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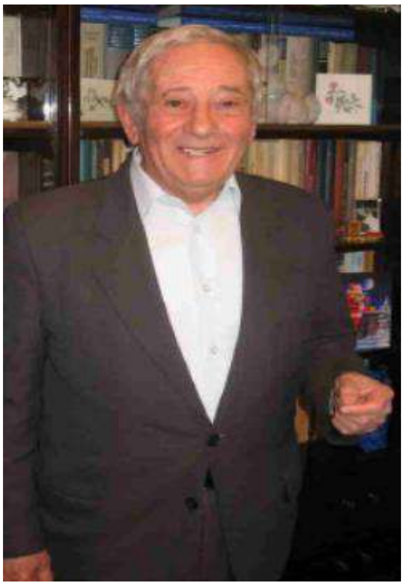
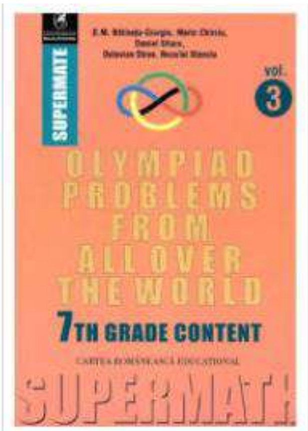
CONTENT

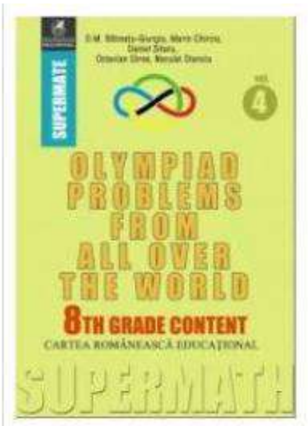
GENERALIZATIONS FOR SOME PUBLISHED PROBLEMS IN THE AMERICAN MATHEMATICAL MONTHLY (AMM),THE PENTAGON MATH JOURNAL AND SCHOOL SCIENCE AND MATHEMATICS JOURNAL (SSMJ)

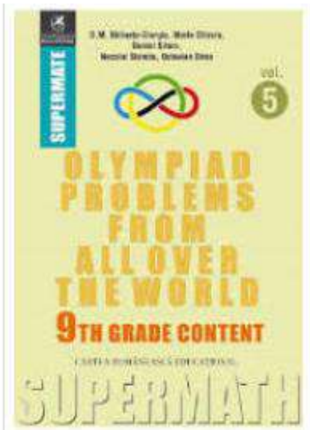
by Daniel Sitaru and Neculai Stanciu-Romania

~ DEDICATED TO PROFESSOR D. M. BĂTINEŢU-GIURGIU ~









I. PROBLEM 692 FROM THE PENTAGON, FALL 2011 & PROBLEM 12360 FROM THE AMERICAN MATHEMATICAL MONTHLY, DECEMBER 2022.

Find $\lim_{n\to\infty}\left(\frac{(n+1)^2}{x_{n+1}}-\frac{n^2}{x_n}\right)$**, where** $x_n=\sqrt[n]{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}}$ **.**

Solution without Stirling' approximation.

$$X_n=\frac{(n+1)^2}{x_{n+1}}-\frac{n^2}{x_n}=\frac{n^2}{x_n}\left(\left(\frac{n+1}{n}\right)^2\cdot\frac{x_n}{x_{n+1}}-1\right)=\frac{n^2}{x_n}\cdot(u_n-1)=$$

$$=\frac{n^2}{x_n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n=\frac{n}{x_n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n,\forall n\ge 2.$$

$$\lim_{n\to\infty}\frac{n}{x_n}=\lim_{n\to\infty}\frac{n}{\sqrt[n]{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}}}=\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}}}=$$

$$=\lim_{n\to\infty}\left(\frac{(n+1)^{n+1}}{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}\cdot\sqrt[n+1]{(n+1)!}}\cdot\frac{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}}{n^n}\right)=\lim_{n\to\infty}\left(\frac{n+1}{\sqrt[n+1]{(n+1)!}}\cdot e_n\right)=e^2,$$

where $e_n=\left(1+\frac{1}{n}\right)^n$ **. Since,** $u_n=\left(\frac{n+1}{n}\right)^2\cdot\frac{x_n}{x_{n+1}},\forall n\ge 2$**,**

$$\lim_{n\to\infty}u_n=\lim_{n\to\infty}\left(\frac{(n+1)^2}{x_{n+1}}\cdot\frac{x_n}{n^2}\right)=\lim_{n\to\infty}\left(\frac{n+1}{x_{n+1}}\cdot\frac{x_n}{n}\cdot\frac{n+1}{n}\right)=e^2\cdot\frac{1}{e^2}\cdot 1=1,\text{ so }\lim_{n\to\infty}\frac{u_n-1}{\ln u_n}=1;$$

$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}e_n^2\cdot\left(\frac{x_n}{x_{n+1}}\right)^n=e^2\cdot\lim_{n\to\infty}\left(\frac{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}}{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}\cdot\sqrt[n+1]{(n+1)!}}\cdot\sqrt[n+1]{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}\cdot\sqrt[n+1]{(n+1)!}}\right)=$$

$$=e^2\cdot\lim_{n\to\infty}\left(\frac{n+1}{\sqrt[n+1]{(n+1)!}}\cdot\frac{x_{n+1}}{n+1}\right)=e^2\cdot\frac{1}{e^2}\cdot e=e.$$

Hence, $\lim_{n\to\infty}X_n=e^2\cdot 1\cdot\ln\left(\lim_{n\to\infty}u_n^n\right)=e^2\cdot 1\cdot\ln e=e^2$**.**

II. PROBLEM 5495 FROM SCHOOL SCIENCE AND MATHEMATICS JOURNAL, APRIL 2018

Let $(x_n)_{n\ge 1}$**,** $x_1=1$**,** $x_n=1\cdot\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\ldots\cdot\sqrt[n]{(2n-1)!!}$**. Find** $\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{x_{n+1}}}-\frac{n^2}{\sqrt[n]{x_n}}\right)$**.**

Solution without Stirling' approximation.

We have $\lim_{n\to\infty}\frac{n}{\sqrt[n]{x_n}}=\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{x_n}}\overset{Cauchy-D'Alembert}{=}\lim_{x\to\infty}\frac{(n+1)^{n+1}}{x_{n+1}}\cdot\frac{x_n}{n^n}=\lim_{n\to\infty}\frac{(n+1)^{n+1}}{\sqrt[n+1]{(2n+1)!!}}\cdot\frac{1}{n^n}=$

$$=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n\cdot\lim_{n\to\infty}\frac{(n+1)}{\sqrt[n+1]{(2n+1)!!}}=e\cdot\lim_{n\to\infty}\frac{n}{\sqrt[n]{(2n-1)!!}}=e\cdot\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{(2n-1)!!}}\overset{C-D'A}{=}$$

$$\overset{C-D'A}{=}e\cdot\lim_{n\to\infty}\frac{(n+1)^{n+1}}{(2n+1)!!}\cdot\frac{(2n-1)!!}{n^n}=e\cdot\lim_{n\to\infty}\frac{n+1}{2n+1}\cdot\left(\frac{n+1}{n}\right)^n=\frac{e^2}{2}\text{ , (1).}$$

We have $\frac{(n+1)^2}{\sqrt[n+1]{x_{n+1}}}-\frac{n^2}{\sqrt[n]{x_n}}=\frac{n^2}{\sqrt[n]{x_n}}\cdot(u_n-1)=\frac{n^2}{\sqrt[n]{x_n}}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n=\frac{n}{\sqrt[n]{x_n}}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n$**, (2).**

Above we denote $u_n=\left(\frac{n+1}{n}\right)^2\cdot\frac{\sqrt[n]{x_n}}{\sqrt[n+1]{x_{n+1}}}$**. We have** $\lim_{n\to\infty}u_n=1\Rightarrow\lim_{n\to\infty}\frac{u_n-1}{\ln u_n}=1$**;**

$$\lim_{n\to\infty}\frac{n+1}{\sqrt[n+1]{(2n+1)!!}}=\lim_{n\to\infty}\frac{n}{\sqrt[n]{(2n-1)!!}}=\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{(2n-1)!!}}\overset{C-D'A}{=}\lim_{n\to\infty}\frac{(n+1)^{n+1}}{(2n+1)!!}\cdot\frac{(2n-1)!!}{n^n}=$$

$$=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n\cdot\frac{n+1}{2n+1}=\frac{e}{2}.$$

Then, $\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{2n}\cdot\frac{x_n}{x_{n+1}}\cdot\sqrt[n+1]{x_{n+1}}=e^2\cdot\lim_{n\to\infty}\frac{1}{\sqrt[n+1]{(2n+1)!!}}\cdot\sqrt[n+1]{x_{n+1}}=$

$$=e^2\cdot\lim_{n\to\infty}\frac{n+1}{\sqrt[n+1]{(2n+1)!!}}\cdot\lim_{n\to\infty}\frac{\sqrt[n+1]{x_{n+1}}}{n+1}=e^2\cdot\frac{e}{2}\cdot\frac{2}{e^2}=e.$$

From (2) and above we obtain that

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{x_{n+1}}}-\frac{n^2}{\sqrt[n]{x_n}}\right)=\frac{e^2}{2}\cdot 1\cdot\ln e=\frac{e^2}{2}$$ **, and we are done!**

III. GENERALIZATION FOR AMM DECEMBER 2022 AND SSMJ APRIL 2018

Let $(a_n)_{n\ge1}$**,** $(b_n)_{n\ge1}$ **be positive real sequences such that** $b_n=a_1\cdot\sqrt{a_2!}\cdot\sqrt[3]{a_3!}\cdot\ldots\cdot\sqrt[n]{a_n!}$ **and** $\lim_{n\to\infty}\frac{a_{n+1}}{a_n\cdot n}=a$**. Find** $\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{b_{n+1}}}-\frac{n^2}{\sqrt[n]{b_n}}\right)$**.**

Solution. We have $\lim_{n\to\infty}\frac{n}{\sqrt[n]{a_n}}=\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{a_n}}\overset{Cauchy-D'Alembert}{=}\lim_{x\to\infty}\frac{(n+1)^{n+1}}{a_{n+1}}\cdot\frac{a_n}{n^n}=$

$$=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{n+1}\cdot\lim_{n\to\infty}\frac{a_n\cdot n}{a_{n+1}}=\frac{e}{a}$$ **and**

$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{b_n}}=\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{b_n}}\overset{Cauchy-D'Alembert}{=}\lim_{x\to\infty}\frac{(n+1)^{n+1}}{b_{n+1}}\cdot\frac{b_n}{n^n}=$$

$$=\lim_{x\to\infty}\left(\frac{n+1}{n}\right)^n\cdot\frac{b_n(n+1)}{b_{n+1}}=e\cdot\lim_{n\to\infty}\frac{n+1}{\sqrt[n+1]{a_{n+1}}}=e\cdot\frac{e}{a}=\frac{e^2}{a}.$$

We have $\frac{(n+1)^2}{\sqrt[n+1]{b_{n+1}}}-\frac{n^2}{\sqrt[n]{b_n}}=\frac{n^2}{\sqrt[n]{b_n}}\cdot(u_n-1)=\frac{n^2}{\sqrt[n]{b_n}}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n=\frac{n}{\sqrt[n]{b_n}}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n$**, (1).**

Above we denote $u_n=\left(\frac{n+1}{n}\right)^2\cdot\frac{\sqrt[n]{b_n}}{\sqrt[n+1]{b_{n+1}}}$**. We have** $\lim_{n\to\infty}u_n=1\Rightarrow\lim_{n\to\infty}\frac{u_n-1}{\ln u_n}=1$**;**

Then, $\lim\limits_{n\to\infty} u_n^n = \lim\limits_{n\to\infty}\left(\frac{n+1}{n}\right)^{2n}\cdot\frac{b_n}{b_{n+1}}\cdot\sqrt[n+1]{b_{n+1}} = e^2\cdot\lim\limits_{n\to\infty}\left(\frac{b_n\cdot(n+1)}{b_{n+1}}\cdot\frac{\sqrt[n+1]{b_{n+1}}}{n+1}\right)=$

$$= e^2\cdot\lim_{n\to\infty}\frac{n+1}{\sqrt[n+1]{a_{n+1}}}\cdot\lim_{n\to\infty}\frac{\sqrt[n+1]{b_{n+1}}}{n+1} = e^2\cdot\frac{e}{a}\cdot\frac{a}{e^2} = e.$$

From (1) and above we obtain that

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{b_{n+1}}}-\frac{n^2}{\sqrt[n]{b_n}}\right)=\frac{e^2}{a}\cdot 1\cdot\ln e=\frac{e^2}{a}$$ **, and we are done!**

IV. GENERALIZATION OF PROBLEM 5710 FROM SCHOOL SCIENCE AND MATHEMATICS JOURNAL (SSMJ), DECEMBER 2022

Let the sequences $(a_n)_{n\geq 1}$ **,** $(b_n)_{n\geq 1}$**:** $a_n = \sum\limits_{k=1}^{n}\arctan\frac{1}{k^2-k+1}$ **and** $\lim\limits_{n\to\infty}\frac{b_{n+1}}{nb_n} = b\in R_+^*$**.**

Compute $\lim\limits_{n\to\infty}\left(\frac{\pi}{2}-a_n\right)\sqrt[n]{b_n}$ **.**

Solution. We have:

$$a_n = \sum_{k=1}^{n}\arctan\frac{1}{k^2-k+1} = \arctan 1+\sum_{k=2}^{n}\left(\arctan\frac{1}{k+1}-\arctan\frac{1}{k}\right)=$$

$= \frac{\pi}{4}+\arctan 1-\arctan\frac{1}{n}=\frac{\pi}{2}-\arctan\frac{1}{n}$**, so** $\lim\limits_{n\to\infty} a_n = \frac{\pi}{2}$**, (1).**

From (1) and Cesaro-Stolz theorem we get:

$$\lim_{n\to\infty}\left(\frac{\pi}{2}-a_n\right)n = \lim_{n\to\infty}\frac{\frac{\pi}{2}-a_n}{\frac{1}{n}} \overset{Cesaro-Stolz}{=} \lim_{n\to\infty}\frac{\frac{\pi}{2}-a_{n+1}-\frac{\pi}{2}+a_n}{\frac{1}{n+1}-\frac{1}{n}} = \lim_{n\to\infty}\frac{a_{n+1}-a_n}{\frac{1}{n}-\frac{1}{n+1}} =$$

$$= \lim_{n\to\infty}(a_{n+1}-a_n)n(n+1) = \lim_{n\to\infty}(n^2+n)\arctan\frac{1}{(n+1)^2-n-1+1} =$$

$$= \lim_{n\to\infty}\frac{n^2+n}{n^2+n+1}(n^2+n+1)\arctan\frac{1}{n^2+n+1} = 1\cdot\lim_{n\to\infty}\frac{\arctan\frac{1}{n^2+n+1}}{\frac{1}{n^2+n+1}} = 1\cdot 1 = 1$$ **, (2).**

$$\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{b_n}{n^n}} \overset{C-d'A}{=} \lim_{n\to\infty}\frac{b_{n+1}}{(n+1)^n}\cdot\frac{n^n}{b_n} = \lim_{n\to\infty}\frac{b_{n+1}}{na_n}\left(\frac{n}{n+1}\right)^{n+1} = \frac{b}{e}$$ **, (3).**

By (2) and (3) we obtain that:

$$\lim_{n\to\infty}\left(\frac{\pi}{2}-a_n\right)\sqrt[n]{b_n}=\lim_{n\to\infty}\left(\frac{\pi}{2}-a_n\right)n\cdot\frac{\sqrt[n]{b_n}}{n}=$$

$$=(\lim_{n\to\infty}(\frac{\pi}{2}-a_n)n)\cdot(\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n})=1\cdot\frac{b}{e}=\frac{b}{e}.$$

Remark. For $b=\pi$ we obtain the problem 5710 from SSMJ.

11 INDEPENDENT SOLUTIONS FOR A JAPANESSE INEQUALITY

By Daniel Sitaru-Romania

Abstract: *In this paper is presented an inequality proposed by Kunihiko Chikaya-Tokyo-Japan and 11 independent solutions for it from Japan,Greece and Romania*

MAIN RESULT: If $a,b,c>0$ then:

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+3abc\geq 6$$

Proposed by Kunihiko Chikaya-Tokyo-Japan

Solution 1 by proposer, Solution 2 by Panagiotis Danousis-Greece, Solution 3 by Lazaros Zachariadis-Greece, Solutions 4,5,6,7,8,9,10,11 and generalization by Daniel Sitaru-Romania

Solution 1 by proposer

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+3abc=\frac{\left(\frac{1}{a}\right)^2}{a}+\frac{\left(\frac{1}{b}\right)^2}{b}+\frac{\left(\frac{1}{c}\right)^2}{c}+3abc\overset{BERGSTROM}{\geq}$$

$$\geq\frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{a+b+c}+3abc\geq\frac{3\left(\frac{1}{a}\cdot\frac{1}{b}+\frac{1}{b}\cdot\frac{1}{c}+\frac{1}{c}\cdot\frac{1}{a}\right)}{a+b+c}+3abc=$$

$$=\frac{3\left(\frac{a+b+c}{abc}\right)}{a+b+c}+3abc=3\left(abc+\frac{1}{abc}\right)\overset{AM-GM}{\geq}3\cdot 2\sqrt{abc\cdot\frac{1}{abc}}=6$$

Equality holds for: $\frac{1}{a}=\frac{1}{b}=\frac{1}{c},abc=\frac{1}{abc}\Leftrightarrow a=b=c=1.$

Solution 2 by Panagiotis Danousis-Greece

Let be $x,y,z>0$ such that: $a=e^x,b=e^y,c=e^z$. It is known that:

$e^x\geq 1+x,\forall x\in\mathbb{R}$ with equality for $x=0$.

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+3abc=e^{-3x}+e^{-3y}+e^{-3z}+3e^{x+y+z}\geq$$

$$\geq 1-3x+1-3y+1-3z+3(1+x+y+z)=6$$

Equality holds for: $x=y=z=0\Leftrightarrow a=b=c=1$.

