_b+h_c)\tan\frac{A}{2}\geq 6r\cdot\sum\tan\frac{A}{2}\geq 6r\cdot\sqrt{3}=6\sqrt{3}r$$

1686. In ΔABC the following relationship holds:

$$a\left(cot\frac{B}{2}+cot\frac{C}{2}\right)+b\left(cot\frac{C}{2}+cot\frac{A}{2}\right)+c\left(cot\frac{A}{2}+cot\frac{B}{2}\right)\geq 36r$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$a\left(cot\frac{B}{2}+cot\frac{C}{2}\right)+b\left(cot\frac{C}{2}+cot\frac{A}{2}\right)+c\left(cot\frac{A}{2}+cot\frac{B}{2}\right)=\sum_{cyc}a\left(cot\frac{B}{2}+cot\frac{C}{2}\right)=$$

$$=\sum_{cyc}a\left(\frac{cos\frac{B}{2}}{sin\frac{B}{2}}+\frac{cos\frac{C}{2}}{sin\frac{C}{2}}\right)=\sum_{cyc}a\cdot\frac{cos\frac{B}{2}sin\frac{C}{2}+cos\frac{C}{2}sin\frac{B}{2}}{sin\frac{B}{2}sin\frac{C}{2}}=$$

$$=\sum_{cyc}a\cdot\frac{sin\left(\frac{B+C}{2}\right)}{sin\frac{B}{2}sin\frac{C}{2}}=\sum_{cyc}a\cdot\frac{sin\left(\frac{\pi-A}{2}\right)}{sin\frac{B}{2}sin\frac{C}{2}}=\sum_{cyc}a\cdot\frac{cos\frac{A}{2}}{sin\frac{B}{2}sin\frac{C}{2}}\overset{AM-GM}{\geq}$$

$$\geq 3\cdot\sqrt[3]{abc\cdot\prod_{cyc}cos\frac{A}{2}\cdot\left(\prod_{cyc}sin\frac{A}{2}\right)^{-2}}=3\cdot\sqrt[3]{abc\cdot\frac{s}{4R}\cdot\left(\frac{r}{4R}\right)^{-2}}=$$

$$=3\cdot\sqrt[3]{abc\cdot\frac{16R^2s}{4Rr^2}}=3\cdot\sqrt[3]{4Rrs\cdot\frac{16R^2s}{4Rr^2}}=3\cdot\sqrt[3]{4Rrs\cdot\frac{4Rs}{r^2}}=$$

$$=3\cdot\sqrt[3]{\frac{16R^2s^2}{r}}\overset{EULER}{\geq}3\cdot\sqrt[3]{\frac{64r^2s^2}{r}}=12\cdot\sqrt[3]{rs^2}\overset{MITRINOVIC}{\geq}12\cdot\sqrt[3]{27r^3}=36r$$

Equality holds for $a=b=c$.

1687. In ΔABC the following relationship holds:

$$r_a\left(tan\frac{B}{2}+tan\frac{C}{2}\right)+r_b\left(tan\frac{C}{2}+tan\frac{A}{2}\right)+r_c\left(tan\frac{A}{2}+tan\frac{B}{2}\right)\ge 6\sqrt{3}r$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$r_a\left(tan\frac{B}{2}+tan\frac{C}{2}\right)+r_b\left(tan\frac{C}{2}+tan\frac{A}{2}\right)+r_c\left(tan\frac{A}{2}+tan\frac{B}{2}\right)=$$

$$=\sum_{cyc} r_a\left(tan\frac{B}{2}+tan\frac{C}{2}\right)=\sum_{cyc} r_a\left(\frac{sin\frac{B}{2}}{cos\frac{B}{2}}+\frac{sin\frac{C}{2}}{cos\frac{C}{2}}\right)=\sum_{cyc} r_a\cdot\frac{sin\left(\frac{B+C}{2}\right)}{cos\frac{B}{2}cos\frac{C}{2}}=$$

$$=\sum_{cyc} r_a\cdot\frac{sin\left(\frac{\pi-A}{2}\right)}{cos\frac{B}{2}cos\frac{C}{2}}=\sum_{cyc} r_a\cdot\frac{cos\frac{A}{2}}{cos\frac{B}{2}cos\frac{C}{2}}\overset{AM-GM}{\ge}$$

$$=3\cdot\sqrt[3]{\prod_{cyc} r_a\cdot\prod_{cyc}\frac{1}{cos\frac{A}{2}}}=3\cdot\sqrt[3]{rs^2\cdot\frac{4R}{s}}\overset{EULER}{\ge}3\cdot\sqrt[3]{8r^2s}\overset{MITRINOVIC}{\ge}$$

$$\ge 6\cdot\sqrt[3]{r^2\cdot 3\sqrt{3}r}=6\cdot\sqrt[3]{\left(\sqrt{3}\cdot r\right)^3}=6\sqrt{3}r$$

Equality holds for $a=b=c$.

1688. In ΔABC the following relationship holds:

$$(b+c)cot\frac{A}{2}+(c+a)cot\frac{B}{2}+(a+b)cot\frac{C}{2}\ge 36r$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$(b+c)cot\frac{A}{2}+(c+a)cot\frac{B}{2}+(a+b)cot\frac{C}{2}=\sum_{cyc}(a+b)cot\frac{C}{2}\overset{AM-GM}{\ge}$$

$$\ge 3\cdot\sqrt[3]{\prod_{cyc}(a+b)\cdot\prod_{cyc}cot\frac{A}{2}}\overset{CESARO}{\ge}3\cdot\sqrt[3]{8abc\cdot\frac{s}{r}}=$$

$$=3\cdot\sqrt[3]{32Rrs\cdot\frac{s}{r}}\overset{EULER}{\ge}3\cdot\sqrt[3]{64rs^2}=12\sqrt[3]{rs^2}\overset{MITRINOVIC}{\ge}12\sqrt[3]{27r^3}=36r$$

Equality holds for $a = b = c$.

1689. In ΔABC the following relationship holds:

$$h_a\left(cot\frac{B}{2}+cot\frac{C}{2}\right)+h_b\left(cot\frac{C}{2}+cot\frac{A}{2}\right)+h_c\left(cot\frac{A}{2}+cot\frac{B}{2}\right)\geq 18\sqrt{3}r$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$h_a\left(cot\frac{B}{2}+cot\frac{C}{2}\right)+h_b\left(cot\frac{C}{2}+cot\frac{A}{2}\right)+h_c\left(cot\frac{A}{2}+cot\frac{B}{2}\right)=$$

$$=\sum_{cyc}h_a\left(cot\frac{B}{2}+cot\frac{C}{2}\right)=\sum_{cyc}\frac{2F}{a}\left(\frac{cos\frac{B}{2}}{sin\frac{B}{2}}+\frac{cos\frac{C}{2}}{sin\frac{C}{2}}\right)=$$

$$=2F\sum_{cyc}\frac{1}{a}\cdot\frac{sin\left(\frac{B+C}{2}\right)}{sin\frac{B}{2}sin\frac{C}{2}}=2F\sum_{cyc}\frac{1}{a}\cdot\frac{sin\left(\frac{\pi-A}{2}\right)}{sin\frac{B}{2}sin\frac{C}{2}}=2F\sum_{cyc}\frac{1}{a}\cdot\frac{cos\frac{A}{2}}{sin\frac{B}{2}sin\frac{C}{2}}\geq$$

$$\overset{AM-GM}{\geq}2F\cdot 3\sqrt[3]{\frac{1}{abc}\cdot\prod_{cyc}cos\frac{A}{2}\cdot\left(\prod_{cyc}sin\frac{A}{2}\right)^{-2}}=$$

$$=6F\sqrt[3]{\frac{1}{4Rrs}\cdot\frac{s}{4R}\cdot\left(\frac{4R}{r}\right)^2}=6F\sqrt[3]{\frac{1}{16R^2r}\cdot\frac{16R^2}{r^2}}=6rs\cdot\sqrt[3]{\frac{1}{r^3}}=6s\overset{MITRINOVIC}{\geq}6\cdot 3\sqrt{3}r$$

$$=18\sqrt{3}r$$

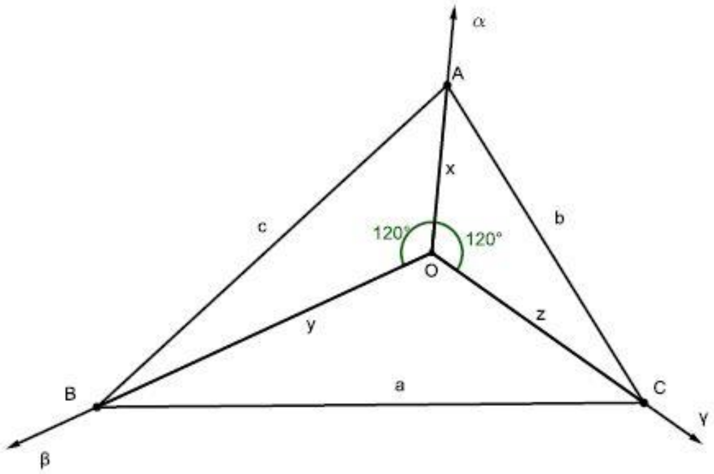
Equality holds for $a = b = c$.

1690. Let O, G —be the Toricelli's point and the centroid in ΔABC. Prove that:

$$OG^2=\frac{a^2+b^2+c^2-4\sqrt{3}F}{18}$$

Proposed by Adil Abdullayev-Azerbaijan

Solution by Daniel Sitaru-Romania



$\boldsymbol{O}$ –Toricelli's point

$$OA = x, OB = y, OC = z,$$
$$\measuredangle AOB = \measuredangle BOC = \measuredangle COA = 120^\circ$$
$$[AOB] + [BOC] + [COA] = \sum_{cyc} \frac{1}{2} \cdot xysin120^\circ$$
$$F = \frac{1}{2}sin(180^\circ - 60^\circ)\sum_{cyc} xy$$
$$\sum_{cyc} xy = \frac{2F}{sin60^\circ} = \frac{4F}{\sqrt{3}} = \frac{4\sqrt{3}F}{3}$$
$$\sum_{cyc} c^2 = \sum_{cyc}(x^2 + y^2 - 2xycos120^\circ)$$
$$a^2 + b^2 + c^2 = 2\sum_{cyc} x^2 + 2cos60^\circ \sum_{cyc} xy$$
$$2\sum_{cyc} x^2 = a^2 + b^2 + c^2 - 2 \cdot \frac{1}{2} \cdot \frac{4\sqrt{3}F}{3}$$
$$\sum_{cyc} OA^2 = \frac{1}{2}(a^2 + b^2 + c^2) - \frac{2\sqrt{3}F}{3}$$
$$3OG^2 + \sum_{cyc} GA^2 = \frac{1}{2}(a^2 + b^2 + c^2) - \frac{2\sqrt{3}F}{3}$$
$$3OG^2 = \frac{1}{2}(a^2 + b^2 + c^2) - \frac{2\sqrt{3}F}{3} - \sum_{cyc} \frac{4}{9}m_a^2$$
$$3OG^2 = \frac{1}{2}(a^2 + b^2 + c^2) - \frac{2\sqrt{3}F}{3} - \frac{4}{9} \cdot \frac{3}{4}(a^2 + b^2 + c^2)$$

$$OG^2=\frac{1}{6}(a^2+b^2+c^2)-\frac{1}{9}(a^2+b^2+c^2)-\frac{2\sqrt{3}F}{9}$$

$$OG^2=\frac{a^2+b^2+c^2-4\sqrt{3}F}{18}$$

Observation: This is a new proof for IONESCU-WEITZENBOCK'S inequality:

$$0\le OG^2=\frac{a^2+b^2+c^2-4\sqrt{3}F}{18}\Rightarrow 0\le a^2+b^2+c^2-4\sqrt{3}F\Rightarrow$$

$$a^2+b^2+c^2\ge 4\sqrt{3}F$$

1691. In any Δ ABC, the following relationship holds :

$$\frac{m_a^2+m_b^2+m_c^2}{m_am_b+m_bm_c+m_cm_a}+\frac{12\sqrt{3}F}{a^2+b^2+c^2}\le 4$$