Solution 3 by Lazaros Zachariadis-Greece

Denote: $x=\frac{1}{a},y=\frac{1}{b},z=\frac{1}{c}$.

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+3abc=\sum_{cyc}x^3+\frac{3}{xyz}\overset{AM-GM}{\geq}3xyz+\frac{3}{xyz}=$$

$$=3\left(xyz+\frac{1}{xyz}\right)\overset{AM-GM}{\geq}3\cdot 2\sqrt{xyz\cdot\frac{1}{xyz}}=6$$

Equality holds for: $x=y=z,xyz=\frac{1}{xyz}\Leftrightarrow x=y=z=1\Leftrightarrow a=b=c=1$.

Solution 4 by Daniel Sitaru-Romania

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+3abc=\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+abc+abc+abc\overset{AM-GM}{\geq}$$

$$\geq 6\sqrt[6]{\frac{1}{a^3}\cdot\frac{1}{b^3}\cdot\frac{1}{c^3}\cdot abc\cdot abc\cdot abc}=6$$

Equality holds for: $\frac{1}{a^3}=\frac{1}{b^3}=\frac{1}{c^3}=abc\Leftrightarrow a=b=c=1$

Solution 5 by Daniel Sitaru-Romania

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+3abc\overset{AM-GM}{\geq}3\sqrt[3]{\frac{1}{a^3}\cdot\frac{1}{b^3}\cdot\frac{1}{c^3}}+3abc=$$

$$=\frac{3}{abc}+3abc=3\left(\frac{1}{abc}+abc\right)\overset{AM-GM}{\geq}3\cdot 2\sqrt{\frac{1}{abc}\cdot abc}=6$$

Equality holds for: $\frac{1}{a^3}=\frac{1}{b^3}=\frac{1}{c^3},\frac{1}{abc}=abc\Leftrightarrow a=b=c=1$

Solution 6 by Daniel Sitaru-Romania

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+3abc=\frac{(bc)^3+(ca)^3+(ab)^3+3(abc)^4}{(abc)^3}=$$

$$= \frac{(bc)^3 + (ca)^3 + (ab)^3 + (abc)^4 + (abc)^4 + (abc)^4}{(abc)^3} \overset{AM-GM}{\geq}$$

$$\geq \frac{6\sqrt[6]{(bc)^3 \cdot (ca)^3 \cdot (ab)^3 \cdot (abc)^4 \cdot (abc)^4 \cdot (abc)^4}}{(abc)^3} =$$

$$= \frac{6\sqrt[6]{(abc)^{18}}}{(abc)^3} = \frac{6(abc)^3}{(abc)^3} = 6$$

Equality holds for: $(bc)^3 = (ca)^3 = (ab)^3 = (abc)^4 \Leftrightarrow a = b = c = 1$

Solution 7 by Daniel Sitaru-Romania

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + 3abc = \frac{1}{a^3} + abc + \frac{1}{b^3} + abc + \frac{1}{c^3} + abc \overset{AM-GM}{\geq}$$

$$\geq 2\sqrt{\frac{1}{a^3} \cdot abc} + 2\sqrt{\frac{1}{b^3} \cdot abc} + 2\sqrt{\frac{1}{c^3} \cdot abc} = 2\sqrt{\frac{bc}{a^2}} + 2\sqrt{\frac{ca}{b^2}} + 2\sqrt{\frac{ab}{c^2}} \geq$$

$$\overset{AM-GM}{\geq} 3\sqrt[3]{2\sqrt{\frac{bc}{a^2}} \cdot 2\sqrt{\frac{ca}{b^2}} \cdot 2\sqrt{\frac{ab}{c^2}}} = 3\sqrt[6]{\frac{64(abc)^2}{(abc)^2}} = 6$$

Equality holds for: $\frac{1}{a^3} = \frac{1}{b^3} = \frac{1}{c^3} = abc \Leftrightarrow a = b = c = 1$

Solution 8 by Daniel Sitaru-Romania

$$\frac{1}{a^3} + \frac{2}{b^3} + \frac{3}{c^3} + 6abc \overset{AM-GM}{\geq} 12\sqrt[12]{\frac{(abc)^6}{a^3 b^6 c^9}} = 12\sqrt[4]{\frac{a}{c}}$$

$$\frac{2}{a^3} + \frac{3}{b^3} + \frac{1}{c^3} + 6abc \overset{AM-GM}{\geq} 12\sqrt[12]{\frac{(abc)^6}{a^6 b^9 c^3}} = 12\sqrt[4]{\frac{c}{b}}$$

$$\frac{3}{a^3} + \frac{1}{b^3} + \frac{2}{c^3} + 6abc \overset{AM-GM}{\geq} 12\sqrt[12]{\frac{(abc)^6}{a^9 b^3 c^6}} = 12\sqrt[4]{\frac{b}{a}}$$

By adding:

$$6\left(\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3}\right) + 18abc \geq 12\left(\sqrt[4]{\frac{a}{c}} + \sqrt[4]{\frac{c}{b}} + \sqrt[4]{\frac{b}{a}}\right) \overset{AM-GM}{\geq} 36\sqrt[12]{\frac{a}{c} \cdot \frac{c}{b} \cdot \frac{b}{a}} = 36$$

$$6\left(\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3}\right) + 18abc \geq 36, \qquad \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + 3abc \geq 6$$

Equality holds for: $\frac{1}{a^3}=\frac{1}{b^3}=\frac{1}{c^3}=abc \Leftrightarrow a=b=c=1$

Solution 9 by Daniel Sitaru-Romania

$$\frac{1}{a^3}+\frac{1}{b^3}+2abc \overset{AM-GM}{\geq} 4\sqrt[4]{\frac{(abc)^2}{a^3b^3}}=4\sqrt[4]{\frac{c^2}{ab}}$$

$$\frac{1}{b^3}+\frac{1}{c^3}+2abc \overset{AM-GM}{\geq} 4\sqrt[4]{\frac{(abc)^2}{b^3c^3}}=4\sqrt[4]{\frac{a^2}{bc}}$$

$$\frac{1}{c^3}+\frac{1}{a^3}+2abc \overset{AM-GM}{\geq} 4\sqrt[4]{\frac{(abc)^2}{c^3a^3}}=4\sqrt[4]{\frac{b^2}{ca}}$$

By adding:

$$2\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)+6abc \geq 4\left(\sqrt[4]{\frac{c^2}{ab}}+\sqrt[4]{\frac{a^2}{bc}}+\sqrt[4]{\frac{b^2}{ca}}\right) \overset{AM-GM}{\geq} 4\cdot 3\sqrt[4]{\frac{c^2}{ab}\cdot\frac{a^2}{bc}\cdot\frac{b^2}{ca}}=12$$

$$2\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)+6abc \geq 12 \Rightarrow \frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+3abc \geq 6$$

Equality holds for: $\frac{1}{a^3}=\frac{1}{b^3}=\frac{1}{c^3}=abc \Leftrightarrow a=b=c=1$

Solution 10 by Daniel Sitaru-Romania

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+3abc=$$

$$=\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^3-3\left(\frac{1}{a}+\frac{1}{b}\right)\left(\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{c}+\frac{1}{a}\right)+3abc \overset{AM-GM}{\geq}$$

$$\geq\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^3-3\cdot\left(\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{b}+\frac{1}{c}+\frac{1}{c}+\frac{1}{a}}{3}\right)^3+3abc=$$

$$=\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^3-\frac{8}{9}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^3+3abc=\frac{1}{9}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^3+3abc \geq$$

$$\overset{AM-GM}{\geq}\frac{1}{9}\cdot\left(3\sqrt[3]{\frac{1}{abc}}\right)^3+3abc=3\left(\frac{1}{abc}+abc\right) \overset{AM-GM}{\geq} 3\cdot 2\sqrt{\frac{1}{abc}\cdot abc}=6$$

Equality holds for: $\frac{1}{a}=\frac{1}{b}=\frac{1}{c}, abc=\frac{1}{abc} \Leftrightarrow a=b=c=1.$

Solution 11 by Daniel Sitaru-Romania

$$f(a,b,c)=\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+3abc$$

$$\begin{cases} f'_a=0 \\ f'_b=0 \\ f'_c=0 \end{cases} \Leftrightarrow \begin{cases} \frac{1}{a^4}=bc \\ \frac{1}{b^4}=ca \\ \frac{1}{c^4}=ab \end{cases} \Rightarrow (abc)^6=1 \Rightarrow abc=1 \Rightarrow bc=\frac{1}{a} \Rightarrow \frac{1}{a^4}=\frac{1}{a}$$

$\Rightarrow a=1$.Analogous: $b=1, c=1$.

$$H_f(1,1,1)=\begin{vmatrix} 12 & 3 & 3 \\ 3 & 12 & 3 \\ 3 & 3 & 12 \end{vmatrix} - \text{positive-definite} \Rightarrow (1,1,1) - \text{minimum point}$$

$$f(a,b,c)\ge f(1,1,1)=6 \Rightarrow \frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+3abc\ge 6$$

Generalization by Daniel Sitaru-Romania

If $a_i>0, i\in\overline{1,n}, n\in\mathbb{N}^*$ then:

$$\sum_{i=1}^{n}\frac{1}{a_i^n}+n\prod_{i=1}^{n}a_i\ge 2n$$

Proof:

$$\sum_{i=1}^{n}\frac{1}{a_i^n}+n\prod_{i=1}^{n}a_i\ge\frac{n}{\prod_{i=1}^{n}a_i}+n\prod_{i=1}^{n}a_i=$$

$$=n\left(\frac{1}{\prod_{i=1}^{n}a_i}+\prod_{i=1}^{n}a_i\right)\overset{AM-GM}{\ge} n\cdot 2\sqrt{\frac{1}{\prod_{i=1}^{n}a_i}\cdot\prod_{i=1}^{n}a_i}=2n$$

Equality holds for: $a_i>0, i\in\overline{1,n}, n\in\mathbb{N}^*$.

Reference: Romanian Mathematical Magazine-www.ssmrmh.ro

A PSEUDO-CEBYSHEV'S INEQUALITY AND APPLICATIONS

By Daniel Sitaru-Romania

ABSTRACT: In this paper it is proved a classical inequality and are gived a few applications.

MAIN PROBLEM: If $0 < a_1 \le a_2 \le \cdots . \le a_n$, $n \in \mathbb{N}^*$ then:

$$\frac{a_1}{a_2}+\frac{a_2}{a_3}+\frac{a_3}{a_4}+\cdots+\frac{a_n}{a_1} \ge \frac{a_2}{a_1}+\frac{a_3}{a_2}+\frac{a_4}{a_3}+\cdots+\frac{a_1}{a_n} \quad (1)$$

Proof:

For $n = 2 \Rightarrow \frac{a_1}{a_2}+\frac{a_2}{a_1} \ge \frac{a_2}{a_1}+\frac{a_1}{a_2}$. True.

For $n = 3$: $\frac{a_1}{a_2}+\frac{a_2}{a_3}+\frac{a_3}{a_1} \ge \frac{a_2}{a_1}+\frac{a_3}{a_2}+\frac{a_1}{a_3} \quad (2)$

$$a_1^2a_3 + a_2^2a_1 + a_3^2a_2 \ge a_2^2a_3 + a_3^2a_2 + a_1^2a_2$$

$$a_1^2(a_3 - a_2) - a_1(a_3^2 - a_2^2) + a_3a_2(a_3 - a_2) \ge 0$$

$$(a_3 - a_2)(a_1^2 - a_1a_3 - a_1a_2 + a_3a_2) \ge 0$$

$$(a_3 - a_2)(a_2 - a_1)(a_3 - a_1) \ge 0 \quad (3)$$

$$\text{By } 0 < a_1 \le a_2 \le a_3 \Rightarrow \begin{cases} a_3 - a_2 \ge 0 \\ a_2 - a_1 \ge 0 \\ a_3 - a_1 \ge 0 \end{cases} \Rightarrow (3)$$

We will use the mathematical induction to prove (1):

$P(n)$: $\frac{a_1}{a_2}+\frac{a_2}{a_3}+\frac{a_3}{a_4}+\cdots+\frac{a_n}{a_1} \ge \frac{a_2}{a_1}+\frac{a_3}{a_2}+\frac{a_4}{a_3}+\cdots+\frac{a_1}{a_n}$ suppose true

$$P(n+1): \frac{a_1}{a_2}+\frac{a_2}{a_3}+\frac{a_3}{a_4}+\cdots+\frac{a_{n+1}}{a_1} \ge \frac{a_2}{a_1}+\frac{a_3}{a_2}+\frac{a_4}{a_3}+\cdots+\frac{a_1}{a_{n+1}}$$

By $0 < a_1 \le a_n \le a_{n+1} \overset{(2)}{\Longrightarrow} \frac{a_1}{a_n}+\frac{a_n}{a_{n+1}}+\frac{a_{n+1}}{a_1} \ge \frac{a_n}{a_1}+\frac{a_{n+1}}{a_n}+\frac{a_1}{a_{n+1}} \quad (4)$

By adding $(1), (4)$:

$$\frac{a_1}{a_2}+\frac{a_2}{a_3}+\frac{a_3}{a_4}+\cdots+\frac{a_n}{a_1}+\frac{a_1}{a_n}+\frac{a_n}{a_{n+1}}+\frac{a_{n+1}}{a_1} \ge$$

$$\ge \frac{a_2}{a_1}+\frac{a_3}{a_2}+\frac{a_4}{a_3}+\cdots+\frac{a_1}{a_n}+\frac{a_n}{a_1}+\frac{a_{n+1}}{a_n}+\frac{a_1}{a_{n+1}}$$

$$\frac{a_1}{a_2}+\frac{a_2}{a_3}+\frac{a_3}{a_4}+\cdots+\frac{a_{n+1}}{a_1}\ge\frac{a_2}{a_1}+\frac{a_3}{a_2}+\frac{a_4}{a_3}+\cdots+\frac{a_1}{a_{n+1}}$$

$$P(n)\Longrightarrow P(n+1)$$

Application 1: If $\mathbf{0}<x\le 1$ then:

$$\boldsymbol{\frac{2x}{x^2+1}+\frac{x^2+1}{2}+\frac{1}{x}\ge\frac{x^2+1}{2x}+\frac{2}{x^2+1}+x}$$

Proof:

$$0<x\le 1\Longrightarrow 0<2x\le 2\Longrightarrow 0<2x<x^2+1\le 2$$

We take in (2): $a_1=2x, a_2=x^2+1, a_3=2$.

$$\frac{2x}{x^2+1}+\frac{x^2+1}{2}+\frac{2}{2x}\ge\frac{x^2+1}{2x}+\frac{2}{x^2+1}+\frac{2x}{2}$$

$$\frac{2x}{x^2+1}+\frac{x^2+1}{2}+\frac{1}{x}\ge\frac{x^2+1}{2x}+\frac{2}{x^2+1}+x$$

Equality holds for $x=1$.

Application 2: If $\mathbf{0}<a\le b\le c$ then in $\Delta\boldsymbol{ABC}$ holds:

a. $\boldsymbol{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge\frac{b}{a}+\frac{c}{b}+\frac{a}{c}}\quad(\mathbf{5})$

b. $\boldsymbol{\frac{sinA}{sinB}+\frac{sinB}{sinC}+\frac{sinC}{sinA}\ge\frac{sinB}{sinA}+\frac{sinC}{sinB}+\frac{sinA}{sinC}}$

c. $\boldsymbol{\frac{\mu(A)}{\mu(B)}+\frac{\mu(B)}{\mu(C)}+\frac{\mu(C)}{\mu(A)}\ge\frac{\mu(B)}{\mu(A)}+\frac{\mu(C)}{\mu(B)}+\frac{\mu(A)}{\mu(C)}}$

d. $\boldsymbol{\frac{m_c}{m_b}+\frac{m_a}{m_c}+\frac{m_b}{m_a}\ge\frac{m_b}{m_c}+\frac{m_c}{m_a}+\frac{m_a}{m_b}}$

e. $\boldsymbol{\frac{h_c}{h_b}+\frac{h_a}{h_c}+\frac{h_b}{h_a}\ge\frac{h_b}{h_c}+\frac{h_c}{h_a}+\frac{h_a}{h_b}}$

Proof:

a. We take in (2): $a_1=a, a_2=b, a_3=c$.
b. We use: $a=2RsinA, b=2RsinB, c=2RsinC$ in (5) and recall:
$$0<sinA\le sinB\le sinC$$
c. We take in (2): $a_1=\mu(A), a_2=\mu(B), a_3=\mu(C)$ and recall:
$$0<\mu(A)\le\mu(B)\le\mu(C)$$
d. We take in (2): $a_1=m_c, a_2=m_b, a_3=m_a$ and recall:
$$0<m_c\le m_b\le m_a$$
e. We take in (2): $a_1=h_c, a_2=h_b, a_3=h_a$ and recall:

$$0 < h_c \le h_b \le h_a$$

Application 3: If $0 < a \le b \le c, n \in \mathbb{N}, n \ge 2$ then in ΔABC holds:

a. $\frac{a+b}{a+c} + \frac{a+c}{b+c} + \frac{b+c}{a+b} \ge \frac{a+c}{a+b} + \frac{b+c}{a+c} + \frac{a+b}{b+c} \quad (6)$

b. $\sqrt{\frac{a+b}{a+c}} + \sqrt{\frac{a+c}{b+c}} + \sqrt{\frac{b+c}{a+b}} \ge \sqrt{\frac{a+c}{a+b}} + \sqrt{\frac{b+c}{a+c}} + \sqrt{\frac{a+b}{b+c}}$

c. $\sqrt[n]{\frac{a+b}{a+c}} + \sqrt[n]{\frac{a+c}{b+c}} + \sqrt[n]{\frac{b+c}{a+b}} \ge \sqrt[n]{\frac{a+c}{a+b}} + \sqrt[n]{\frac{b+c}{a+c}} + \sqrt[n]{\frac{a+b}{b+c}}$

d. $\sum_{cyc} \frac{sinA+sinB}{sinA+sinC} \ge \sum_{cyc} \frac{sinA+sinC}{sinA+sinB}$