Proposed by Adil Abdullayev-Azerbaijan

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{We shall prove : }\frac{a^2+b^2+c^2}{ab+bc+ca}+\frac{9\sqrt{3}F}{m_a^2+m_b^2+m_c^2}\le 4\to(1)$$

$$\text{LHS of }(1)\overset{\text{Mitrinovic}}{\le}\frac{2(s^2-4Rr-r^2)}{s^2+4Rr+r^2}+\frac{3s^2}{\frac{3}{2}(s^2-4Rr-r^2)}\overset{?}{\le}4$$

$$\Leftrightarrow 2-\frac{s^2-4Rr-r^2}{s^2+4Rr+r^2}\overset{?}{\ge}\frac{s^2}{s^2-4Rr-r^2}\Leftrightarrow\frac{s^2+12Rr+3r^2}{s^2+4Rr+r^2}\overset{?}{\ge}\frac{s^2}{s^2-4Rr-r^2}$$

$$\Leftrightarrow(4R+r)s^2\overset{?}{\ge}3r(4R+r)^2\Leftrightarrow s^2-12Rr-3r^2\overset{?}{\ge}0$$

$$\Leftrightarrow s^2-16Rr+5r^2+4r(R-2r)\overset{?}{\ge}0\to\text{true}\because s^2-16Rr+5r^2\overset{\text{Gerretsen}}{\ge}0$$

$$\text{and }R-2r\overset{\text{Euler}}{\ge}0\therefore\frac{a^2+b^2+c^2}{ab+bc+ca}+\frac{9\sqrt{3}F}{m_a^2+m_b^2+m_c^2}\le 4\text{ and implementing }(1)$$

$$\text{on a triangle with sides }\frac{2m_a}{3},\frac{2m_b}{3},\frac{2m_c}{3}\text{ whose area}$$

$$\text{via elementary calculations}=\frac{F}{3}\text{ and medians}=\frac{a}{2},\frac{b}{2},\frac{c}{2}\text{ we get :}$$

$$\frac{\frac{4}{9}(m_a^2+m_b^2+m_c^2)}{\frac{4}{9}(m_am_b+m_bm_c+m_cm_a)}+\frac{\frac{9\sqrt{3}F}{3}}{\frac{1}{4}(a^2+b^2+c^2)}\le 4$$

$$\Rightarrow \frac{m_a^2+m_b^2+m_c^2}{m_am_b+m_bm_c+m_cm_a}+\frac{12\sqrt{3}F}{a^2+b^2+c^2}\le 4$$

$\forall\,\Delta\, ABC,''='' \text{ iff } \Delta\, ABC$ is equilateral (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have :

$$m_am_b+m_bm_c+m_cm_a \overset{AM-GM}{\geq} 3\sqrt[3]{(m_am_bm_c)^2} \overset{Leuenberger}{\geq} 3\sqrt[3]{(s^2r)^2} \overset{Mitrinovic}{\geq} 3\sqrt{3}F,$$

and since $a^2+b^2+c^2=\frac{4}{3}({m_a}^2+{m_b}^2+{m_c}^2)$, then it suffices to prove that

$$\frac{{m_a}^2+{m_b}^2+{m_c}^2}{m_am_b+m_bm_c+m_cm_a}+\frac{3(m_am_b+m_bm_c+m_cm_a)}{{m_a}^2+{m_b}^2+{m_c}^2}\le 4 \qquad (1)$$

$$\text{Let } x:=\frac{{m_a}^2+{m_b}^2+{m_c}^2}{m_am_b+m_bm_c+m_cm_a}.$$

Since m_a, m_b, m_c can be the sides of a triangle, we have $1\le x<2$.

$$\Rightarrow\ (1)\ \Leftrightarrow x+\frac{3}{x}\le 4\ \Leftrightarrow\ \frac{(x-1)(3-x)}{x}\ge 0,$$

which is true. So the proof is complete. Equality holds iff ΔABC is equilateral.

1692. In any Δ ABC, the following relationship holds :

$$\frac{6(a^2b^2+b^2c^2+c^2a^2)}{(a^2+b^2+c^2)^2}\ge 1+\frac{4\sqrt{3}F}{a^2+b^2+c^2}$$

Proposed by Adil Abdullayev-Azerbaijan

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{6(a^2b^2+b^2c^2+c^2a^2)}{(a^2+b^2+c^2)^2}-1$$
$$=\frac{3\left((s^2+4Rr+r^2)^2-16Rrs^2\right)-2(s^2-4Rr-r^2)^2}{2(s^2-4Rr-r^2)^2}$$
$$=\frac{s^4-(8Rr-10r^2)s^2+r^2(4R+r)^2}{2(s^2-4Rr-r^2)^2}\ge\frac{4\sqrt{3}F}{a^2+b^2+c^2}=\frac{2\sqrt{3}rs}{s^2-4Rr-r^2}$$

$$\Leftrightarrow \frac{\left(s^4-(8Rr-10r^2)s^2+r^2(4R+r)^2\right)^2}{4(s^2-4Rr-r^2)^4} \ge \frac{12r^2s^2}{(s^2-4Rr-r^2)^2}$$

$$\Leftrightarrow \left(s^4-(8Rr-10r^2)s^2+r^2(4R+r)^2\right)^2 \overset{(*)}{\boxed{\ge}} 48r^2s^2\left(s^2-4Rr-r^2\right)^2 \text{ and}$$