Proof:

a. $a \le b \le c \Rightarrow \begin{cases} a+c \le b+c \\ a+b \le c+b \\ b+a \le c+a \end{cases} \Rightarrow a+b \le a+c \le b+c \quad (7)$

We take in (2): $a_1 = a+b, a_2 = a+c, a_3 = b+c$.

b. By (7): $a+b \le a+c \le b+c \Rightarrow \sqrt{a+b} \le \sqrt{a+c} \le \sqrt{b+c}$

We take in (2): $a_1 = \sqrt{a+b}, a_2 = \sqrt{a+c}, a_3 = \sqrt{b+c}$.

c. By (7): $a+b \le a+c \le b+c \Rightarrow \sqrt[n]{a+b} \le \sqrt[n]{a+c} \le \sqrt[n]{b+c}$

We take in (2): $a_1 = \sqrt{a+b}, a_2 = \sqrt{a+c}, a_3 = \sqrt{b+c}$.

d. We replace in (6): $a = 2RsinA, b = 2RsinB, c = 2RsinC$

Application 4: If $0 < a \le b \le c, n \in \mathbb{N}, n \ge 2$ then in ΔABC holds:

a. $\sqrt[n]{\frac{a}{b}} + \sqrt[n]{\frac{b}{c}} + \sqrt[n]{\frac{c}{a}} \ge \sqrt[n]{\frac{b}{a}} + \sqrt[n]{\frac{c}{b}} + \sqrt[n]{\frac{b}{c}} \quad (8)$

b. $\sqrt[n]{\frac{sinA}{sinB}} + \sqrt[n]{\frac{sinB}{sinC}} + \sqrt[n]{\frac{sinC}{sinA}} \ge \sqrt[n]{\frac{sinB}{sinA}} + \sqrt[n]{\frac{sinC}{sinB}} + \sqrt[n]{\frac{sinB}{sinC}}$

Proof: $a \le b \le c \Rightarrow \sqrt[n]{a} \le \sqrt[n]{b} \le \sqrt[n]{c}$

We take in (2): $a_1 = \sqrt[n]{a}, a_2 = \sqrt[n]{b}, a_3 = \sqrt[n]{c}$.

a. We take in (8): $a = 2RsinA, b = 2RsinB, c = 2RsinC$ and recall:

$$0 < sinA \le sinB \le sinC \Rightarrow \sqrt[n]{sinA} \le \sqrt[n]{sinB} \le \sqrt[n]{sinC}$$

Application 5: If $0 < a \le b \le c, n \in \mathbb{N}, n \ge 2$ then in acute ΔABC holds:

a. $\frac{cosC}{cosB} + \frac{cosA}{cosC} + \frac{cosB}{cosA} \ge \frac{cosB}{cosC} + \frac{cosC}{cosA} + \frac{cosA}{cosB}$

b. $\sqrt[n]{\frac{cosC}{cosB}} + \sqrt[n]{\frac{cosA}{cosC}} + \sqrt[n]{\frac{cosB}{cosA}} \ge \sqrt[n]{\frac{cosB}{cosC}} + \sqrt[n]{\frac{cosC}{cosA}} + \sqrt[n]{\frac{cosA}{cosB}}$

Proof: $a \le b \le c \Rightarrow cosC \le cosB \le cosA$.

We take in (2): $a_1 = cosC, a_2 = cosB, a_3 = cosA$.

a. $a \le b \le c \Rightarrow \sqrt[n]{cosC} \le \sqrt[n]{cosB} \le \sqrt[n]{cosA}$

We take in (2): $a_1 = \sqrt[n]{cosC}, a_2 = \sqrt[n]{cosB}, a_3 = \sqrt[n]{cosA}$.

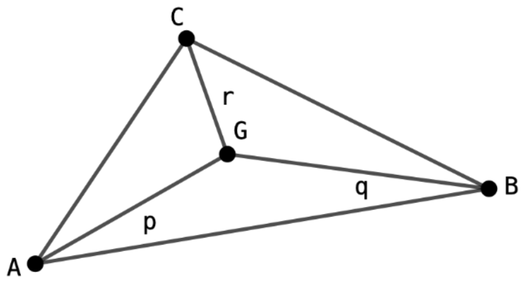
REFERENCE: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

A MINI HERON FORMULA IN A DANIEL HARDISKY'S VARIANT

By Daniel Sitaru-Romania

Abstract: The content of this article is illustrated in this figure:

MINI HERON- G: centroid of triangle ABC



If $s = \frac{1}{2}(p+q+r)$ then: $Area\,[ABC] = 3\sqrt{s(s-p)(s-q)(s-r)}$

Daniel Hardisky – USA

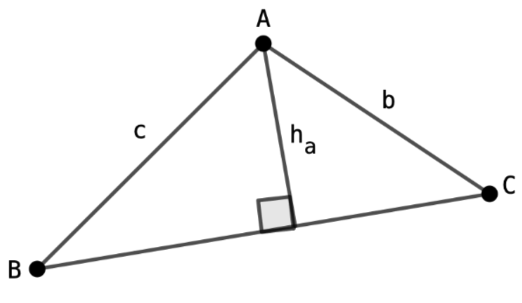
We will prove this formula.

Lemma 1: In ΔABC the following relationship holds:

$$F = \sqrt{s(s-a)(s-b)(s-c)} \qquad \textbf{(HERON)}$$

where $s = \frac{a+b+c}{2}$ – semiperimeter.

Proof.



$$\sin(\widehat{ABC}) = \sin B = \frac{h_a}{c} \Rightarrow c\sin B$$

$$F = \frac{1}{2}ah_a = \frac{1}{2}ac\sin B = \frac{1}{2}ac\sin\left(\frac{2B}{2}\right) =$$

$$=\frac{1}{2}ac\cdot 2\sin\frac{B}{2}\cos\frac{B}{2}=ac\sqrt{\frac{(s-a)(s-c)}{ac}}\cdot\sqrt{\frac{s(s-b)}{ac}}$$

$$F=\sqrt{s(s-a)(s-b)(s-c)}\qquad \text{(HERON)}$$

Lemma 2: In ΔABC the following relationship holds:

$$\mathbf{16F^2=2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}$$

Proof.

By lemma 1:

$$F=\sqrt{s(s-a)(s-b)(s-c)},\qquad F^2=s(s-a)(s-b)(s-c)$$

$$F^2=\frac{a+b+c}{2}\cdot\frac{b+c-a}{2}\cdot\frac{a+c-b}{2}\cdot\frac{a+b-c}{2}$$

$$16F^2=((b+c)^2-a^2)(a^2-(b-c)^2)$$

$$16F^2=(b^2+c^2-a^2+2bc)(a^2-b^2-c^2+2bc)$$

$$16F^2=\left(2bc+(b^2+c^2-a^2)\right)\left(2bc-(b^2+c^2-a^2)\right)$$

$$16F^2=4b^2c^2-(b^2+c^2-a^2)^2$$

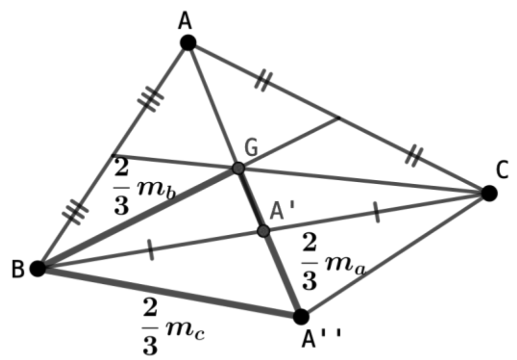
$$16F^2=4b^2c^2-b^4-c^4-a^4-2b^2c^2+2a^2b^2+2c^2a^2$$

$$16F^2=2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)$$

Lemma 3. Let be m_a, m_b, m_c – medians in ΔABC.

In these conditions: m_a, m_b, m_c can be sides in another triangle. This triangle is called "Bager's triangle".

See: www.ssmrmh.ro/2022/12/12/proof-without-words-bagers-triangle/



$$BG+BA''>GA''\Leftrightarrow\frac{2}{3}m_b+\frac{2}{3}m_c>\frac{2}{3}m_a\Leftrightarrow m_b+m_c>m_a$$

and analogs.

$$GA' = A'A'' = \frac{1}{3}m_a \Rightarrow GA'' = \frac{2}{3}m_a$$

Lemma 4: Let be $\Delta A'B'C'$ with area F' and sides $a' = m_a; b' = m_b; c' = m_c$. In these conditions:

$$\boldsymbol{F' = \frac{3F}{4}}$$

Proof.

$$2m_b^2m_c^2 - m_a^4 = 2 \cdot \frac{2(a^2+c^2)-b^2}{4} \cdot \frac{2(a^2+b^2)-c^2}{4} - \frac{(2(b^2+c^2)-a^2)^2}{16}$$

$$2m_b^2m_c^2 - m_a^4 = \frac{1}{16}\left((4a^2+4c^2-2b^2)(2a^2+2b^2-c^2)-(2b^2+2c^2-a^2)^2\right)$$

$$2m_b^2m_c^2 - m_a^4 = \frac{1}{16}(7a^4-8b^4-8c^4+8a^2b^2+8b^2c^2+8c^2a^2)$$

$$2\sum_{cyc} m_b^2m_c^2 - \sum_{cyc} m_a^4 = \frac{1}{16}\sum_{cyc}(7-8-8)\,a^4 + \frac{1}{16}\sum_{cyc}(8+8+2)\,a^2b^2$$

By lemmas 2;3:

$$\frac{1}{16}F'^2 = \frac{1}{16}(-9)\sum_{cyc} a^4 + \frac{1}{16}\cdot 18\sum_{cyc} a^2b^2, \qquad F'^2 = 9\left(2\sum_{cyc} a^2b^2 - \sum_{cyc} a^4\right)$$

$$F'^2 = 9 \cdot \frac{F^2}{16}, \qquad 16F'^2 = 9F^2, \qquad F'^2 = \frac{9F^2}{16} \Rightarrow F' = \frac{3F}{4}$$

Back to the problem: By lemmas 2;3:

$$16F'^2 = 2\sum_{cyc} m_a^2m_b^2 - \sum_{cyc} m_a^4$$

By lemma 4:

$$16 \cdot \frac{9F^2}{16} = 2\sum_{cyc} m_a^2m_b^2 - \sum_{cyc} m_a^4$$

$$F^2 = \frac{1}{9}\left(2\textstyle\sum_{cyc} m_a^2m_b^2 - \sum_{cyc} m_a^4\right) \quad (1)$$

Let's consider the triangle with sides:

$$p = \frac{2}{3}m_a; q = \frac{2}{3}m_b; r = \frac{2}{3}m_c$$

By multiplying (1) with $\frac{16}{81}$:

$$\frac{16}{81}F^2=\frac{1}{9}\left(2\sum_{cyc}\left(\frac{2}{3}m_a\cdot\frac{2}{3}m_b\right)^2-\sum_{cyc}\left(\frac{2}{3}m_a\right)^4\right)$$

$$\frac{16}{9}F^2=2\sum_{cyc}p^2q^2-\sum_{cyc}p^4,\qquad \frac{1}{9}F^2=\frac{1}{16}\left(2\sum_{cyc}p^2q^2-\sum_{cyc}p^4\right)$$

By lemma 2:

$$\frac{1}{9}F^2=\frac{p+q+r}{2}\cdot\frac{q+r-p}{2}\cdot\frac{r+p-q}{2}\cdot\frac{p+q-r}{2}$$

If we use Daniel Hardisky's notations:

$$s=\frac{p+q+r}{2};s-p=\frac{q+r-p}{2}$$

$$s-q=\frac{p+r-q}{2};s-r=\frac{p+q-r}{2}$$

We obtain:

$$\frac{1}{9}F^2=s(s-p)(s-q)(s-r)$$

$$F^2=9s(s-p)(s-q)(s-r)$$

$$F=3\sqrt{s(s-p)(s-q)(s-r)}$$

References:

[1] – www.ssmrmh.ro/2022/12/12/proof-without-words-bagers-triangle/

[2] – www.ssmrmh.ro/2021/09/29/famous-inequalities-redesigned-in-the-triangle-with-medians-as-sides/

10 INDEPENDENT SOLUTIONS FOR A SSMA MATH PROBLEM

By Daniel Sitaru-Romania

5687. Find complex numbers u,v such that:

$$\begin{cases}\dfrac{|u|^2}{3}+\dfrac{|v|^2}{4}=\dfrac{|u+v|^2}{7}\\ 8u+v=7+7i\end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Hyunbin Yoo, South Korea

Let $u = a + bi$ and $v = c + di$ where c and d are real numbers. Substitution gives

$$\frac{a^2+b^2}{3}+\frac{c^2+d^2}{4}=\frac{(a+c)^2+(b+d)^2}{7}$$

$$8a+c=7, 8b+d=7$$

We substitute $c = 7 - 8a$ and $d = 7 - 8b$ to get:

$$\frac{a^2+b^2}{3}+\frac{(7-8a)^2+(7-8b)^2}{4}=\frac{(7-7a)^2+(7-7b)^2}{7}$$

$$\Leftrightarrow \frac{a^2+b^2}{3}+\frac{(64a^2-112a+49)+(64b^2-112b+49)}{4}=7((a-1)^2+(b-1)^2)$$

$$\Leftrightarrow \frac{49}{3}a^2-28a+\frac{49}{3}b^2-28b+\frac{49}{2}=7(a^2-2+b^2-2b+2)$$

$$\Leftrightarrow \frac{7}{3}a^2-4a+\frac{7}{3}b^2-4b+\frac{7}{2}=a^2-2a+b^2-2b+2$$

$$\Leftrightarrow \frac{4}{3}a^2-2a+\frac{4}{3}b^2-2b+\frac{3}{2}=0 \Leftrightarrow 8a^2-12a+8b^2-12b+9=0$$

$\Leftrightarrow 2\left(2a-\frac{3}{2}\right)^2+2\left(2b-\frac{3}{2}\right)^2=0$. In conclusion, $a=\frac{3}{4}, b=\frac{3}{4}, c=1, d=1$.

Solution 2 by Andrew Siefker-USA

Let $u = a + ib$ and $v = c + id$ where $a, b, c, d \in \mathbb{R}$. Then the given equations become

$$\frac{a^2+b^2}{3}+\frac{c^2+d^2}{4}=\frac{(a+c)^2+(b+d)^2}{7} \quad (1)$$

and

$$\begin{pmatrix} 8a+c=7 \\ 8b+d=7 \end{pmatrix} \quad (2)$$

respectively. Expanding the right hand side of equation (1), clearing the denominators, and moving everything to one side yields

$$(16a^2-24ac+9c^2)+(16b^2-24bd+9d^2)=0$$

Factoring the grouped terms yields: $(4a-3c)^2+(4b-3d)^2=0$.

This equation is true iff $4a - 3c = 0$ and $4b - 3d = 0$. Solving for c in terms of a and for d in terms of b in equations (2) and substituting into $4a - 3c = 0$ and $4b - 3d = 0$ respectively results in

$4a = 3(7-8a) \Rightarrow a = \frac{3}{4} \Rightarrow c = 1$ and $4b = 3(7-8b) \Rightarrow b = \frac{3}{4} \Rightarrow d = 1$

$$\therefore u = \frac{3}{4} + i\frac{3}{4} \quad \text{and} \quad v = 1 + i$$

Solution 3 by The Eagle Problem Solvers-USA

The unique solution is $u = \frac{3}{4}(1+i)$ and $v = 1 + i$.

Let $u = a + bi$ and $v = c + di$, where a, b, c and d are real numbers. Multiplying each side of the first equation by 84 gives

$$28(a^2+b^2) + 21(c^2+d^2) = 12((a+c)^2 + (b+d)^2)$$

$$16(a^2+b^2) + 9(c^2+d^2) = 24(ac+bd)$$

$$(16a^2 - 24ac + 9c^2) + (16b^2 - 24bd + 9d^2) = 0$$

$$(4a-3c)^2 + (4b-3d)^2 = 0$$

from which we see that $4a = 3c$ and $4b = 3d$.

From the second equation, $8a + c = 7 = 8b + d$, so that $8a + c = 7c = 7$ and

$$8b + d = 7d = 7.$$

Consequently, $c = d = 1$ and $a = b = \frac{3}{4}$; thus $u = \frac{3}{4}(1+i)$ and $v = 1 + i$.

Solution 4 by Dionne Bailey, Elsie Campbell, and Charles Diminnie-USA

Let $u = a + bi$ and $v = c + di$ with $a, b, c, d \in \mathbb{R}$. Then, the equation

$$8u + v = 7 + 7i \text{ becomes } (8a + c) + (8b + d)i = 7 + 7i$$

and hence,

$$8a + c = 7 \qquad (1)$$

$$8b + d = 7 \qquad (2)$$

Further,

$$|u|^2 = a^2 + b^2, |v|^2 = c^2 + d^2, \text{ and } |u+v|^2 = (a+c)^2 + (b+d)^2.$$

Therefore, the equation $\frac{|u|^2}{3} + \frac{|v|^2}{4} = \frac{|u+v|^2}{7}$ becomes

$$\frac{a^2+b^2}{3} + \frac{c^2+d^2}{4} = \frac{(a+c)^2 + (b+d)^2}{7}.$$

If we clear the denominators, we obtain

$$28(a^2+b^2) + 21(c^2+d^2) = 12(a+c)^2 + 12(b+d)^2$$

which reduces to $(16a^2 - 24ac + 9c^2) + (16b^2 - 24bd + 9d^2) = 0$,

or $(4a-3c)^2 + (4b-3d)^2 = 0. \quad (3)$

Equation (3) yields

$$4a - 3c = 0 \qquad (4)$$

and

$$4b - 3d = 0. \qquad (5)$$

If we clear the denominators, we obtain

$28(a^2 + b^2) + 21(c^2 + d^2) = 12(a + c)^2 + 12(b + d)^2$ which reduces to

$(16a^2 - 24ac + 9c^2) + (16b^2 - 24bd + 9d^2) = 0$, or

$$(4a - 3c)^2 + (4b - 3d)^2 = 0. \qquad (3)$$

Equation (3) yields

$$4a - 3c = 0 \qquad (4)$$

and

$$4b - 3d = 0. \qquad (5)$$

By combining (1) and (4), we get $4a - 3(7 - 8a) = 0$

which reduces to $a = \frac{21}{28} = \frac{3}{4}$. Then: $c = 7 - 8a = 7 - 6 = 1$.

Similar steps using (2) and (5) lead to

$b = \frac{3}{4}$ and $d = 1$.

Therefore, our result is $u = a + bi = \frac{3}{4} + \frac{3}{4}i = \frac{3}{4}(1 + i)$ and $v = c + di = 1 + i$. It is easily seen that these numbers satisfy

$$8u + v = 8\left[\frac{3}{4}(1 + i)\right] + (1 + i) = 7(1 + i).$$

Further, since $|u|^2 = \frac{9}{16}|1 + i|^2 = \frac{9}{16}(2) = \frac{9}{8}, |v|^2 = |1 + i|^2 = 2$, and

$|u + v|^2 = \left|\frac{7}{4}(1 + i)\right|^2 = \frac{49}{16}(2) = \frac{49}{8}$, we have

$$\frac{|u|^2}{3} + \frac{|v|^2}{4} = \frac{3}{8} + \frac{1}{2} = \frac{7}{8} = \frac{1}{7}\left(\frac{49}{8}\right) = \frac{|u + v|^2}{7}$$

as well.

Solution 5 by Sean M. Stewart-Saudi Arabia

We shall show that: $u = \frac{3}{4}(1 + i)$ and $v = 1 + i$.

Since: $|u+v|^2=(u+v)\overline{(u+v)}=(u+v)(\overline{u}+\overline{v})=|u|^2+|v|^2=u\overline{v}+v\overline{u}$,

the first of the given equations $\frac{|u|^2}{3}+\frac{|v|^2}{4}=\frac{|u+v|^2}{7}$, becomes

$$\frac{4}{21}|u|^2+\frac{3}{28}|v|^2=\frac{1}{7}u\overline{v}+\frac{1}{7}v\overline{u}. \qquad (6)$$

From the second of the given equations, we have: $v=7+7i-8u$ and $\overline{v}=7-7i-8\overline{u}$.

So we see that

$$u\overline{v}+v\overline{u}=7(u+\overline{u})-7i(u-\overline{u})-16|u|^2. \qquad (7)$$

Recalling that if z is a complex number, then

$Re(z)=\frac{z+\overline{z}}{2}$ and $Im(z)=\frac{z-\overline{z}}{2i}$.

So one can rewrite the expression in (7) as

$$u\overline{v}+v\overline{u}=14Re(u)+14Im(u)+16|u|^2. \qquad (8)$$

Combining (6) with (8), after simplifying we arrive at

$\frac{2}{3}|u|^2+\frac{3}{4}=Re(u)+Im(u)$. If we now let $u=x+iy$ where $x,y\in\mathbb{R}$, then

$\frac{2}{3}(x^2+y^2)+\frac{3}{4}=x+y$, or after rearranging $\left(x-\frac{3}{4}\right)^2+\left(y-\frac{3}{4}\right)^2=0$.

This equation can only be true provided $x=y=\frac{3}{4}$. So

$u=x+iy=\frac{3}{4}(1+i)$ and $v=7+7i-8u=7+7i-8\cdot\frac{3}{4}(1+i)=1+i$,

as announced.

Solution 6 by Albert Stadler-Switzerland

We write $u=a+ib, v=c+id$ with a,b,c,d real. The system of equation is then equivalent to

$$\begin{Bmatrix} \frac{1}{3}(a^2+b^2)+\frac{1}{4}(c^2+d^2)=\frac{1}{7}((a+c)^2+(b+d)^2) \\ 8a+c=7 \\ 8b+d=7 \end{Bmatrix}.$$

The first equation is a quadratic equation with respect to the variable a with discriminant $-\frac{4}{441}(4b-3d)^2$ and at the same time a quadratic equation with respect to the variable b with discriminant $-\frac{4}{441}(4a-3c)^2$.

However the two discriminants must be ≥ 0, since a and b are both real. So $4b=3d$ and $4a=3c$, giving the solution $(a,b,c,d)=\left(\frac{3}{4},\frac{3}{4},1,1\right)$ or equivalently

$$u=\frac{3}{4}(1+i),v=1+i.$$

Solution 7 by Brian D. Beasley-USA

We write $u=a+bi$ and $v=c+di$ for real numbers a,b,c and d. Then the given system of equations becomes

$$\begin{Bmatrix}\dfrac{a^2+b^2}{3}+\dfrac{c^2+d^2}{4}=\dfrac{(a+c)^2+(b+d)^2}{7}\\(8a+c)+(8b+d)i=7+7i\end{Bmatrix}.$$

This implies $c=7-8a$ and $d=7-8b$, so we have

$$\frac{a^2+b^2}{3}+\frac{(7-8a)^2+(7-8b)^2}{4}=\frac{(7-7a)^2+(7-7b)^2}{7}$$

and thus

$$\frac{28}{3}\left(a-\frac{3}{4}\right)^2+\frac{28}{3}\left(b-\frac{3}{4}\right)^2=0.$$

Since a and b are real, we conclude that $a=b=\frac{3}{4}$. Hence $c=d=1$, so

$$u=\tfrac{3}{4}+\tfrac{3}{4}i=\tfrac{3}{4}(1+i)\quad\text{and}\quad v=1+i.$$

Solution 8 by David A. Huckaby-USA

Let $u=a+bi$ and $v=c+di$, with a,b,c, and d real numbers. Then the second equation above becomes $8(a+bi)+c+di=7+7i$, which gives $8a+c=7$ and $8b+d=7$, so that $c=7-8a$ and $d=7-8b$. The first equation above becomes

$$\frac{|a+bi|^2}{3}+\frac{|c+di|^2}{4}=\frac{|(a+c)+(b+d)i|^2}{7}$$

$$\frac{a^2+b^2}{3}+\frac{c^2+d^2}{4}=\frac{(a+c)^2+(b+d)^2}{7}$$

$$\frac{a^2+b^2}{3}+\frac{(7-8a)^2+(7-8b)^2}{4}=\frac{(a+7-8a)^2+(b+7-8b)^2}{7}.$$

Simplifying gives the equation $112a^2+112b^2-168a-168b+126=0$. Solving for a yields

$$a=\frac{3\pm\sqrt{-(4b-3)^2}}{4}.$$

Since a is real, $4b-3=0$, so that $b=\frac{3}{4}$ and $a=\frac{3}{4}$. Then $c=7-8a=7-8\left(\frac{3}{4}\right)=1$; likewise

$d = 7 - 8b = 7 - 8\left(\frac{3}{4}\right) = 1$. So $u = \frac{3}{4} + \frac{3}{4}i$ and $v = 1 + i$.

Solution 9 by Michel Bataille-France

Let u, v satisfying the equations. Note that $u \neq 0$ and $v \neq 0$ (if, say, $u = 0$, then the first equation implies that $v = 0$ as well, contradicting the second equation).

Since $|u + v|^2 = (u + v)(\overline{u} + \overline{v}) = |u|^2 + |v|^2 + (u\overline{v} + \overline{u}v)$, the first equation shows that the real number $u\overline{v} + \overline{u}v$ is equal to $\frac{4}{3}|u|^2 + \frac{3}{4}|v|^2$. It follows that

$$u\overline{v} + \overline{u}v \geq \left(\frac{4}{3}|u|^2 \cdot \frac{3}{4}|v|^2\right)^{\frac{1}{2}} = 2|u||v|$$

so that

$$2|u||v| \leq u\overline{v} + \overline{u}v \leq |u\overline{v} + \overline{u}v| \leq |u\overline{v}| + |\overline{u}v| = 2|u||v|.$$

Thus, $|u\overline{v} + \overline{u}v| = |u\overline{v}| + |\overline{u}v|$, which implies that $u\overline{v} = \lambda\overline{u}v$ for some positive real number λ. Taking conjugates, we also have $\overline{u}v = \lambda u\overline{v} = \lambda^2\overline{u}v$, hence $\lambda = 1$, that is $\overline{u}v = u\overline{v}$ is a nonzero real number.

Since $u\overline{v} + \overline{u}v = 2|u||v|$, we have $\left(\frac{2}{\sqrt{3}}|u| - \frac{\sqrt{3}}{2}|v|\right)^2 = \frac{4}{3}|u|^2 + \frac{3}{4}|v|^2 - (u\overline{v} + \overline{u}v) = 0$, hence $|u| = \frac{3}{4}|v|$.

Now, the second equation gives $(8u + v)(8\overline{u} + \overline{v}) = 98$ and it follows that

$$64|u|^2 + |v|^2 + 16|u||v| = 98.$$

With $|u| = \frac{3}{4}|v|$, this leads to $|v|^2 = 2$, so that $|v| = \sqrt{2}, |u| = \frac{3\sqrt{2}}{4}$. Let us set $u = \frac{3\sqrt{2}}{4}e^{i\alpha}$, $v = \sqrt{2}e^{i\beta}$ where $\alpha, \beta \in \mathbb{R}$. Since $u\overline{v} = |u||v|$ is a positive real number, we must have $\alpha \equiv \beta (mod\ 2\pi)$ and therefore $7 + 7i = 8u + v = 7\sqrt{2}e^{i\alpha}$. Thus $\alpha = \frac{\pi}{4}$ and

$$u = \frac{3\sqrt{2}}{4} \cdot \frac{1 + i}{\sqrt{2}} = \frac{3}{4}(1 + i), v = \sqrt{2} \cdot \frac{1 + i}{\sqrt{2}} = 1 + i.$$

Conversely, it is easily checked that these complex numbers satisfy the two equations.

We conclude that the system has a unique solution

$$(u, v) = \left(\frac{3}{4}(1 + i), 1 + i\right).$$

Solution 10 by proposer

$$\frac{|u|^2}{3} + \frac{|v|^2}{4} - \frac{|u + v|^2}{7} = 0 \Rightarrow 28|u|^2 + 21|v|^2 - 12|u + v|^2 = 0$$

$$28u\overline{u}+21v\overline{v}-12(u+v)\left(\overline{u+v}\right)=0,\qquad 28u\overline{u}+21v\overline{v}-12(u+v)(\overline{u}+\overline{v})=0$$

$$28u\overline{u}+21v\overline{v}-12u\overline{u}-12v\overline{v}-12\overline{u}v-12\overline{u}v=0$$

$$16u\overline{v}+9v\overline{v}-12u\overline{v}-12\overline{u}v=0,\qquad 4u(4\overline{u}-3\overline{v})-3v(4\overline{u}-3\overline{v})=0$$

$$(4u-3v)(4\overline{u}-3\overline{v})=0,\qquad (4u-3v)\left(\overline{4u-4v}\right)=0$$

$$|4u-3v|^2=0\Rightarrow 4u-3v=0\Rightarrow u=\frac{3}{4}v$$

$$8u+v=7+7i\Rightarrow 8\cdot\frac{3}{4}v+v=7+7i$$

$$6v+v=7+7i\Rightarrow 7v=7+7i\Rightarrow v=1+i$$

$$u=\frac{3}{4}v\Rightarrow u=\frac{3}{4}(1+i)\Rightarrow u=\frac{3}{4}+\frac{3i}{4}$$

Observation: we use the facts: $a\in\mathbb{C}\Rightarrow a\cdot\overline{a}=|a|^2$

$$a=x+iy;\overline{a}=x-iy;|a|=\sqrt{x^2+y^2},\overline{a+b}=\overline{a}+\overline{b}$$

RMM-SOLVED PROBLEMS

By Marin Chirciu – Romania

01. Solve in $\mathbb{R}$:

$$(\mathbf{2^x+3^x})\sqrt{\mathbf{6^{1-x}}}=\mathbf{5}$$

Daniel Sitaru-Romania

Solution:

$$(2^x+3^x)\sqrt{6^{1-x}}=5\Leftrightarrow(2^x+3^x)^2 6^{1-x}=25\Leftrightarrow\frac{2^{2x}+3^{2x}+2\cdot 6^x}{6^x}=\frac{25}{6}\Leftrightarrow$$

$$\Leftrightarrow\frac{2^{2x}}{2^x3^x}+\frac{3^{2x}}{2^x3^x}+2=\frac{25}{6}\Leftrightarrow\frac{2^x}{3^x}+\frac{3^x}{2^x}=\frac{13}{6}\Leftrightarrow\left(\frac{2}{3}\right)^x+\left(\frac{3}{2}\right)^x=\frac{2}{3}+\frac{3}{2}\Leftrightarrow x=\pm1$$

The set of the solutions is $S=\{-1,1\}$. **Remark:** The problem can be developed.

Let $a>b>1$ fixed. Solve in $\mathbb{R}$:

$$(\boldsymbol{a^x}+\boldsymbol{b^x})\sqrt{(\boldsymbol{ab})^{\mathbf{1-x}}}=\boldsymbol{a}+\boldsymbol{b}$$

Marin Chirciu-Romania

Solution: $(a^x+b^x)\sqrt{(ab)^{1-x}}=a+b\Leftrightarrow(a^x+b^x)^2(ab)^{1-x}=(a+b)^2\Leftrightarrow$

$$\Leftrightarrow \frac{a^{2x}+b^{2x}+2\cdot(ab)^x}{(ab)^x}=\frac{(a+b)^2}{ab}\Leftrightarrow$$

$$\frac{a^{2x}}{a^xb^x}+\frac{b^{2x}}{a^xb^x}+2=\frac{a^2+b^2+2ab}{ab}\Leftrightarrow\frac{a^x}{b^x}+\frac{b^x}{a^x}=\frac{a^2+b^2}{ab}\Leftrightarrow$$

$\Leftrightarrow\left(\frac{a}{b}\right)^x+\left(\frac{b}{a}\right)^x=\frac{a}{b}+\frac{b}{a}\Leftrightarrow x=\pm1$. The set of the solution is $S=\{-1,1\}$.

Note: For $a=3, b=2$ we obtain the proposed problem by Daniel Sitaru in RMM 11/2022.

02. In ΔABC holds:

$$\sum(\boldsymbol{a}+\boldsymbol{2b})(\boldsymbol{a}+\boldsymbol{2c})\leq\boldsymbol{81R^2}$$

Daniel Sitaru-Romania

Solution: Lemma: If $x,y,z>0$: $(x+2y)(x+2z)\leq(x+y+z)^2$.

Proof:

$$(x+2y)(x+2z)\overset{AM-GM}{\leq}\left[\frac{(x+2y)+(x+2z)}{2}\right]=(x+y+z)^2$$

with equality for $(x+2y)=(x+2z)\Leftrightarrow y=z$. Let's get back to the main problem.

Using the Lemma for $(x,y,z)=(a,b,c)$ we obtain:

$$LHS=\sum(a+2b)(a+2c)\overset{Lemma}{\leq}\sum(a+b+c)^2=\sum(2p)^2=12p^2\overset{Mitrinovic}{\leq}$$

$$\overset{Mitrinovic}{\leq}12\cdot\frac{27R^2}{4}=81R^2=RHS$$

The equality holds if and only if the triangle is equilateral.

Remark: In the same way: **In ΔABC holds:**

$$\sum(\boldsymbol{m_a}+\boldsymbol{2m_b})(\boldsymbol{m_a}+\boldsymbol{2m_c})\leq\boldsymbol{3(4R+r)^2}$$

Marin Chirciu-Romania

Solution: Using the Lemma for $(x,y,z)=(m_a,m_b,m_c)$ we obtain:

$$LHS=\sum(m_a+2m_b)(m_a+2m_c)\overset{Lemma}{\leq}\sum(m_a+m_b+m_c)^2\overset{Leuenberger}{\leq}\sum(4R+r)^2$$

$$=3(4R+r)^2=RHS$$

Equality holds if and only if the triangle is equilateral.

Remark: In the same way: **In ΔABC holds:**

$$\sum(\boldsymbol{h_a}+\boldsymbol{2h_b})\,(\boldsymbol{h_a}+\boldsymbol{2h_c})\le\frac{\boldsymbol{243R^2}}{\boldsymbol{4}}$$

Marin Chirciu-Romania

Solution: Using the Lemma for $(x,y,z)=(h_a,h_b,h_c)$ we obtain:

$$LHS=\sum(h_a+2h_b)(h_a+2h_c)\overset{Lemma}{\le}\sum(h_a+h_b+h_c)^2\overset{Santalo}{\le}\sum\left(p\sqrt{3}\right)^2=9p^2\le$$

$$\overset{Mitrinovic}{\le}9\cdot\frac{27R^2}{4}=\frac{243R^2}{4}=RHS$$

Equality holds if and only if the triangle is equilateral. **Remark:** In the same way.

In ΔABC holds:

$$\sum(\boldsymbol{w_a}+\boldsymbol{2w_b})(\boldsymbol{w_a}+\boldsymbol{2w_c})\le\frac{\boldsymbol{243R^2}}{\boldsymbol{4}}$$

Marin Chirciu-Romania

Solution: Using the Lemma for $(x,y,z)=(w_a,w_b,w_c)$ we obtain:

$$LHS=\sum(w_a+2w_b)\,(w_a+2w_c)\overset{Lemma}{\le}\sum(w_a+w_b+w_c)^2\overset{Santalo}{\le}\sum\left(p\sqrt{3}\right)^2=$$

$$=9p^2\overset{Mitrinovic}{\le}9\cdot\frac{27R^2}{4}=\frac{243R^2}{4}=RHS$$

Equality holds if and only if the triangle is equilateral. **Remark:** In the same way.

In ΔABC holds:

$$\sum(\boldsymbol{s_a}+\boldsymbol{2s_b})\,(\boldsymbol{s_a}+\boldsymbol{2s_c})\le\frac{\boldsymbol{243R^2}}{\boldsymbol{4}}.$$

Marin Chirciu-Romania

Solution: Using the Lemma for $(x,y,z)=(s_a,s_b,s_c)$ we obtain:

$$LHS=\sum(s_a+2s_b)\,(s_a+2s_c)\overset{Lemma}{\le}\sum(s_a+s_b+s_c)^2\overset{s_a\le h_a}{\le}\sum(h_a+h_b+h_c)^2\le$$

$$\overset{Santalo}{\le}\sum\left(p\sqrt{3}\right)^2=9p^2\overset{Mitrinovic}{\le}9\cdot\frac{27R^2}{4}=\frac{243R^2}{4}=RHS$$

Equality holds if and only if the triangle is equilateral.