$$\because \left(s^2-16Rr+5r^2\right)^4 \overset{\text{Gerretsen}}{\ge} 0 \therefore \text{in order to prove } (*), \text{it suffices to prove :}$$

$$\text{LHS of } (*) \ge \left(s^2-16Rr+5r^2\right)^4$$

$$\Leftrightarrow (R-r)s^6-r(30R^2-25Rr-r^2)s^4+r^2(336R^3-332R^2r+95Rr^2-11r^3)s^2$$

$$-r^3(1360R^4-1712R^3r+798R^2r^2-167Rr^3+13r^4) \overset{(**)}{\boxed{\ge}} 0$$

$$\text{and} \because (R-r)\left(s^2-16Rr+5r^2\right)^3 \overset{\text{Gerretsen}}{\ge} 0 \therefore \text{in order to prove } (**),$$

$$\text{it suffices to prove : LHS of } (**) \ge (R-r)\left(s^2-16Rr+5r^2\right)^3$$

$$\Leftrightarrow (9R^2-19Rr+8r^2)s^4-r(216R^3-458R^2r+230Rr^2-32r^3)s^2$$

$$+r^2(1368R^4-3112R^3r+2121R^2r^2-579Rr^3+56r^4) \overset{(***)}{\boxed{\ge}} 0 \text{ and}$$

$$\because (9R^2-19Rr+8r^2)\left(s^2-16Rr+5r^2\right)^2 \overset{\text{Gerretsen}}{\ge} 0 \therefore \text{in order to prove } (***),$$

$$\text{it suffices to prove : LHS of } (***) \ge (9R^2-19Rr+8r^2)\left(s^2-16Rr+5r^2\right)^2$$

$$\Leftrightarrow (3R^3-10R^2r+9Rr^2-2r^3)s^2 \ge r(39R^4-133R^3r+133R^2r^2-49Rr^3+6r^4)$$

$$\Leftrightarrow (R-2r)(3R^2-4Rr+r^2)s^2 \ge r(R-2r)(39R^3-55R^2r+23Rr^2-3r^3)$$

$$\Leftrightarrow (3R^2-4Rr+r^2)s^2 \overset{(****)}{\boxed{\ge}} r(39R^3-55R^2r+23Rr^2-3r^3) \left(\because R-2r \overset{\text{Euler}}{\ge} 0\right)$$

$$\text{Now, } (3R^2-4Rr+r^2)s^2 \overset{\text{Gerretsen}}{\ge} (3R^2-4Rr+r^2)(16Rr-5r^2) \overset{?}{\ge}$$

$$r(39R^3-55R^2r+23Rr^2-3r^3) \Leftrightarrow 9t^3-24t^2+13t-2 \overset{?}{\ge} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(3t-1)^2 \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (****) \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{6(a^2b^2+b^2c^2+c^2a^2)}{(a^2+b^2+c^2)^2} \ge 1+\frac{4\sqrt{3}F}{a^2+b^2+c^2}$$

$$\forall \Delta ABC, ''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since

$$16F^2 = 2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4),$$

then the given inequality is equivalent to

$$\frac{3[2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)]}{2(a^2+b^2+c^2)^2}+\frac{1}{2} \ge \frac{4\sqrt{3}F}{a^2+b^2+c^2}$$

$$\text{or } \frac{24F^2}{(a^2+b^2+c^2)^2}+\frac{1}{2} \ge \frac{4\sqrt{3}F}{a^2+b^2+c^2} \text{ or } \frac{1}{2}\left(\frac{4\sqrt{3}F}{a^2+b^2+c^2}-1\right)^2 \ge 0,$$

which is true. Equality holds iff $a^2+b^2+c^2=4\sqrt{3}F$ or ΔABC is equilateral.

1693. In ΔABC the following relationship holds:

$$\frac{18R^2}{a^2+b^2+c^2}\ge 1+\frac{4\sqrt{3}F}{a^2+b^2+c^2}$$

Proposed by Adil Abdullyev-Azerbaijan

Solution by Daniel Sitaru-Romania

$$\frac{18R^2}{a^2+b^2+c^2}\ge 1+\frac{4\sqrt{3}F}{a^2+b^2+c^2}\Leftrightarrow a^2+b^2+c^2+4\sqrt{3}F\le 18R^2$$

Nakajima(1925): In ΔABC the following relationship holds:

$$a^2+b^2+c^2\le 8R^2+\frac{4}{3\sqrt{3}}F$$

Proof:

$$a^2+b^2+c^2=(a+b+c)^2-2(ab+bc+ca)=$$
$$=4s^2-2(s^2+r^2+4Rr)=2s^2-2r^2-8Rr\overset{GERRETSEN}{\le}$$
$$\le 2(4R^2+4Rr+3r^2)-2r^2-8Rr=8R^2+4r^2=8R^2+4r\cdot r\le$$
$$\overset{MITRINOVIC}{\le}\ 8R^2+4r\cdot\frac{s}{3\sqrt{3}}=8R^2+\frac{4}{3\sqrt{3}}F$$

Back to the problem:

$$a^2+b^2+c^2+4\sqrt{3}F\overset{Nakajima}{\le}8R^2+\frac{4}{3\sqrt{3}}F+4\sqrt{3}F$$

Remains to prove:

$$8R^2+\frac{4}{3\sqrt{3}}F+4\sqrt{3}F\le 18R^2\Leftrightarrow\left(\frac{4}{3\sqrt{3}}+4\sqrt{3}\right)F\le 10R^2$$
$$\frac{40F}{3\sqrt{3}}\le 10R^2\Leftrightarrow 4F\le 3\sqrt{3}R^2\ (to\ prove)$$
$$4F=4rs\overset{EULER}{\le}2Rs\overset{MITRINOVIC}{\le}2R\cdot\frac{3\sqrt{3}R}{2}=3\sqrt{3}R^2$$

Equality holds for $a=b=c$.