Remark: In the same way: **In ΔABC holds:** $\sum(\boldsymbol{r_a}+\boldsymbol{2r_b})(\boldsymbol{r_a}+\boldsymbol{2r_c})\le\frac{\boldsymbol{243R^2}}{\boldsymbol{4}}$

Marin Chirciu-Romania

Solution: Using Lemma for $(x,y,z)=(r_a,r_b,r_c)$ we obtain:

$$LHS=\sum(r_a+2r_b)\,(r_a+2r_c)\overset{Lemma}{\leq}\sum(r_a+r_b+r_c)^2=\sum(4R+r)^2=RHS$$

Equality holds if and only if the triangle is equilateral. **Remark:** In the same way.

In ΔABC holds:

$$\sum\left(\tan\frac{A}{2}+2\tan\frac{B}{2}\right)\left(\tan\frac{A}{2}+2\tan\frac{C}{2}\right)\leq 3\left(1+\frac{R}{r}\right)$$

Marin Chirciu-Romania

Solution: Using the Lemma for $(x,y,z)=\left(\tan\frac{A}{2},\tan\frac{B}{2},\tan\frac{C}{2}\right)$ we obtain:

$$LHS=\sum\left(\tan\frac{A}{2}+2\tan\frac{B}{2}\right)\left(\tan\frac{A}{2}+2\tan\frac{C}{2}\right)\overset{Lemma}{\leq}\sum\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)^2=$$

$$=\sum\left(\frac{4R+r}{p}\right)^2=3\cdot\frac{(4R+r)^2}{p^2}\overset{Gerretsen}{\leq}3\cdot\frac{(4R+r)^2}{r(4R+r)^2}=3\cdot\frac{R+r}{r}=3\left(1+\frac{R}{r}\right)=RHS$$

Equality holds if and only if the triangle is equilateral. **Remark:** In the same way.

In ΔABC holds:

$$\sum\left(\cot\frac{A}{2}+2\cot\frac{B}{2}\right)\left(\cot\frac{A}{2}+2\cot\frac{C}{2}\right)\leq 81\left(\frac{R}{2r}\right)^2$$

Marin Chirciu-Romania

Solution: Using the Lemma for $(x,y,z)=\left(\cot\frac{A}{2},\cot\frac{B}{2},\cot\frac{C}{2}\right)$ we obtain:

$$LHS=\sum\left(\cot\frac{A}{2}+2\cot\frac{B}{2}\right)\left(\cot\frac{A}{2}+2\cot\frac{C}{2}\right)\overset{Lemma}{\leq}\sum\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right)^2=$$

$$=\sum\left(\frac{p}{r}\right)^2=3\cdot\frac{p^2}{r^2}\overset{Mitrinovic}{\leq}3\cdot\frac{\frac{27R^2}{4}}{r^2}=\frac{81R^2}{4r^2}=RHS$$

Equality holds if and only if the triangle is equilateral. **Remark:**In the same way.

In ΔABC holds:

$$\sum\left(\sin\frac{A}{2}+2\sin\frac{B}{2}\right)\left(\sin\frac{A}{2}+2\sin\frac{C}{2}\right)\leq\frac{27}{4}$$

Marin Chirciu-Romania

Solution: Using the Lemma for $(x,y,z)=\left(\sin\frac{A}{2},\sin\frac{B}{2},\sin\frac{C}{2}\right)$ we obtain:

$$LHS = \sum\left(\sin\frac{A}{2} + 2\sin\frac{B}{2}\right)\left(\sin\frac{A}{2} + 2\sin\frac{C}{2}\right) \overset{Lemma}{\leq} \sum\left(\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}\right)^2 \overset{Jensen}{\leq}$$

$$\overset{Jensen}{\leq} \sum\left(\frac{3}{2}\right)^2 = 3\cdot\left(\frac{3}{2}\right)^2 = \frac{27}{4} = RHS$$

Equality holds if and only if the triangle is equilateral.

Remark: In the same way. **In ΔABC holds:**

$$\sum\left(\cos\frac{A}{2} + 2\cos\frac{B}{2}\right)\left(\cos\frac{A}{2} + 2\cos\frac{C}{2}\right) \leq \frac{81}{4}.$$

Marin Chirciu-Romania

Solution: Using the Lemma for $(x, y, z) = \left(\cos\frac{A}{2}, \cos\frac{B}{2}, \cos\frac{C}{2}\right)$ we obtain:

$$LHS = \sum\left(\cos\frac{A}{2} + 2\cos\frac{B}{2}\right)\left(\cos\frac{A}{2} + 2\cos\frac{C}{2}\right) \overset{Lemma}{\leq} \sum\left(\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}\right)^2 \overset{Jensen}{\leq}$$

$$\overset{Jensen}{\leq} \sum\left(\frac{3\sqrt{3}}{2}\right)^2 = 3\cdot\left(\frac{3\sqrt{3}}{2}\right)^2 = \frac{81}{4} = RHS$$

Equality holds if and only if the triangle is equilateral.

Remark: In the same way: **In ΔABC holds:**

$$\sum\left(\sec\frac{A}{2} + 2\sec\frac{B}{2}\right)\left(\sec\frac{A}{2} + 2\sec\frac{C}{2}\right) \leq \frac{9R^2}{r^2}$$

Marin Chirciu -Romania

Solution: Using Lemma for $(x, y, z) = \left(\sec\frac{A}{2}, \sec\frac{B}{2}, \sec\frac{C}{2}\right)$ we obtain:

$$LHS = \sum\left(\sec\frac{A}{2} + 2\sec\frac{B}{2}\right)\left(\sec\frac{A}{2} + 2\sec\frac{C}{2}\right) \overset{Lemma}{\leq} \sum\left(\sec\frac{A}{2} + \sec\frac{B}{2} + \sec\frac{C}{2}\right)^2 \overset{Jensen}{\leq}$$

$$\overset{Jensen}{\leq} \sum\left(\frac{2p}{3r}\right)^2 = 3\cdot\frac{4p^2}{9r^2} \overset{Mitrinovic}{\leq} 3\cdot\frac{4\cdot\frac{27R^2}{4}}{9r^2} = \frac{9R^2}{r^2} = RHS.$$

Equality holds if and only if the triangle is equilateral. **Remark:** In the same way.

In ΔABC holds:

$$\sum\left(\csc\frac{A}{2} + 2\csc\frac{B}{2}\right)\left(\csc\frac{A}{2} + 2\csc\frac{C}{2}\right) \leq \frac{27R^2}{r^2}$$

Marin Chirciu-Romania

Solution: Using the Lemma for $(x, y, z) = \left(\csc\frac{A}{2}, \csc\frac{B}{2}, \csc\frac{C}{2}\right)$ we obtain:

$$LHS=\sum\left(\csc\frac{A}{2}+2\csc\frac{B}{2}\right)\left(\csc\frac{A}{2}+2\csc\frac{C}{2}\right)\overset{Lemma}{\leq}\sum\left(\csc\frac{A}{2}+\csc\frac{B}{2}+\csc\frac{C}{2}\right)^2\overset{Jensen}{\leq}$$

$$\overset{Jensen}{\leq}\sum\left(\frac{3R}{r}\right)^2=3\cdot\frac{9R^2}{r^2}=\frac{27R^2}{r^2}=RHS.$$

Equality holds if and only if the triangle is equilateral.

03. If $a,b,c>0$ then:

$$2^{a-b}+2^{b-c}+2^{c-a}\geq\frac{2^a+2^b+2^c}{\sqrt[3]{2^{a+b+c}}}$$

Daniel Sitaru-Romania

Solution:

$$2^{a-b}+2^{b-c}+2^{c-a}\geq\frac{2^a+2^b+2^c}{\sqrt[3]{2^{a+b+c}}}\Leftrightarrow\frac{2^a}{2^b}+\frac{2^b}{2^c}+\frac{2^c}{2^a}\geq\frac{2^a+2^b+2^c}{\sqrt[3]{2^a}\sqrt[3]{2^b}\sqrt[3]{2^c}}$$

With the substitution $\left(\sqrt[3]{2^a},\sqrt[3]{2^b},\sqrt[3]{2^c}\right)=(x,y,z)$ the conclusion can be written:

$$\frac{x^3}{y^3}+\frac{y^3}{z^3}+\frac{z^3}{x^3}\geq\frac{x^3+y^3+z^3}{xyz}\Leftrightarrow\frac{x^6z^3+y^6x^3+z^6y^3}{x^3y^3z^3}\geq\frac{x^3+y^3+z^3}{xyz}\Leftrightarrow$$

$\Leftrightarrow x^6z^3+y^6x^3+z^6y^3\geq x^2y^2z^2(x^3+y^3+z^3)$, (*) which follows from means inequality:

$x^6z^3+x^6z^3+y^6x^3\overset{AM-GM}{\geq}3\sqrt[3]{x^6z^3\cdot x^6z^3\cdot y^6x^3}=3\sqrt[3]{x^{15}y^6z^6}=3x^5y^2z^2$ (1)

we write the other two analog inequalities $y^6x^3+y^6x^3+z^6y^3\geq 3y^5z^2x^2$ (2),

$z^6y^3+z^6y^3+z^6z^3\geq 3z^5x^2y^2$ (3). We add the inequalities (1), (2), (3), we divide by 3 and we obtain (*). Equality holds if and only if $x=y=z\Leftrightarrow a=b=c$.

04. If $x,y,z>0$ then in ΔABC

$$\sum\frac{x}{y+z}\cdot\frac{a\sqrt{a}}{\sqrt{h_a}}\geq\sqrt{6F}$$

D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

Solution:

$$LHS=\sum\frac{x}{y+z}\cdot\frac{a\sqrt{a}}{\sqrt{h_a}}=\sum\frac{x}{y+z}\cdot\frac{a^2}{\sqrt{ah_a}}=\sum\frac{x}{y+z}\cdot\frac{a^2}{\sqrt{2F}}=\frac{1}{\sqrt{2F}}\sum\frac{x}{y+z}\cdot a^2\geq$$

$$\overset{Tsintsifas}{\geq}\frac{1}{\sqrt{2F}}\cdot 2\sqrt{3}F=\sqrt{6F}=RHS$$

Lemma (G. Tsintsifas): In ΔABC holds:

$$\frac{x}{y+z}\boldsymbol{a^2}+\frac{y}{z+x}\boldsymbol{b^2}+\frac{z}{x+y}\boldsymbol{c^2}\ge \boldsymbol{2\sqrt{3}S}, \text{ where } \boldsymbol{x,y,z}>0$$

G. Tsintsifas

Solution:

We have $\sum\frac{x}{y+z}a^2=\sum\left(\frac{x}{y+z}+1-1\right)a^2=\sum\frac{x+y+z}{y+z}a^2-\sum a^2\overset{Bergstrom}{\ge}$

$$\ge (x+y+z)\frac{(\sum a)^2}{\sum(y+z)}-\sum a^2=(x+y+z)\frac{(2p)^2}{2(x+y+z)}-2(p^2-r^2-4Rr)=$$

$$=2p^2-2(p^2-r^2-4Rr)=2(r^2+4Rr)$$

Above, we've used the known identities in triangle: $\sum a=2p$ and

$$\sum a^2=2(p^2-r^2-4Rr).$$

It remains to prove that $2(r^2+4Rr)\ge 2\sqrt{3}S\Leftrightarrow r^2+4Rr\ge\sqrt{3}rp\Leftrightarrow 4R+r\ge p\sqrt{3}$, witch is Doucet's inequality.Equality holds if and only if $a=b=c$ and $x=y=z$.

05. In ΔABC holds:

$$\sum\frac{\boldsymbol{a^3}}{\boldsymbol{b+c}}\ge \boldsymbol{2\sqrt{3}F}$$

D.M. Bătinețu-Giurgiu, Dan Nănuți – Romania

Solution:

$$LHS=\sum\frac{a^3}{b+c}\overset{Holder}{\ge}\frac{(\sum a)^3}{3\sum(b+c)}=\frac{(\sum a)^3}{3\cdot 2\sum a}=\frac{(\sum a)^2}{6}=\frac{(2p)^2}{6}=\frac{4p^2}{6}\overset{Mitrinovic}{\ge}2\sqrt{3}pr$$

$=2\sqrt{3}F=RHS$. Equality holds if and only if the triangle is equilateral. **Remark:** The problem can be developed. **In ΔABC holds:**

$$\sum\frac{\boldsymbol{a^3}}{\boldsymbol{b+\lambda c}}\ge\frac{\boldsymbol{4\sqrt{3}}}{\boldsymbol{\lambda+1}}\boldsymbol{F},\boldsymbol{\lambda}\ge\boldsymbol{0}$$

Marin Chirciu-Romania

Solution:

$$LHS=\sum\frac{a^3}{b+\lambda c}\overset{Holder}{\ge}\frac{(\sum a)^3}{3\sum(b+\lambda c)}=\frac{(\sum a)^3}{3(\lambda+1)\sum a}=\frac{(\sum a)^2}{3\cdot(\lambda+1)}=\frac{(2p)^2}{3\cdot(\lambda+1)}=$$

$$=\frac{4p^2}{3\cdot(\lambda+1)}\overset{Mitrinovic}{\ge}\frac{4\sqrt{3}}{\lambda+1}pr=\frac{4\sqrt{3}}{\lambda+1}F=RHS.$$

Equality holds if and only if the triangle is equilateral

Note: For $\lambda = 1$ we obtain the problem J.2104 from RMM-40 Spring Edition 2024, proposed by D.M. Bătinețu-Giurgiu, Dan Nănuți.

In ΔABC holds:

$$\sum \frac{a^3}{b+c} \geq 2\sqrt{3}F.$$

D.M. Bătinețu-Giurgiu, Dan Nănuți – Romania

Remark: The problem can be developed. **In ΔABC holds:**

$$\sum \frac{a^n}{b+\lambda c} \geq \left(\frac{2p}{3}\right)^{n-1} \cdot \frac{3}{\lambda+1}, \lambda \geq 0.$$

Marin Chirciu-Romania

Solution:

$$LHS = \sum \frac{a^n}{b+\lambda c} \overset{Holder}{\geq} \frac{(\sum a)^n}{3^{n-2}\sum(b+\lambda c)} = \frac{(\sum a)^n}{3^{n-2}(\lambda+1)\sum a} = \frac{(\sum a)^{n-1}}{3^{n-2}(\lambda+1)} =$$

$$= \frac{(2p)^{n-1}}{3^{n-2}(\lambda+1)} = \left(\frac{2p}{3}\right)^{n-1} \frac{3}{\lambda+1} = \frac{2^{n-1}p^{n-1}}{3^{n-2}(\lambda+1)} = \frac{2^{n-1}p^{n-3}p^2}{3^{n-2}(\lambda+1)} \overset{Mitrinovic}{\geq}$$

$$\overset{Mitrinovic}{\geq} \frac{2^{n-1}p^{n-3} \cdot 3\sqrt{3}pr}{3^{n-2}(\lambda+1)} = \frac{2^{n-1}p^{n-3} \cdot 3\sqrt{3}F}{3^{n-2}(\lambda+1)} = RHS$$

Equality holds if and only if the triangle is equilateral.

Note: For $\lambda = 1$ and $n = 3$ we obtain the problem J.2104 from RMM-40 Spring Edition 2024, proposed by D.M. Bătinețu-Giurgiu, Dan Nănuți.

Reference: Romanian Mathematical Magazine-www.ssmrmh.ro

A SURPRISING PROOF FOR A TRIGONOMETRIC INEQUALITY

Prove that:

$$-\sqrt{2} \leq \sin x + \cos x \leq \sqrt{2}, \forall x \in \mathbb{R}$$

By Tran Quoc Anh-Vietnam

Solution by proposer

Consider $f(t) = t^2 + t(\sin x + \cos x) + \frac{1}{2}$, we have discriminant is:

$$\Delta = (\sin x + \cos x)^2 - 4 \cdot \frac{1}{2} = \sin^2 x + \cos^2 x + 2\sin x \cos x - 2$$

$$= 2\sin x\cos x - 1 = \sin 2x - 1 \le 0$$

Thus

$$f(t) = t^2 + t(\sin x + \cos x) + \frac{1}{2} \ge 0, \forall t \in \mathbb{R}$$

Choose $t = -\frac{\sqrt{2}}{2}$, we have:

$$f\left(-\frac{\sqrt{2}}{2}\right) = \left(-\frac{\sqrt{2}}{2}\right)^2 - \frac{\sqrt{2}}{2}\cdot(\sin x + \cos x) + \frac{1}{2} \ge 0$$

$$\Leftrightarrow -\frac{\sqrt{2}}{2}\cdot(\sin x + \cos x) + 1 \ge 0 \Leftrightarrow \sin x + \cos x \le \sqrt{2}$$

Choose $t = \frac{\sqrt{2}}{2}$, we have:

$$f\left(\frac{\sqrt{2}}{2}\right) = \left(\frac{\sqrt{2}}{2}\right)^2 + \frac{\sqrt{2}}{2}\cdot(\sin x + \cos x) + \frac{1}{2} \ge 0 \Leftrightarrow \frac{\sqrt{2}}{2}\cdot(\sin x + \cos x) + 1 \ge 0$$

$$\Leftrightarrow -\sqrt{2} \le \sin x + \cos x$$

100 OLD AND NEW INEQUALITIES AND IDENTITIES IN TRIANGLE

By Bogdan Fuștei-Romania

In memory of TRAN HONG-Vietnam

We consider the triangle ABC with well-known results:

$$\cos^2\frac{A}{2} = \frac{1+\cos A}{2} \text{ (and analogs);}$$

$$\cos^2\frac{A}{2} = \frac{r_b + r_c}{4R} \text{ (and analogs);}$$

$r_a = \frac{S}{p-a}$ (and analogs);

a=2Rsin A (and analogs);

$\sin\frac{A}{2} = \sqrt{\frac{r}{2R}}\sqrt{\frac{r_a}{h_a}} = \sqrt{\frac{r_a - r}{4R}}$ (and analogs);