1694. In any Δ ABC, the following relationship holds :

$$\frac{r_a^3+r_b^3+r_c^3}{r_ar_br_c}+5\ge\frac{4R}{r}$$

Proposed by Adil Abdullayev-Azerbaijan

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{4R}{r}=\frac{4R+r-\frac{rs^2}{s^2}}{\frac{rs^2}{s^2}}=\frac{\sum_{cyc} r_a-\frac{r_ar_br_c}{\sum_{cyc} r_ar_b}}{\frac{r_ar_br_c}{\sum_{cyc} r_ar_b}}=\frac{(\sum_{cyc} x)(\sum_{cyc} xy)-xyz}{xyz}$$

$$(x=r_a, y=r_b, z=r_c)\overset{?}{\leq}\frac{r_a^3+r_b^3+r_c^3}{r_ar_br_c}+5=\frac{\sum_{cyc} x^3+5xyz}{xyz}$$

$$\Leftrightarrow \sum_{cyc} x^3+5xyz\overset{?}{\geq}\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right)-xyz=\sum_{cyc} x^2y+\sum_{cyc} xy^2+2xyz$$

$$\Leftrightarrow \sum_{cyc} x^3+3xyz\overset{?}{\geq}\sum_{cyc} x^2y+\sum_{cyc} xy^2 \to \text{true via Schur}$$

$$\therefore \frac{r_a^3+r_b^3+r_c^3}{r_ar_br_c}+5\geq\frac{4R}{r}\ \forall\ \Delta\ ABC,''=''\ \text{iff}\ \Delta\ ABC \text{ is equilateral (QED)}$$

Solution 2 by Tapas Das-India

$$\sum r_a^3=\left(\sum r_a\right)^3-3\left[\left(\sum r_a\right)\left(\sum r_ar_b\right)-\prod r_a\right]=(4R+r)^3-12s^2R.$$

$$Now\ \frac{\sum r_a^3}{r_ar_br_c}+5=\frac{(4R+r)^3-12s^2R}{s^2r}+5\overset{Gerrestn}{\geq}\frac{(4R+r)^3}{(4R^2+4Rr+3r^2)r}-\frac{12R}{r}+5,$$

we need to show

$$\frac{(4R+r)^3}{(4R^2+4Rr+3r^2)r}-\frac{12R}{r}+5\geq\frac{4R}{r}\ or\ \frac{(4R+r)^3}{(4R^2+4Rr+3r^2)r}+5\geq\frac{16R}{r}$$

$$or\ \frac{(4x+1)^3}{4x^2+4x+3}+5\geq 16x\ \left(where\ \frac{R}{r}=x\geq 2\ Euler\right) or$$

$$4x^2-16x+16\geq 0\ or\ (2x-4)^2\geq 0\ True$$

1695. In ΔABC the following relationship holds:

$$2\sum_{cyc}\frac{m_a}{h_a}\geq\frac{2s}{\sqrt[3]{abc}}+\frac{1}{\sqrt[3]{h_ah_bh_c}}\sum_{cyc} h_a$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Tereshin and AM − GM inequalities, we have

$$2\sum_{cyc}\frac{m_a}{h_a}\ge 2\sum_{cyc}\frac{b^2+c^2}{4Rh_a}=\sum_{cyc}\frac{b^2+c^2}{bc}=\sum_{cyc}\frac{a}{b}+\sum_{cyc}\frac{h_a}{h_b}=$$

$$=\sum_{cyc}\frac{1}{3}\left(\frac{a}{b}+\frac{a}{b}+\frac{b}{c}\right)+\sum_{cyc}\frac{1}{3}\left(\frac{h_a}{h_b}+\frac{h_a}{h_b}+\frac{h_b}{h_c}\right)\ge\sum_{cyc}\sqrt[3]{\frac{a^2}{bc}}+\sum_{cyc}\sqrt[3]{\frac{{h_a}^2}{h_bh_c}}=$$

$$=\frac{2s}{\sqrt[3]{abc}}+\frac{1}{\sqrt[3]{h_ah_bh_c}}\sum_{cyc}h_a.$$

Equality holds iff ΔABC is equilateral.

1696. In ΔABC the following relationship holds:

$$\left(1+e^{\tan\frac{A}{2}}\right)\left(1+e^{\tan\frac{B}{2}}\right)\left(1+e^{\tan\frac{C}{2}}\right)\ge\left(1+e^{\frac{\sqrt{3}}{3}}\right)^3$$

Proposed by Khaled Abd Imouti-Syria

Solution by Tapas Das-India

$$\left(1+e^{\tan\frac{A}{2}}\right)\left(1+e^{\tan\frac{B}{2}}\right)\left(1+e^{\tan\frac{C}{2}}\right)\overset{Holder}{\ge}$$

$$\ge\left((1\cdot1\cdot1)^{\frac{1}{3}}+\left(e^{\sum\tan\frac{A}{2}}\right)^{\frac{1}{3}}\right)^3=\left(1+e^{\frac{4R+r}{3s}}\right)^3\ge\left(1+e^{\frac{\sqrt{3}}{3}}\right)^3$$

$$\left(since\ s^2\le(4R+r)^2\frac{1}{3}\right)-Doucet's\ inequality$$

Equality holds for $a=b=c$.