$r = \frac{S}{p}$; $2p = a + b + c$;

$$\frac{1+\cos A}{2}=\frac{r_b+r_c}{4R} \rightarrow \cos A=\frac{r_b+r_c-2R}{2R}=\frac{r_a+r_b+r_c-2R-r_a}{2R}$$

$$r_a+r_b+r_c=4R+r\ ;$$

We obtain the next identity : $\cos A=\frac{2R+r-r_a}{2R}$ (and analogs);

$$\rightarrow \mathbf{\cos B+\cos C=\frac{r_a+r}{2R}} \rightarrow \mathbf{\frac{\cos B+\cos C}{\sin A}=\frac{r_a+r}{a}} \textbf{ (and analogs) (1)}$$

$$\mathbf{r_a+r=\frac{S}{p-a}\frac{(2p-a)}{p}=r_a\frac{b+c}{p}} \rightarrow \mathbf{\frac{r_a+r}{r_a}=\frac{b+c}{p}} \textbf{ (and analogs) (2)}$$

$$2S=ah_a=bh_b=ch_c=2pr\ ;\ r_a=\frac{ah_a}{2(p-a)}=\frac{ah_a}{b+c-a} \rightarrow$$

$$\mathbf{\frac{r_a}{h_a}=\frac{a}{b+c-a}} \textbf{ (and analogs) (3)}$$

We will obtain

$$\mathbf{\frac{b+c}{a}=1+\frac{h_a}{r_a}} \textbf{ (and analogs) (4)}$$

From (2) and (4) we obtain

$$\mathbf{\frac{r_a+r}{a}=\frac{r_a+h_a}{p}} \textbf{ (and analogs) (5)}$$

From (1) and (5) we obtain

$$\mathbf{\frac{\cos B+\cos C}{\sin A}=\frac{r_a+h_a}{p}} \ (\textbf{and analogs})(\mathbf{6})$$

We know that : $\frac{R}{r} \geq \frac{n_a+h_a}{h_a}$ (and analogs)**([1])** ;

$$\frac{R}{r}=\frac{a}{2r}\frac{1}{\sin A}.$$

After simplification we obtain this new inequality :

$$\mathbf{\frac{1}{\sin A} \geq \frac{n_a+h_a}{p}} \textbf{ (7)} \times(\cos B+\cos C) \rightarrow \mathbf{\frac{\cos B+\cos C}{\sin A} \geq \frac{n_a+h_a}{p}(\cos B+\cos C)} \textbf{ (8)}$$

From (6) and (8) we obtain :

$$\mathbf{\frac{r_a+h_a}{n_a+h_a} \geq \cos B+\cos C}\ (\mathbf{9}) \text{ (and analogs)}$$

Its easy to see that $\cos A+\cos B+\cos C=\frac{R+r}{R}$ and using (9) we will obtain a new inequality :

$$\mathbf{\sum\frac{r_a+h_a}{n_a+h_a} \geq \frac{2(R+r)}{R}} \textbf{ (10)}$$

Now we use the inequality: $\frac{R}{r} \geq \frac{m_b}{h_c}+\frac{m_c}{h_b}$ **([2])** and $\frac{R}{r}=\frac{a}{2r}\frac{1}{\sin A}$ (and analogs) ×($\cos B+$ $+\cos C$) and we obtain : $\frac{a}{2r}\frac{r_a+h_a}{p} \geq (\cos B+\cos C)(\frac{m_b}{h_c}+\frac{m_c}{h_b})$

2S=$ah_a=bh_b=ch_c$=2pr ,we get the next relationship :

$\frac{a}{a}\frac{r_a+h_a}{h_a} \geq (\cos B+\cos C)(\frac{m_b}{h_c}+\frac{m_c}{h_b})$, so we will get another inequality:

$$\mathbf{\frac{r_a+h_a}{h_a} \geq (\cos B+\cos C)(\frac{m_b}{h_c}+\frac{m_c}{h_b})} \textbf{(11)}\text{(and analogs)}$$

We know that: $\sin^2\frac{A}{2}=\frac{r_a-r}{4R}=\frac{r}{2R}\frac{r_a}{h_a}$ (and analogs)→ $\frac{r_a}{h_a}=\frac{r_a-r}{2r}$ (and analogs)

After sumation we obtain $\sum\frac{r_a}{h_a}=\frac{2R-r}{r}$ →3+$\sum\frac{r_a}{h_a}=2(1+\frac{R}{r})$

From (11) we obtain this inequality:

$$\mathbf{1+\frac{R}{r} \geq \frac{1}{2}\sum(\cos B+\cos C)(\frac{m_b}{h_c}+\frac{m_c}{h_b})} \textbf{ (12)}$$

Also $\frac{r_ar_br_c}{h_ah_bh_c}=\frac{R}{2r}$ and from (11) we obtain a new inequality:

$$\mathbf{\frac{R}{2r} \geq \prod[(\cos B+\cos C)\left(\frac{m_b}{h_c}+\frac{m_c}{h_b}\right)-1]} \textbf{(13)}$$

$\frac{R}{2r} \geq \frac{m_a}{h_a}$ (and analogs)(Panaitopol); $\frac{R}{r}=\frac{a}{2r}\frac{1}{\sin A}$,after some simple manipulations we obtain a new inequality :

$$\mathbf{r_a+h_a \geq 2m_a(\cos B+\cos C)} \textbf{(and analogs) (14)}$$

From (4) and (11) we obtain a new inequality :

$$\mathbf{\frac{b+c}{a} \geq \frac{2m_a}{r_a}(\cos B+\cos C)(and\ analogs)} \textbf{(15)}$$

From (14) we have $\frac{r_a+h_a}{2m_a} \geq \cos B+\cos C$ and after sumation and using

$\cos A++\cos B+\cos C=\frac{R+r}{R}$,we will obtain this next inequality :

$$\mathbf{\sum\frac{r_a+h_a}{m_a} \geq \frac{4(R+r)}{R}} \textbf{ (16)}$$

From $\frac{r_a+h_a}{2m_a} \geq \cos B+\cos C$ and (9) we have :

$$\mathbf{(r_a+h_a)(\frac{1}{2m_a}+\frac{1}{n_a+h_a}) \geq 2(\cos B+\cos C)} \textbf{ (and analogs) (17)}$$

2S=$ah_a=bh_b=ch_c$=2pr=(a+b+c)r→ $\frac{h_a}{r}=1+\frac{b+c}{a}$ (and analogs) and using (15) we obtain a new inequality:

$$\frac{h_a+h_b+h_c-3r}{2r} \geq \sum \frac{m_a}{r_a}(\cos B+\cos C) \ \textbf{(18)}$$

From $r_a+r=\frac{S}{p-a}\frac{(2p-a)}{p}=r_a\frac{b+c}{p}$ and $r_a-r=\frac{r_a}{p}a$, we obtain this new relationship

$$\frac{r_a+r}{r_a-r}=\frac{b+c}{a} \ \textbf{(and analogs) (19)}$$

and using (15) we obtain the next inequality:

$$\frac{r_a+r}{r_a-r} \geq \frac{2m_a}{r_a}(\cos B+\cos C)(20)$$

$\frac{h_a}{r}=1+\frac{b+c}{a}$ (and analogs) and using (15) we obatin another inequality:

$$\frac{h_a-r}{r} \geq \frac{2m_a}{r_a}(\cos B+\cos C)(21)$$

$$\rightarrow \frac{r_a}{2r} \geq \frac{m_a}{h_a-r}(\cos B+\cos C) \ \textbf{(22)}$$

and after sumation we will get a fresh inequality:

$$\frac{4R+r}{2r} \geq \sum \frac{m_a}{h_a-r}(\cos B+\cos C)\textbf{(23)}$$

We know that this next relation is true :$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}$, $\cos B+\cos C=\frac{r_a+r}{2R} \rightarrow \frac{\cos B+\cos C}{r_a}$ $=\frac{1}{2R}\frac{b+c}{p}$ (and analogs) and after sumation we obtain this new identity:

$$\sum \frac{\cos B+\cos C}{r_a}=\frac{2}{R} \ \textbf{(24)}$$

from $\frac{h_a-r}{m_a r} \geq \frac{2(\cos B+\cos C)}{r_a}$ (and analogs) and after sumation we will obtain a new relationship :

$$\sum \frac{h_a-r}{m_a} \geq \frac{4r}{R} \ \textbf{(25)}$$

Now we will use Tereshin inequality: $m_a \geq \frac{b^2+c^2}{4R}$ (and analogs) and the identity $bc=2Rh_a$ (and analogs) and we will obtain a new inequality:

$\frac{m_a}{h_a} \geq \frac{1}{2}(\frac{b}{c}+\frac{b}{c})$ (and analogs) ,after sumation we obtain the inequality:

$\sum \frac{m_a}{h_a} \geq \frac{1}{2}\sum \frac{b+c}{a}$ and from (15) we get:

$$\frac{1}{2}\sum \frac{b+c}{a} \geq \sum \frac{m_a}{r_a}(\cos B+\cos C) \ \textbf{(26)}$$

From these last two inequalitys we obtain another inequality:

$$\sum \frac{m_a}{h_a} \geq \sum \frac{m_a}{r_a}(\cos B+\cos C) \ \textbf{(27)}$$

We know that $\cos\frac{B-C}{2}=\frac{h_a}{l_a}$ (and analogs) and $\cos\frac{B-C}{2}=\frac{b+c}{a}\sin\frac{A}{2}$ (and analogs) we obtain $\frac{h_a}{l_a}=\frac{b+c}{a}\sin\frac{A}{2}$ and from $\sin\frac{A}{2}=\sqrt{\frac{r}{2R}}\sqrt{\frac{r_a}{h_a}}$ (and analogs) we obtain an identity**:**

$$\mathbf{\frac{b+c}{a}=\sqrt{\frac{2R}{r}}\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}}\,(and\ analogs)(28)}$$

and from (19) we obtain the next identity :

$$\mathbf{\frac{r_a+r}{r_a-r}=\sqrt{\frac{2R}{r}}\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}}\,(and\ analogs)(29)}$$

$$\rightarrow \mathbf{\frac{r_a+r}{h_a}=\sqrt{\frac{2R}{r}}\frac{(r_a-r)}{l_a}\sqrt{\frac{h_a}{r_a}}}\ \textbf{(30)}$$

We will use the well known identity: $:\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r},\sum\frac{r_a}{h_a}=\frac{2R-r}{r}$

and we will obtain a new result:

$$\sum\mathbf{\frac{r_a+r}{h_a}=\frac{2R-r}{r}+1=\frac{2R}{r}}\ \textbf{(31)}$$

From (30) and (31) we get :

$$\sum\mathbf{\frac{(r_a-r)}{l_a}\sqrt{\frac{h_a}{r_a}}=\sqrt{\frac{2R}{r}}}\ \textbf{(32)}$$

From (4) and (28) we have $:\frac{r_a+h_a}{r_a}=\sqrt{\frac{2R}{r}}\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}}$ and after some simple manipulation we obtain a new result :

$$\mathbf{1+\frac{r_a}{h_a}=\sqrt{\frac{2R}{r}}\frac{\sqrt{r_ah_a}}{l_a}}\ \textbf{(33)}\ \text{(and analogs)}$$

From (14) we get $1+\frac{r_a}{h_a}\geq\frac{2m_a}{h_a}(\cos B+\cos C)$ and from (33) we obtain:

$$\mathbf{\sqrt{\frac{R}{2r}}\frac{\sqrt{r_ah_a}}{l_a}\geq\frac{m_a}{h_a}(\cos B+\cos C)}\ \textbf{(34)}\ .$$

From (11) and (33) we obtain a new result:

$$\mathbf{\sqrt{\frac{2R}{r}}\frac{\sqrt{r_ah_a}}{l_a}\geq(\cos B+\cos C)\left(\frac{m_b}{h_c}+\frac{m_c}{h_b}\right)}\ \textbf{(35)}\text{(and analogs).}$$

From $1+\frac{r_a}{h_a}\geq\frac{2m_a}{h_a}(\cos B+\cos C)$(and analogs) after sumation we obtain:

$3+\sum\frac{r_a}{h_a}=2(1+\frac{R}{r})\geq2\sum\frac{m_a}{h_a}(\cos B+\cos C)$ and after simplification we get:

$$1+\frac{R}{r}\geq\sum\frac{m_a}{h_a}(\cos B+\cos C)\ (36).$$

From (33) we have :

$3+\sum\frac{r_a}{h_a}=2\left(1+\frac{R}{r}\right)=\sqrt{\frac{2R}{r}}\sum\frac{\sqrt{r_ah_a}}{l_a}$ and after simplification we obtain the identity: $\sum\frac{\sqrt{r_ah_a}}{l_a}=$

$$\left(1+\frac{R}{r}\right)\sqrt{\frac{2r}{R}}=\sqrt{2}\left(\sqrt{\frac{R}{r}}+\sqrt{\frac{r}{R}}\right)\ (37)$$

From (37) and $h_a\leq l_a$ (and analogs) $\sin\frac{A}{2}=\sqrt{\frac{r}{2R}}\sqrt{\frac{r_a}{h_a}}$ (and analogs), we obtain the well known inequality:

$$\sum\sin\frac{A}{2}\geq\cos A+\cos B+\cos C=1+\frac{r}{R}$$

From (4) and (15) we obtain the inequality:

$$3+\sum\frac{h_a}{r_a}\geq 2\sum\frac{m_a}{r_a}(\cos B+\cos C)\ (38)$$

(Van-Aubel) If AD, BE and CF are three cevianes concurent in a point

P inside to triangle ABC, then: $\frac{AP}{PD}=\frac{AF}{FB}+\frac{AE}{EC}$**.**

If N_a is the Nagel point in triangle ABC ,applying Van-Aubel theorem,we have:

$\frac{AN_a}{n_a-AN_a}=\frac{p-c}{p-a}+\frac{p-b}{p-a}=\frac{a}{p-a}\rightarrow\frac{n_a-AN_a}{AN_a}=\frac{p-a}{a}\rightarrow\frac{n_a}{AN_a}=\frac{p}{a}$ **(and analogs)**

$$AN_a=\frac{an_a}{p}\text{ (and analogs)}\rightarrow 1+\frac{b+c}{a}=\frac{2n_a}{AN_a}\text{ (and analogs) (39)}$$

$\frac{h_a}{r}=1+\frac{b+c}{a}$ (and analogs)and useing (39) we obtain:

$$\frac{n_a}{h_a}=\frac{AN_a}{2r}\text{ (and analogs)(40)}$$

$p^2=n_a^2+2r_ah_a$(and analogs)([**1**]), $2S=ah_a=bh_b=ch_c=2pr$

$\frac{a}{2r}=\frac{p}{h_a}$ (and analogs) $\rightarrow\frac{a^2}{4r^2}=\frac{n_a^2}{h_a^2}+2\frac{r_a}{h_a}$ (and analogs), $\frac{r_a}{h_a}=\frac{r_a-r}{2r}$

But $(r_a-r)r=(p-b)(p-c)-r^2$

$$4p(p-a)=\frac{4(a+b+c)(b+c-a)}{4}=(a+b+c)(b+c-a)=(b+c)^2-a^2$$

$$a^2=(b+c)^2-4p(p-a)\text{ (and analogs).}$$

$\frac{a^2}{4r^2}=\frac{n_a^2}{h_a^2}+2\frac{r_a}{h_a}$ becomes $\frac{(b+c)^2-4p(p-a)}{4r^2}=\frac{n_a^2}{h_a^2}+\frac{4(p-b)(p-c)-4r^2}{4r^2}$

$$\frac{n_a^2}{h_a^2}-1=\frac{(b+c)^2-4p(p-a)-4(p-b)(p-c)}{4r^2}\text{(and analogs)}$$

$$4(p-b)(p-c)=(a+c-b)(a+b-c)=a^2+2bc-b^2-c^2,$$

$$4p(p-a)+4(p-b)(p-c)=(b+c)^2-a^2+a^2+2bc-b^2-c^2=4bc$$

$$\frac{n_a^2}{h_a^2}-1=\frac{(b+c)^2-4bc}{4r^2}=\frac{(b-c)^2}{4r^2}\rightarrow$$

$$\mathbf{\frac{n_a^2}{h_a^2}=1+\frac{(b-c)^2}{4r^2}\ (and\ analogs)(41)}$$

From (40) and (41) we obtain :

$$\mathbf{AN_a=\sqrt{4r^2+(b-c)^2}}\ \textbf{(and analogs) (42)}$$

From (39) and (42) we obtain :

$$\mathbf{1+\frac{b+c}{a}=\frac{2n_a}{\sqrt{4r^2+(b-c)^2}}}\ \textbf{(and analogs) (43)}$$

From (40) and (42) we obtain :

$$\mathbf{\frac{n_a}{h_a}=\frac{\sqrt{4r^2+(b-c)^2}}{2r}=\sqrt{1+\frac{(b-c)^2}{4r^2}}}\ \textbf{(and analogs) (44)}$$

From (43) after sumation we obtain the identity:

$$\mathbf{3+\sum\frac{b+c}{a}=\sum\frac{2n_a}{\sqrt{4r^2+(b-c)^2}}}\ \textbf{(45)}$$

From (4) and (45) after sumation we obtain :

$$\mathbf{6+\sum\frac{h_a}{r_a}=\sum\frac{2n_a}{\sqrt{4r^2+(b-c)^2}}}\ \textbf{(46)}$$

From (4) and (43) we obtain :

$$\mathbf{\frac{h_a}{r_a}=\frac{b+c-a}{a}=2\left(\frac{n_a}{\sqrt{4r^2+(b-c)^2}}-1\right)}\ \textbf{(and analogs) (47)}$$

From (47) we obtain :

$$\mathbf{\sqrt{4r^2+(b-c)^2}=\frac{2n_ar_a}{2r_a+h_a}}\ \textbf{(and analogs) (48)}$$

From (48) after sumation we obtain:

$$\mathbf{\sum\frac{\sqrt{4r^2+(b-c)^2}}{r_a}=\sum\frac{2n_a}{2r_a+h_a}}\ \textbf{(49)}$$

We start from well known inequality : $2m_a\ge(b+c)\cos\frac{A}{2}$ and after some manipulation we obtain : $1+\frac{2m_a}{a\cos\frac{A}{2}}\ge1+\frac{b+c}{a}=\frac{2n_a}{\sqrt{4r^2+(b-c)^2}}$,and we obtain:

$$\frac{1}{2} \geq \frac{n_a}{\sqrt{4r^2+(b-c)^2}} - \frac{m_a}{a\cos\frac{A}{2}}$$ **(and analogs) (50)** and we obtain:

$$\frac{1}{8} \geq \prod\left(\frac{n_a}{\sqrt{4r^2+(b-c)^2}} - \frac{m_a}{a\cos\frac{A}{2}}\right) \quad (51)$$

From (19) we have :$1+\frac{b+c}{a} = \frac{2r_a}{r_a-r}$ and from (43) we obtain :

$$\frac{n_a}{r_a} = \frac{\sqrt{4r^2+(b-c)^2}}{r_a-r}$$ **(and analogs) (52)**

and after sumation we obtain:

$$\sum\frac{n_a}{r_a} = \sum\frac{\sqrt{4r^2+(b-c)^2}}{r_a-r} \quad (53)$$

but $r_a - r = 4R\sin^2\frac{A}{2}$ (and analogs) and we obtain:

$$4R\left(\frac{n_a}{r_a}+\frac{n_b}{r_b}+\frac{n_c}{r_c}\right) = \sum\frac{\sqrt{4r^2+(b-c)^2}}{\sin^2\frac{A}{2}} \quad (54)$$

$r_a - r = \frac{r_a\sqrt{4r^2+(b-c)^2}}{n_a}$ (and analogs); $r_a + r_b + r_c = 4R + r$

We obtain a new relationship :

$$2R - r = \frac{1}{2}\sum\frac{r_a\sqrt{4r^2+(b-c)^2}}{n_a} \quad (55)$$

$1+\frac{b+c}{a} \geq 1+\frac{2m_a}{r_a}(\cos B+\cos C)$, and using (43) we obtain:

$$\frac{2n_a}{\sqrt{4r^2+(b-c)^2}} \geq 1+\frac{2m_a}{r_a}(\cos B+\cos C)$$ **(and analogs) (56)**

$\sum\frac{m_a}{h_a} \geq \frac{1}{2}\sum\frac{b+c}{a}$, and useing (43) we obtain a new inequality :

$$\frac{3}{2}+\sum\frac{m_a}{h_a} \geq \sum\frac{n_a}{\sqrt{4r^2+(b-c)^2}} \quad (57)$$

From $\frac{m_a}{h_a} \geq \frac{1}{2}\left(\frac{b}{c}+\frac{b}{c}\right)$ (and analogs) after sumation we obtain:

$2\left(\frac{m_b}{h_b}+\frac{m_c}{h_c}\right) \geq \frac{b+c}{a}+\frac{a}{c}+\frac{a}{b}$ and $\frac{a}{c} = \frac{h_c}{h_a}$ (and analogs) we obtain:

$2\left(\frac{m_b}{h_b}+\frac{m_c}{h_c}\right) \geq \frac{b+c}{a}+\frac{h_b+h_c}{h_a}$ and using (43) we obtain :

$$2\left(\frac{m_b}{h_b}+\frac{m_c}{h_c}-\frac{n_a}{\sqrt{4r^2+(b-c)^2}}\right) \geq \frac{h_b+h_c-h_a}{h_a}$$ **(and analogs) (58)**

$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = 3+\sum\frac{b+c}{a}$ and using (43) we obtain:

$$\mathbf{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=\sum\frac{2n_a}{\sqrt{4r^2+(b-c)^2}}\ (59)}$$

From $\frac{h_a}{r}=1+\frac{b+c}{a}$ (and analogs), $1+\frac{b+c}{a}=\frac{2n_a}{\sqrt{4r^2+(b-c)^2}}$ (and analogs), $\frac{1}{h_a}+\frac{1}{h_b}++\frac{1}{h_c}=\frac{1}{r}$ we obtain a new identity:

$$\mathbf{(h_a+h_b+h_c)\left(\frac{1}{h_a}+\frac{1}{h_b}++\frac{1}{h_c}\right)=\sum\frac{2n_a}{\sqrt{4r^2+(b-c)^2}}\ (60)}$$

From $p^2=n_a^2+2r_ah_a$(and analogs) $\rightarrow p^2-n_a^2=(p+n_a)(p-n_a)=2r_ah_a$

$\frac{a}{2r}=\frac{p}{h_a}$ (and analogs) $\rightarrow\frac{a}{2r}=\frac{n_a}{h_a}+\frac{2r_a}{n_a+p}$ and $\frac{n_a}{h_a}=\frac{\sqrt{4r^2+(b-c)^2}}{2r}$ (and analogs) we obtain $\mathbf{\frac{r_a}{n_a+p}=\frac{a-\sqrt{4r^2+(b-c)^2}}{4r}}$ **(and analogs) (61)**

and after sumation we obtain:

$$\mathbf{\frac{p}{2r}=\sum\frac{r_a}{n_a+p}+\sum\sqrt{1+\frac{(b-c)^2}{4r^2}}\ (62)}$$

from (61) we have :

$\mathbf{\frac{n_a+p}{4r}=\frac{r_a}{a-\sqrt{4r^2+(b-c)^2}}}$ **(and analogs)** and after sumation we obtain a new result :

$$\mathbf{\frac{n_a+n_b+n_c+3p}{4r}=\sum\frac{r_a}{a-\sqrt{4r^2+(b-c)^2}}\ (63)}$$

We use the well know inequality $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}$ and $\mathbf{\frac{n_a+p}{r_a}=\frac{4r}{a-\sqrt{4r^2+(b-c)^2}}}$ **(and analogs)** we obtain a new result:

$$\mathbf{\frac{n_a}{r_a}+\frac{n_b}{r_b}+\frac{n_c}{r_c}+\frac{p}{r}=4\sum\frac{r}{a-\sqrt{4r^2+(b-c)^2}}\ (64)}$$

$$\mathbf{\frac{(n_a+p)(n_b+p)(n_c+p)}{64r^3}=\prod\frac{r_a}{a-\sqrt{4r^2+(b-c)^2}}\ (65)}$$

but $\sqrt{r_br_c}\geq l_a$ (and analogs)$\rightarrow r_ar_br_c\geq l_al_bl_c$ and we obtain:

$$\mathbf{\frac{(n_a+p)(n_b+p)(n_c+p)}{64r^3}\geq\prod\frac{l_a}{a-\sqrt{4r^2+(b-c)^2}}\ (66)}$$

$p\geq3\sqrt{3}r$ (Mitrinovic inequality) and with (64) we obtain a new inequality:

$$\mathbf{3\sqrt{3}+\frac{n_a}{r_a}+\frac{n_b}{r_b}+\frac{n_c}{r_c}\leq4\sum\frac{r}{a-\sqrt{4r^2+(b-c)^2}}\ (67)}$$

$\mathbf{\frac{p}{r}\geq\sqrt{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}$**(** ***Ion Cristian Miu-rafinament of Mitrinovic inequality*****)**

and from (64) we obtain a new inequality:

$$\mathbf{\frac{n_a}{r_a}+\frac{n_b}{r_b}+\frac{n_c}{r_c}+\sqrt{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\leq4\sum\frac{r}{a-\sqrt{4r^2+(b-c)^2}}\ (68)}$$

Now we use :

$m_a l_a \ge r_b r_c = p(p-a)$(and analogs)(Panaitopol)$\rightarrow \sqrt{m_a m_b m_c l_a l_b l_c} \ge r_a r_b r_c$ and with (65) we obtain a new inequality :

$$\frac{(n_a+p)(n_b+p)(n_c+p)}{64r^3} \le \prod \frac{\sqrt{m_a l_a}}{a-\sqrt{4r^2+(b-c)^2}} \quad \textbf{(69)}$$

$\frac{(n_b+p)(n_c+p)}{16r^2} = \frac{r_b r_b}{[b-\sqrt{(a-c)^2+4r^2}][c-\sqrt{(a-b)^2+4r^2}}$ (and analogs) and using $m_a l_a \ge r_b r_c$ we obtain :

$$\frac{(n_b+p)(n_c+p)}{16r^2} \le \frac{m_a}{[b-\sqrt{(a-c)^2+4r^2}]} \frac{l_a}{[c-\sqrt{(a-b)^2+4r^2}]} \quad \textbf{(70)}$$

Is easy to proof that :$(b-c)^2 = 4(m_a^2 - r_b r_c)$ (and analogs)

$AN_a = \sqrt{4r^2+(b-c)^2}$ (and analogs)$\rightarrow AN_a = \sqrt{4r^2+4(m_a^2-r_b r_c)}$

$$AN_a = 2\sqrt{m_a^2+r^2-r_b r_c} \textbf{(and analogs)(71)}$$

$$1+\frac{b+c}{a} = \frac{2n_a}{\sqrt{4r^2+(b-c)^2}} \rightarrow 1+\frac{b+c}{a} = \frac{n_a}{\sqrt{m_a^2+r^2-r_b r_c}} \textbf{ (and analogs) (72)}$$

$$\frac{n_a}{h_a} = \frac{\sqrt{4r^2+(b-c)^2}}{2r} \rightarrow \frac{n_a}{h_a} = \frac{\sqrt{m_a^2+r^2-r_b r_c}}{r} \textbf{ (and analogs) (73)}$$

$$3+\sum\frac{b+c}{a} = \sum\frac{2n_a}{\sqrt{4r^2+(b-c)^2}} \rightarrow 3+\sum\frac{b+c}{a} = \sum\frac{n_a}{\sqrt{m_a^2+r^2-r_b r_c}} \textbf{ (and analogs) (74)}$$

$$6+\sum\frac{h_a}{r_a} = \sum\frac{2n_a}{\sqrt{4r^2+(b-c)^2}} \rightarrow 6+\sum\frac{h_a}{r_a} = \sum\frac{n_a}{\sqrt{m_a^2+r^2-r_b r_c}} \textbf{ (and analogs) (75)}$$

$\frac{h_a}{r_a} = \frac{b+c-a}{a} = 2\left(\frac{n_a}{\sqrt{4r^2+(b-c)^2}} - 1\right)$

$$\frac{h_a}{r_a} = \frac{b+c-a}{a} = \frac{n_a}{\sqrt{m_a^2+r^2-r_b r_c}} - 2 \text{ (and analogs)(76)}$$

$$\sqrt{4r^2+(b-c)^2} = \frac{2n_a r_a}{2r_a+h_a} \rightarrow \sqrt{m_a^2+r^2-r_b r_c} = \frac{n_a r_a}{2r_a+h_a} \textbf{(and analogs) (77)}$$

$$\sum\frac{\sqrt{4r^2+(b-c)^2}}{r_a} = \sum\frac{2n_a}{2r_a+h_a} \rightarrow \sum\frac{\sqrt{m_a^2+r^2-r_b r_c}}{r_a} = \sum\frac{n_a}{2r_a+h_a} \textbf{ (and analogs) (78)}$$

If triangle ABC is acuteangled then we have ERDOS Inequality :

R+r ≤ max (h_a, h_b, h_c) (RMM-Famous Inequalitys Marathon 1-100,inequality 31)**[3]**

If $h_a = \max(h_a, h_b, h_c) \rightarrow \frac{h_a}{r} = 1 + \frac{b+c}{a} \ge \frac{R+r}{r}$ and $1+\frac{b+c}{a} = \frac{2n_a}{\sqrt{4r^2+(b-c)^2}}$ we obtain a new inequality :

$$\frac{2n_a}{\sqrt{4r^2+(b-c)^2}} \geq \frac{R+r}{r} \ \mathbf{(79)}$$

and using (72) we obtain an equivalent form :

$$\frac{n_a}{\sqrt{m_a^2+r^2-r_b r_c}} \geq \frac{R+r}{r} \ \mathbf{(80)}$$

Now we will use this inequality,true for every triangle ABC:

$m_a + m_b + m_c \leq 2(R - 2r) + h_a + h_b + h_c$ (Jian Liu **[4]**)

$\frac{h_a}{r} = 1 + \frac{b+c}{a}$ (and analogs) $\rightarrow \frac{m_a+m_b+m_c}{r} \leq 2(\frac{R}{r} - 2) + 3 + \sum\frac{b+c}{a}$

$1+\frac{b+c}{a} = \frac{2n_a}{\sqrt{4r^2+(b-c)^2}}$ (analogs)$\rightarrow \frac{m_a+m_b+m_c}{r} \leq 2(\frac{R}{r} - 2) + 2\sum\frac{n_a}{\sqrt{4r^2+(b-c)^2}}$ and we obtain a new inequality :

$$\frac{\mathbf{m_a + m_b + m_c}}{\mathbf{2r}} \leq \frac{\mathbf{R}}{\mathbf{r}} - \mathbf{2} + \sum\frac{\mathbf{n_a}}{\sqrt{\mathbf{4r^2 + (b-c)^2}}} \ (\mathbf{81})$$

Now we start from the next inequality:$n_a + g_a \geq 2m_a$ (and analogs)**[5]**

And we obtain: $2n_a \geq 2(2m_a - g_a)$

$$\rightarrow \mathbf{1+\frac{b+c}{a}} \geq \frac{\mathbf{2(2m_a-g_a)}}{\sqrt{\mathbf{4r^2+(b-c)^2}}} \ \textbf{(and analogs) (82)}$$

$$1+\frac{b+c}{a} = \frac{n_a}{\sqrt{m_a^2+r^2-r_b r_c}} \rightarrow \mathbf{1+\frac{b+c}{a}} \geq \frac{\mathbf{2m_a-g_a}}{\sqrt{\mathbf{m_a^2+r^2-r_b r_c}}} \ \textbf{(and analogs) (83)}$$

We know that :$l_a = \frac{2\sqrt{bcp(p-a)}}{b+c}$,$r_b r_c = p(p-a) \rightarrow l_a = \frac{2\sqrt{bc}}{b+c}\sqrt{r_b r_c}$ (and analogs)

and we obtain :$\frac{r_a r_b r_c}{l_a l_b l_b} = \frac{(a+b)(b+c)(a+c)}{8abc}$, $\frac{b+c}{a} = \frac{2n_a}{\sqrt{4r^2+(b-c)^2}} - 1$(and analogs) so we

obtain : $\frac{(a+b)(b+c)(a+c)}{abc} = \prod(\frac{2n_a}{\sqrt{4r^2+(b-c)^2}} - 1)$

From what we prove ,we obtain a new identity:

$$\frac{\mathbf{8r_a r_b r_c}}{\mathbf{l_a l_b l_b}} = \prod\left(\frac{\mathbf{2n_a}}{\sqrt{\mathbf{4r^2+(b-c)^2}}} - \mathbf{1}\right)(\mathbf{84})$$

$1+\frac{b+c}{a} = \frac{n_a}{\sqrt{m_a^2+r^2-r_b r_c}}$ (and analogs) $\rightarrow \frac{\mathbf{8r_a r_b r_c}}{\mathbf{l_a l_b l_b}} = \prod(\frac{\mathbf{n_a}}{\sqrt{\mathbf{m_a^2+r^2-r_b r_c}}} - \mathbf{1})$ **(85)**

From $m_a l_a \geq r_b r_c$ (and analogs)$\rightarrow m_a m_b m_c l_a l_b l_c \geq (r_a r_b r_c)^2$

$\sqrt{\frac{m_a m_b m_c}{l_a l_b l_c}} \geq \frac{r_a r_b r_c}{l_a l_b l_c}$,and useing (84) and (85) we obtain:

$$\sqrt{\frac{\mathbf{m_am_bm_c}}{\mathbf{l_al_bl_c}}} \geq \frac{\mathbf{1}}{\mathbf{8}}\prod\left(\frac{\mathbf{2n_a}}{\sqrt{\mathbf{4r^2+(b-c)^2}}}-\mathbf{1}\right)\mathbf{(86)}$$

$$\sqrt{\frac{\mathbf{m_am_bm_c}}{\mathbf{l_al_bl_c}}} \geq \frac{\mathbf{1}}{\mathbf{8}}\prod\left(\frac{\mathbf{n_a}}{\sqrt{\mathbf{m_a^2+r^2-r_br_c}}}-\mathbf{1}\right)\mathbf{(87)}$$

If ABC is acuteangled triangle with a=min(a,b,c) → r_a= min(r_a, r_b, r_c)

$\frac{r_a+r}{r_a-r}=\frac{b+c}{a}\geq\frac{R}{r}$ (from ERDOS INEQUALITY)→ $r_a+r\geq\frac{R}{r}(r_a-r)\rightarrow$

$R+r\geq(\frac{R}{r}-1)\,r_a$→ $\frac{R+r}{R-r}\ \geq\frac{r_a}{r}$, in the end we have the next result:

If ABC is acuteangled triangle then:

$$\frac{\mathbf{R+r}}{\mathbf{R-r}}\ \geq\frac{\mathbf{min(r_a,r_b,r_c)}}{\mathbf{r}}\ \mathbf{(88)}$$

$R+r\geq\left(\frac{R}{r}-1\right)r_a$, but $\frac{R}{r}-1=\frac{n_a^2+r_a^2}{2r_ah_a}$ (and analogs)(**[1]**) $R+r\geq\frac{n_a^2+r_a^2}{2r_ah_a}r_a\rightarrow \mathbf{2h_a(R+r)}\geq$ $\mathbf{n_a^2+r_a^2}$ **(89)**

$n_a^2+r_a^2\geq 2n_ar_a$ and useing (89)→$\sqrt{\mathbf{h_a(R+r)}}\geq\frac{\mathbf{n_a+r_a}}{\mathbf{2}}$ **(90)**

We know that : $\frac{b+c}{a}=\frac{\cos\frac{B-C}{2}}{\sin\frac{A}{2}}$ (and analogs) (Mollweide's formula)

1+ $\frac{b+c}{a}=\frac{2n_a}{\sqrt{4r^2+(b-c)^2}}=\frac{n_a}{\sqrt{m_a^2+r^2-r_br_c}}$(and analogs) ,se obtain a new result:

$$\frac{\mathbf{2n_a}}{\sqrt{\mathbf{4r^2+(b-c)^2}}}=\frac{\mathbf{n_a}}{\sqrt{\mathbf{m_a^2+r^2-r_br_c}}}=\mathbf{1}+\frac{\mathbf{\cos\frac{B-C}{2}}}{\mathbf{\sin\frac{A}{2}}}\ (\textbf{and analogs})\mathbf{(91)}$$