1697. In ΔABC the following relationship holds:

$$(a^n+b^n+c^n)(a^n\cos^2A+b^n\cos^2B+c^n\cos^2C)\ge 4F^2.\sum_{cyc}b^{n-2}c^{n-2},\ \ n\in\mathbb{N}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let H be the orthocenter of ΔABC. We have $\left(a^n.\overrightarrow{HA}+b^n.\overrightarrow{HB}+c^n.\overrightarrow{HC}\right)^2\ge 0$, then

$$a^{2n}.HA^2+b^{2n}.HB^2+c^{2n}.HC^2+a^nb^n.2\overrightarrow{HA}.\overrightarrow{HB}+b^nc^n.2\overrightarrow{HB}.\overrightarrow{HC}+c^na^n.2\overrightarrow{HC}.\overrightarrow{HA}\geq 0$$

$$\Leftrightarrow a^{2n}.HA^2+b^{2n}.HB^2+c^{2n}.HC^2+a^nb^n.\left(HA^2+HB^2-c^2\right)+b^nc^n.\left(HB^2+HC^2-a^2\right)+$$

$$+c^na^n.\left(HC^2+HA^2-b^2\right)\geq 0$$

$$\Leftrightarrow (a^n+b^n+c^n)\left(a^nHA^2+b^nHB^2+c^nHC^2\right)\geq a^2b^2c^2\left(a^{n-2}b^{n-2}+b^{n-2}c^{n-2}+c^{n-2}a^{n-2}\right)$$

Since $HA=2R|\cos A|$ (and analogs) and $abc=4RF$, then

$$(a^n+b^n+c^n)\left(a^n\cos^2 A+b^n\cos^2 B+c^n\cos^2 C\right)\geq 4F^2\left(a^{n-2}b^{n-2}+b^{n-2}c^{n-2}+c^{n-2}a^{n-2}\right),$$

as desired. Equality holds iff ΔABC is equilateral.

1698. **In ΔABC the following relationship holds:**

$$\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\leq\frac{2}{3}\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum\frac{1}{\sin A}=2R\sum\frac{1}{a}=\frac{2R(ab+bc+ca)}{abc}=\frac{s^2+r^2+4Rr}{2rs}\ and$$

$$\frac{2}{3}\sum\cot\frac{A}{2}=\frac{2}{3}\frac{s}{r}$$

We need to show:

$$\frac{2s}{3r}\geq\frac{s^2+r^2+4Rr}{2rs}\ or\ 4s^2\geq 3s^2+3r^2+12Rr$$

$$or\ s^2\geq 3r^2+12Rr\ or\ 16Rr-5r^2\overset{Gerretsen}{\geq}12Rr+3r^2$$

$$or\ R\geq 2r\ (Euler)$$

1699.In any Δ ABC, the following relationship holds :

$$\cos A\cos B\cos C\leq\frac{1}{8}\cos(A-B)\cos(B-C)\cos(C-A)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\cos(A-B)\cos(B-C)\cos(C-A)$$
$$=\left(2\cos^2\frac{A-B}{2}-1\right)\left(2\cos^2\frac{B-C}{2}-1\right)\left(2\cos^2\frac{C-A}{2}-1\right)$$
$$=8\prod_{cyc}\cos^2\frac{B-C}{2}-4\left(\prod_{cyc}\cos^2\frac{B-C}{2}\right)\left(\sum_{cyc}\sec^2\frac{B-C}{2}\right)+2\sum_{cyc}\cos^2\frac{B-C}{2}-1\to(a)$$
$$\text{Now, }\sum_{cyc}\cos^2\frac{B-C}{2}=\sum_{cyc}\frac{(b+c)^2\sin^2\frac{A}{2}}{16R^2\sin^2\frac{A}{2}\cos^2\frac{A}{2}}=\frac{1}{16R^2s}\cdot\sum_{cyc}\frac{bc(b+c)^2}{s-a}$$
$$=\frac{1}{16R^2s}\cdot\sum_{cyc}\frac{bc(s+s-a)^2}{s-a}=\frac{1}{16R^2s}\cdot\sum_{cyc}\left(\frac{bcs^2}{s-a}+2sbc+bc(s-a)\right)$$
$$=\frac{1}{16R^2s}\left(s^3\sum_{cyc}\sec^2\frac{A}{2}+3s\sum_{cyc}ab-3abc\right)$$
$$=\frac{1}{16R^2s}\left(s^3\left(\frac{s^2+(4R+r)^2}{s^2}\right)+3s\left(s^2+4Rr+r^2\right)-12Rrs\right)$$
$$=\frac{4s^2+(4R+r)^2+3r^2}{16R^2}\Rightarrow\sum_{cyc}\cos^2\frac{B-C}{2}=\frac{4s^2+(4R+r)^2+3r^2}{16R^2}\to(1)$$
$$\text{Again, }\sum_{cyc}\sec^2\frac{B-C}{2}=\sum_{cyc}\frac{16R^2\sin^2\frac{A}{2}\cos^2\frac{A}{2}}{(b+c)^2\sin^2\frac{A}{2}}=\sum_{cyc}\frac{16R^2s(s-a)a}{4Rrs(b+c)^2}$$
$$=\frac{2R}{r}\sum_{cyc}\frac{a(b+c-a)}{(b+c)^2}=\frac{2R}{r}\left(\sum_{cyc}\frac{a}{b+c}-\sum_{cyc}\frac{a^2}{(b+c)^2}\right)\to(2)$$
$$\text{Now, }\sum_{cyc}\frac{a}{b+c}=\frac{\sum_{cyc}(a(c+a)(a+b))}{\prod_{cyc}(b+c)}=\frac{\sum_{cyc}\left(a\left(\sum_{cyc}ab+a^2\right)\right)}{2s(s^2+2Rr+r^2)}$$
$$=\frac{2s(s^2+4Rr+r^2)+2s(s^2-6Rr-3r^2)}{2s(s^2+2Rr+r^2)}=\frac{2s^2-2Rr-2r^2}{s^2+2Rr+r^2}\to(3)$$
$$\text{and, }\sum_{cyc}\frac{a^2}{(b+c)^2}=\sum_{cyc}\frac{\left(2s-(b+c)\right)^2}{(b+c)^2}=\sum_{cyc}\frac{4s^2-4s(b+c)+(b+c)^2}{(b+c)^2}$$
$$\overset{(i)}{=}4s^2\left(\frac{\sum_{cyc}((c+a)^2(a+b)^2)}{\left(\prod_{cyc}(b+c)\right)^2}\right)-4s\left(\frac{\sum_{cyc}(c+a)(a+b)}{\prod_{cyc}(b+c)}\right)+3$$
$$\text{We have : }\sum_{cyc}((c+a)^2(a+b)^2)=\sum_{cyc}\left(\sum_{cyc}ab+a^2\right)^2$$
$$=\sum_{cyc}\left(\left(\sum_{cyc}ab\right)^2+2a^2\sum_{cyc}ab+a^4\right)$$