Also $\cos\frac{B-C}{2}=\frac{h_a}{l_a}$ (and analogs) and AI = $\frac{r}{\sin\frac{A}{2}}$ (and analogs) →

$\frac{AI}{r}=\frac{b+c}{a}\,\frac{l_a}{h_a}$ (and analogs) , 1+ $\frac{b+c}{a}=\frac{2n_a}{\sqrt{4r^2+(b-c)^2}}=\frac{n_a}{\sqrt{m_a^2+r^2-r_br_c}}$(and analogs)

We obtain a new identity:

$$\frac{\mathbf{AI}}{\mathbf{r}}\ =\left(\frac{\mathbf{2n_a}}{\sqrt{\mathbf{4r^2+(b-c)^2}}}-\mathbf{1}\right)\frac{\mathbf{l_a}}{\mathbf{h_a}}=\left(\frac{\mathbf{n_a}}{\sqrt{\mathbf{m_a^2+r^2-r_br_c}}}-\mathbf{1}\right)\frac{\mathbf{l_a}}{\mathbf{h_a}}\ \textbf{(and analogs) (92)}$$

$\frac{AI}{r}\geq\frac{b+c}{a}$ (and analogs)→ $\frac{\mathbf{AI}}{\mathbf{r}}\geq\frac{\mathbf{2n_a}}{\sqrt{\mathbf{4r^2+(b-c)^2}}}-\mathbf{1}=\frac{\mathbf{n_a}}{\sqrt{\mathbf{m_a^2+r^2-r_br_c}}}-\mathbf{1}$ **(93)**

Also from AI = $\frac{r}{\sin\frac{A}{2}}$ (and analogs) , $\sin\frac{A}{2}=\sqrt{\frac{r}{2R}}\sqrt{\frac{r_a}{h_a}}$ and

$\frac{h_a}{r_a} = \frac{b+c-a}{a} = 2(\frac{n_a}{\sqrt{4r^2+(b-c)^2}} - 1)$ we obtain a new relationship :

$$\frac{AI}{2r} = \sqrt{\frac{R}{r}}\sqrt{\frac{n_a}{\sqrt{4r^2+(b-c)^2}} - 1} \text{ (and analogs) (94)}$$

From $\frac{R}{r} - 1 = \frac{n_a^2+r_a^2}{2r_a h_a}$ (and analogs) → $2\frac{h_a}{n_a}(\frac{R}{r} - 1) = \frac{n_a}{r_a} + \frac{r_a}{n_a}$ (and analogs)

$\frac{n_a}{h_a} = \frac{\sqrt{4r^2+(b-c)^2}}{2r}$ (and analogs)→

$$\frac{n_a}{r_a} + \frac{r_a}{n_a} = \frac{4(R-r)}{\sqrt{4r^2+(b-c)^2}} \text{ (and analogs) (95)}$$

From R ≥ 2r (Euler) →

$$\frac{\sqrt{R^2+(b-c)^2}}{2r} \geq \frac{n_a}{h_a} \text{ (and analogs) (96)}$$

From Euler inequality R ≥ 2r and (95) we obtain a new inequality:

$$\frac{n_a}{r_a} + \frac{r_a}{n_a} \geq \frac{4(R-r)}{\sqrt{R^2+(b-c)^2}} \text{ (and analogs) (97)}$$

$\frac{n_a}{r_a} = \frac{\sqrt{4r^2+(b-c)^2}}{r_a - r}$ (and analogs) and $R \geq 2r$ →

$$\frac{\sqrt{R^2+(b-c)^2}}{r_a - r} \geq \frac{n_a}{r_a} \text{ (and analogs) (98)}$$

From $\frac{(a+b)(b+c)(a+c)}{abc} = \prod\left(\frac{2n_a}{\sqrt{4r^2+(b-c)^2}} - 1\right)$ and $R \geq 2r$ we obtain a new result:

$$\frac{(a+b)(b+c)(a+c)}{abc} \geq \prod(\frac{2n_a}{\sqrt{R^2+(b-c)^2}} - 1) \text{ (99)}$$

$$\text{and } \frac{8r_a r_b r_c}{l_a l_b l_b} \geq \prod(\frac{2n_a}{\sqrt{R^2+(b-c)^2}} - 1) \text{(100)}$$

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A CASE OF AM-GM

If $a, b > 0$ and $ab = 1$ then $a + b \geq 2$

By Tran Quoc Anh-Vietnam

Solution by proposer

Consider $f: \mathbb{R} \to \mathbb{R}, f(t) = -abt^2 + (a+b)t - 1$

The discriminant is $\Delta = (a+b)^2 - 4(-ab) \cdot (-1) = (a-b)^2 \geq 0$.

If $\Delta = (a-b)^2 = 0$ then: $f(t) \geq 0, \forall t \in \mathbb{R}$. Choose $t = 1$ we have: $a + b \geq 2$

If $\Delta = (a-b)^2 > 0$, we can assume $a > b$, hence $a^2 > ab$, thus $a > 1$. We have

$$t_1 = \frac{-(a+b) + \sqrt{\Delta}}{-2ab} = \frac{-(a+b) + a - b}{-2ab} = \frac{-2b}{-2ab} = \frac{1}{a}$$

$$t_2 = \frac{-(a+b) - \sqrt{\Delta}}{-2ab} = \frac{-(a+b) - (a-b)}{-2ab} = \frac{-2a}{-2ab} = \frac{1}{b}$$

Because of $a > b$ we have $\frac{1}{a} < \frac{1}{b}$. Therefore: $f(t) > 0, \forall t \in \left(\frac{1}{a}, \frac{1}{b}\right) = \left(\frac{1}{a}, a\right)$

Because of $a > 1$ we have $a > 1 > \frac{1}{a}$ i.e $1 \in \left(\frac{1}{a}, a\right)$. thus:

$$f(1) > 0 \Leftrightarrow -ab + (a+b) - 1 > 0 \Leftrightarrow a + b > 2$$

A NEW SEQUENCE OF PRIME NUMBERS

By Mohammed Bouras-Morocco

ABSTRACT. In this paper, we discovered a new sequence of prime numbers, every term of this sequence is either a prime number or equal to 1.

Keywords. Prime numbers, sequence.

INTRODUCTION

A number is said to be a prime number if the number is divisible only by 1 and itself; otherwise it's composite. In this paper, we present two new sequences related with the continued fraction.

THE SEQUENCE $b(n)$

The sequence $b(n)$ satisfy the following recursive formula

$$b(n) = (n-1)b(n-1) - nb(n-2)$$

With the starting conditions $b(3) = 1$, and $b(4) = 7$

Table 1. The first few values of $b(n)$

n	3	4	5	6	7	8	9	10	11
$b(n)$	1	7	23	73	277	1355	8347	61573	523913

Theorem 2.1 For $n \geq 3$.

$i)$
$$\frac{b(n)}{n^2-n-1} = \cfrac{1}{2-\cfrac{3}{3-\cfrac{4}{4-5\ddots_{(n-1)-\cfrac{n}{n-(n+1)}}}}}$$

For $n \geq 5$.

$ii)$ $$b(n) = (2n^2-6n+3).A051403(n-5) - (2n^2-5n+2).A051403(n-6)$$

Proof. By using some simplification of the denominator of the continued fraction.

THE SEQUENCE a(n)

In this section, we present our sequence of prime numbers defined in the conjecture as follows

Conjecture 3.1. The sequence $a(n)$ of the prime numbers satisfy the following formula

$$a(n) = \frac{n^2-n-1}{gcd(b(n),\ n^2-n-1)}$$

Table 2. The first few values of $a(n)$

n	3	4	5	6	7	8	9	10	11
$a(n)$	5	11	19	29	41	11	71	89	109

Also we have: $a(37) = a(43) = a(48) = a(53) = 1$

Conjecture 3.2. every term of this sequence is either a prime number or equal to 1.

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(Concerned with the sequence $A051403$)

THE ENDLESS DESERT OF DECIMALS OF PI - RESULTS WITHOUT WORDS

By Neculai Stanciu-Romania

I. The perimeter of the circle with diameter equal to unity is equal to π.

I.1. The perimeter of the regular n-sided polygon inscribed in the circle of radius R is

$$p_n = 2nR\sin\frac{\pi}{n}.$$

For $2R = 1 \Rightarrow p_n = n\sin\frac{\pi}{n}$.

$$p_3 = 3\sin\frac{\pi}{3} = \textbf{2,598076211...;}$$

$$p_4 = 4\sin\frac{\pi}{4} = \textbf{2,828427124...;}$$

$$p_6 = 6\sin\frac{\pi}{6} = \textbf{3;}$$

$$p_{57} = 57\sin\frac{\pi}{57} = \textbf{3,140002340...;}$$

$$p_{94} = 94\sin\frac{\pi}{94} = \textbf{3,141007838...;}$$

....

$$p_{2022} = 2022\sin\frac{\pi}{2022} = \textbf{3.1415913896219860251241667227170803134677 1...}$$

$$p_{2023} = 2023\sin\frac{\pi}{2023} = \textbf{3.14159139087127446356375816093428982806031...}$$

$p_n = n\sin\frac{\pi}{n} < \pi$ **(** $2R = 1$**); if** $n \to \infty$**, then** $p_n \to \pi$.

$p_n = 2nR\sin\frac{\pi}{n}$**, if** $n \to \infty$**, then** $p_n \to 2\pi R$.

I.2. The perimeter of the regular polygon with n sides circumscribed in the circle of radius r is $P_n = 2nrtg\frac{\pi}{n}$.

For $2r = 1 \Rightarrow P_n = ntg\frac{\pi}{n}$.

$$P_3 = 3tg\frac{\pi}{3} = \textbf{5,196152422...;}$$

$$P_4 = 4tg\frac{\pi}{4} = \textbf{4;}$$

$$P_6 = 6tg\frac{\pi}{6} = \textbf{3,464101615...;}$$

$$P_{36} = 36tg\frac{\pi}{36}$$ **=3,149591886...;**

$$P_{160} = 160tg\frac{\pi}{160}$$ **=3,141996443...;**

...

$P_{2022} = 2022tg\frac{\pi}{2022}$ **=3.1415951815281537654164667715000820945114989 51...**

$P_{2023} = 2023tg\frac{\pi}{2023}$ **=3.1415951790295714628013085083324069828895939 98...**

$2r = 1$**,** $P_n = ntg\frac{\pi}{n} > \pi$ **; if** $n \to \infty$**, then** $P_n \to \pi$ **.**

$P_n = 2nrtg\frac{\pi}{n}$ **, if** $n \to \infty$**, then** $P_n \to 2\pi r$ **.**

II. The area of the circle with radius equal to unity is equal to π **.**

II.1. The area of the regular polygon with n sides inscribed in the circle of radius R is

$$\Delta_n = \frac{nR^2}{2}\sin\frac{2\pi}{n}.$$

For $R = 1 \Rightarrow \Delta_n = \frac{n}{2}\sin\frac{2\pi}{n}$ **.**

$$\Delta_3 = \frac{3}{2}\sin\frac{2\pi}{3}$$ **=1,299038105...;**

$$\Delta_4 = \frac{4}{2}\sin\frac{2\pi}{4}$$ **= 2;**

$$\Delta_6 = \frac{6}{2}\sin\frac{2\pi}{6}$$ **=2,598076211...;**

$$\Delta_{114} = \frac{114}{2}\sin\frac{2\pi}{114}$$ **=3,140002340...;**

$$\Delta_{187} = \frac{187}{2}\sin\frac{2\pi}{187}$$ **=3,141001567...;**

...

$\Delta_{2022} = 1011\sin\frac{\pi}{1011}$ **=3.1415875977203951165221778257305989304030992...;**

$\Delta_{2023} = \frac{2023}{2}\sin\frac{2\pi}{2023}$ **= 3.1415876027175452531294605853511644610544438...**

$R = 1$**,** $\Delta_n = \frac{n}{2}\sin\frac{2\pi}{n} < \pi$ **; then for** $n \to \infty$ **we have** $\Delta_n \to \pi$ **.**

$\Delta_n = \frac{nR^2}{2}\sin\frac{2\pi}{n}$ **; if** $n \to \infty$**, then** $\Delta_n \to \pi R^2$ **.**

II.2. The area of the regular polygon with n sides circumscribed in the circle of radius r is

$$A_n = nr^2 tg\frac{\pi}{n}.$$

For $r = 1 \Rightarrow A_n = ntg\frac{\pi}{n}$**.**

$A_3 = 3tg\frac{\pi}{3}$ **=5,196152422…;**

$A_4 = 4tg\frac{\pi}{4}$ **=4;**

$A_6 = 6tg\frac{\pi}{6}$ **=3,464101615…;**

$A_{36} = 36tg\frac{\pi}{36}$ **=3,149591886…;**

$A_{160} = 160tg\frac{\pi}{160}$ **=3,141996443…;**

………………………………………………………………………………………………

$A_{2022} = 2022tg\frac{\pi}{2022}$ **=3.1415951815281537654164667715000820945114989 51…;**

$A_{2023} = 2023tg\frac{\pi}{2023}$ **=3.141595179029571462801308508332406982889593998…**

$r = 1$**,** $A_n = ntg\frac{\pi}{n} > \pi$ **; for** $n \to \infty$ **we have** $A_n \to \pi$**.**

$A_n = nr^2 tg\frac{\pi}{n}$ **; if** $n \to \infty$**, then** $A_n \to \pi r^2$ **.**

Remark: The best results are obtained when calculating the perimeter of regular polygons inscribed in the circle with a diameter equal to the unit. In the case of circumscribed polygons, the results for perimeter and area are identical.

7 OUTSTANDING LIMITS

By D.M. Bătineţu-Giurgiu, Daniel Sitaru and Neculai Stanciu-Romania

ABSTRACT: In this paper we present 7 new limits of sequences and functions.

Theorem 1. $\lim\limits_{n\to\infty} \frac{\ln(1+\sqrt[n]{n!})}{\sqrt[n]{(2n-1)!!}} = 0$.

Proof. $\lim\limits_{x\to\infty} \ln(1+x)^{\frac{1}{x}} = \lim\limits_{x\to\infty} \frac{1}{x}\ln(1+x) \overset{L'H}{=} \lim\limits_{x\to\infty} \frac{\frac{1}{1+x}}{1} = 0$;

$$\lim_{n\to\infty} \frac{\ln(1+\sqrt[n]{n!})}{\sqrt[n]{(2n-1)!!}} = \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{\sqrt[n]{(2n-1)!!}} \ln(1+\sqrt[n]{n!})^{\frac{1}{\sqrt[n]{n!}}} = 0 \cdot \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{\sqrt[n]{(2n-1)!}} =$$

$$=0\cdot\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}\cdot\frac{n}{\sqrt[n]{(2n-1)!!}}=0\cdot\frac{1}{e}\cdot\frac{e}{2}=0.$$

Theorem 2. If $a>0$, then $\lim_{n\to\infty}(\sqrt[n]{a}-1)\sqrt[n]{n!}=\frac{\ln a}{e}$.

Proof. $(\sqrt[n]{a}-1)\sqrt[n]{n!}=\frac{\sqrt[n]{n!}}{n}\cdot n\cdot(\sqrt[n]{a}-1)=\frac{\sqrt[n]{n!}}{n}\cdot\frac{e^{\frac{1}{n}\ln a}-1}{\frac{1}{n}\ln a}\cdot\ln a$, $\forall n\geq 2$.

$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{e}$ şi $\lim_{n\to\infty}\frac{e^{\frac{1}{n}\ln a}-1}{\frac{1}{n}\ln a}=1$. So, $\lim_{n\to\infty}(\sqrt[n]{a}-1)\sqrt[n]{n!}=\frac{1}{e}\cdot 1\cdot\ln a=\frac{\ln a}{e}$.

Theorem 3. If $a>0$, $(b_n)_{n\geq 1}>0$ are positive real sequences such that $\lim_{n\to\infty}\frac{b_{n+1}}{nb_n}=b>0$, then $\lim_{n\to\infty}(\sqrt[n]{a}-1)\sqrt[n]{b_n}=\frac{b\ln a}{e}$.

Proof. $(\sqrt[n]{a}-1)\sqrt[n]{b_n}=\frac{\sqrt[n]{b_n}}{n}\cdot n\cdot(\sqrt[n]{a}-1)=\frac{\sqrt[n]{b_n}}{n}\cdot\frac{e^{\frac{1}{n}\ln a}-1}{\frac{1}{n}\ln a}\cdot\ln a$, $\forall n\geq 2$.

$$\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{b_n}{n^n}}=\lim_{n\to\infty}\frac{b_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{b_n}=\lim_{n\to\infty}\frac{b_{n+1}}{nb_n}\left(\frac{n}{n+1}\right)^{n+1}=\frac{b}{e};\ \lim_{n\to\infty}\frac{e^{\frac{1}{n}\ln a}-1}{\frac{1}{n}\ln a}=1.$$

$$\lim_{n\to\infty}(\sqrt[n]{a}-1)\sqrt[n]{b_n}=\frac{b}{e}\cdot 1\cdot\ln a=\frac{b\ln a}{e}.$$

Theorem 4. If $a>0$, then $\lim_{n\to\infty}(\sqrt[n]{a}-1)\sqrt[n]{(2n-1)!!}=\frac{2\ln a}{e}$.

Proof. $(\sqrt[n]{a}-1)\sqrt[n]{(2n-1)!!}=\frac{\sqrt[n]{(2n-1)!!}}{n}\cdot n\cdot(\sqrt[n]{a}-1)=\frac{\sqrt[n]{(2n-1)!!}}{n}\cdot\frac{e^{\frac{1}{n}\ln a}-1}{\frac{1}{n}\ln a}\cdot\ln a$, $\forall n\geq 2$.

$$\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!}{n^n}}=\lim_{n\to\infty}\frac{(2n+1)!!}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!}=\lim_{n\to\infty}\frac{2n+1}{n+1}\left(\frac{n}{n+1}\right)^{n+1}=\frac{2}{e};$$