$$= 3\left(\sum_{cyc} ab\right)^2 + 2\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2\right) + \left(\sum_{cyc} a^2\right)^2 - 2\sum_{cyc} a^2b^2$$

$$= \left(\sum_{cyc} ab\right)^2 + 2\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2\right) + \left(\sum_{cyc} a^2\right)^2 + 2\sum_{cyc} a^2b^2 + 4abc(2s)$$

$$-2\sum_{cyc} a^2b^2 = \left(\sum_{cyc} ab + \sum_{cyc} a^2\right)^2 + 32Rrs^2 = \left(3s^2 - 4Rr - r^2\right)^2 + 32Rrs^2 \to (ii)$$

$$\text{Again, } \sum_{cyc}(c+a)(a+b) = \sum_{cyc}\left(\sum_{cyc} ab + a^2\right) = 3\sum_{cyc} ab + \sum_{cyc} a^2$$

$$= \sum_{cyc} a^2 + 2\sum_{cyc} ab + \sum ab = 4s^2 + s^2 + 4Rr + r^2$$

$$\therefore \sum_{cyc}(c+a)(a+b) = 5s^2 + 4Rr + r^2 \to (iii)$$

$$\because \prod_{cyc}(b+c) = s^2 + 2Rr + r^2 \therefore (i), (ii), (iii) \Rightarrow \sum_{cyc} \frac{a^2}{(b+c)^2} =$$

$$\frac{4s^2\left(\left(3s^2 - 4Rr - r^2\right)^2 + 32Rrs^2\right)}{4s^2(s^2 + 2Rr + r^2)^2} - \frac{4s\left(5s^2 + 4Rr + r^2\right)}{2s\left(s^2 + 2Rr + r^2\right)} + 3$$

$$= \frac{\left(3s^2 - 4Rr - r^2\right)^2 + 32Rrs^2 - 2\left(5s^2 + 4Rr + r^2\right)\left(s^2 + 2Rr + r^2\right) + 3\left(s^2 + 2Rr + r^2\right)^2}{(s^2 + 2Rr + r^2)^2}$$

$$= \frac{2s^4 - s^2\left(8Rr + 12r^2\right) + 12R^2r^2 + 8Rr^3 + 2r^4}{(s^2 + 2Rr + r^2)^2} \Rightarrow$$

$$\sum_{cyc} \frac{a^2}{(b+c)^2} = \frac{2s^4 - s^2\left(8Rr + 12r^2\right) + 12R^2r^2 + 8Rr^3 + 2r^4}{(s^2 + 2Rr + r^2)^2} \to (4);$$

$$(2), (3), (4) \Rightarrow \sum \sec^2\frac{B-C}{2} =$$

$$\frac{2R}{r}\left(\frac{2s^2 - 2Rr - 2r^2}{s^2 + 2Rr + r^2} - \frac{2s^4 - s^2\left(8Rr + 12r^2\right) + 12R^2r^2 + 8Rr^3 + 2r^4}{(s^2 + 2Rr + r^2)^2}\right)$$

$$= \frac{2R}{r}\left(\frac{\left(2s^2 - 2Rr - 2r^2\right)\left(s^2 + 2Rr + r^2\right) - \left(2s^4 - s^2\left(8Rr + 12r^2\right) + 12R^2r^2 + 8Rr^3 + 2r^4\right)}{(s^2 + 2Rr + r^2)^2}\right)$$

$\to (5)$

$$\text{Also, } 8\prod_{cyc}\cos^2\frac{B-C}{2} = 8\prod_{cyc}\frac{(b+c)^2\sin^2\frac{A}{2}}{a^2} = 8\left(\frac{4s^2\left(s^2 + 2Rr + r^2\right)^2}{16R^2r^2s^2}\right)\left(\frac{r^2}{16R^2}\right)$$

$$= \frac{\left(s^2 + 2Rr + r^2\right)^2}{8R^4} \to (6) \therefore (a), (1), (5), (6) \Rightarrow \cos(A-B)\cos(B-C)\cos(C-A)$$

$$= \frac{\left(s^2 + 2Rr + r^2\right)^2}{8R^4}$$

$$-\left(\frac{\left(s^2+2Rr+r^2\right)^2}{16R^4}\right)\cdot\frac{2R}{r}\left(\frac{\begin{matrix}\left(2s^2-2Rr-2r^2\right)\left(s^2+2Rr+r^2\right)-\\\left(2s^4-s^2\left(8Rr+12r^2\right)+12R^2r^2+8Rr^3+2r^4\right)\end{matrix}}{\left(s^2+2Rr+r^2\right)^2}\right)$$

$$+\frac{4s^2+(4R+r)^2+3r^2}{8R^2}-1$$

$$\Rightarrow \cos(A-B)\cos(B-C)\cos(C-A)$$

$$=\frac{r\left(s^2+2Rr+r^2\right)^2-R\sigma+R^2r\left(4s^2+(4R+r)^2+3r^2\right)-8R^4r}{8R^4r}$$

$$\left(\begin{matrix}\text{where } \sigma=\left(2s^2-2Rr-2r^2\right)\left(s^2+2Rr+r^2\right)-\\\left(2s^4-s^2\left(8Rr+12r^2\right)+12R^2r^2+8Rr^3+2r^4\right)\end{matrix}\right)$$

$$\geq 8\cos A\cos B\cos C=\frac{2\left(s^2-4R^2-4Rr-r^2\right)}{R^2}$$

$$\Leftrightarrow s^4-\left(22R^2+8Rr-2r^2\right)s^2+72R^4+88R^3r+38R^2r^2+8Rr^3+r^4 \boxed{\overset{(*)}{\geq}} 0 \text{ and}$$

$$\because \left(s^2-4R^2-4Rr-3r^2\right)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (*),$$

$$\text{it suffices to prove : LHS of } (*) \geq \left(s^2-4R^2-4Rr-3r^2\right)^2$$

$$\Leftrightarrow \left(7R^2-4r^2\right)s^2 \boxed{\overset{(**)}{\leq}} 28R^4+28R^3-R^2r^2-8Rr^3-4r^4$$

$$\text{Again, } \left(7R^2-4r^2\right)s^2 \overset{\text{Rouche}}{\leq} \left(7R^2-4r^2\right)\left(\begin{matrix}2R^2+10Rr-r^2\\+2(R-2r).\sqrt{R^2-2Rr}\end{matrix}\right)$$

$$\overset{?}{\leq} 28R^4+28R^3-R^2r^2-8Rr^3-4r^4$$

$$\Leftrightarrow \left(7R^2-4r^2\right)(R-2r).\sqrt{R^2-2Rr} \boxed{\underset{(***)}{\overset{?}{\leq}}} (R-2r)\left(7R^3-7R^2r-7Rr^2+2r^3\right) \text{ and}$$

$$\because R-2r \overset{\text{Euler}}{\geq} 0 \therefore \text{in order to prove } (***), \text{it suffices to prove :}$$

$$7R^3-7R^2r-7Rr^2+2r^3>\left(7R^2-4r^2\right).\sqrt{R^2-2Rr}$$

$$\Leftrightarrow \left(7R^3-7R^2r-7Rr^2+2r^3\right)^2>\left(R^2-2Rr\right)\left(7R^2-4r^2\right)^2$$

$$\Leftrightarrow r^2\left(7R^4+14R^3+5R^2r^2+4Rr^3+4r^4\right)>0 \rightarrow \textit{true} \Rightarrow (***) \Rightarrow (**) \Rightarrow (*)$$

$$\text{is true} \therefore \cos A\cos B\cos C \leq \frac{1}{8}\cos(A-B)\cos(B-C)\cos(C-A)$$

$$\forall\, \Delta\, ABC, ''='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

Solution 2 by Tapas Das-India

$$\textit{case}1$$

$$\textit{For acute triangle}, 2\sin A\cos(B-C)=\sin 2B+\sin 2C \geq 2\sqrt{\sin 2B\sin 2C}$$
$$=4\sqrt{\sin B\sin C\,\cos B\cos C},$$

$$\textit{now} \prod 2\sin A\cos(B-C) \geq 64\prod \sin A\prod \cos A \textit{ or}, \frac{1}{8}\prod \cos(B-C)$$
$$\geq \prod \cos A,$$

$$case2.\,for\ non\ acute\ \prod\cos A<0<\frac{1}{8}\prod\cos(B-C),\ equality\ A=B=C=\frac{\pi}{3}$$

1700. In any acute triangle ABC, the following relationship holds :

$$\frac{a}{b+c}\sqrt{\cos A}+\frac{b}{c+a}\sqrt{\cos B}+\frac{c}{a+b}\sqrt{\cos C}<\sqrt{2}$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$a\cos A+b\cos B+c\cos C=R(\sin 2A+\sin 2B+\sin 2C)$$
$$=R(2\sin A\cos A+2\sin A\cos(B-C))=2R\sin A\,(\cos(B-C)-\cos(B+C))$$
$$=4R\sin A\sin B\sin C=4R.\frac{4Rrs}{8R^3}\Rightarrow a\cos A+b\cos B+c\cos C=\frac{2rs}{R}\to(1)$$
$$\text{Now, } b+c=2s-a=s+s-a>s\Rightarrow b+c>s\ and\ analogs\to(2)$$
$$\therefore\frac{a}{b+c}\sqrt{\cos A}+\frac{b}{c+a}\sqrt{\cos B}+\frac{c}{a+b}\sqrt{\cos C}\overset{via\ (2)}{<}\frac{1}{s}\sum_{cyc}(\sqrt{a\cos A}.\sqrt{a})$$
$$\overset{CBS}{\le}\frac{1}{s}.\sqrt{\sum_{cyc}a\cos A}.\sqrt{\sum_{cyc}a}\overset{via\ (1)}{=}\frac{1}{s}.\sqrt{\frac{2rs}{R}}.\sqrt{2s}=\sqrt{2}.\sqrt{\frac{2r}{R}}\overset{Euler}{\le}\sqrt{2}$$
$$\therefore\frac{a}{b+c}\sqrt{\cos A}+\frac{b}{c+a}\sqrt{\cos B}+\frac{c}{a+b}\sqrt{\cos C}<\sqrt{2}\ \forall\ acute\ \Delta\ ABC\ (QED)$$

Solution 2 by Tapas Das-India

$$We\ know\ that\ in\ any\ triangle\ a+b>c\ or\ a+b>\frac{1}{2}(a+b+c),$$
$$similarly\ b+c>\frac{1}{2}(a+b+c)and\ c+a>\frac{1}{2}(a+b+c).$$
$$Now\ we\ have\sum\frac{a}{b+c}<\frac{2(a+b+c)}{a+b+c}=2\,.$$

Back to the main problem,

$$LHS\overset{Chebyshev}{\le}\frac{1}{3}\sum\frac{a}{b+c}\sum\sqrt{\cos A}\overset{CBS}{\le}\frac{1}{3}.2.\sqrt{3\sum\cos A}<\frac{2}{3}\sqrt{\frac{9}{2}}=\sqrt{2}$$
$$Note{:}\sum\cos A=1+\frac{r}{R}\le\frac{3}{2}(Euler)and\ WLOG\ a\ge b\ge c,$$
$$so\cos A\le\cos B\le\cos C\ and\frac{a}{b+c}\ge\frac{b}{c+a}\ge\frac{c}{a+b}$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru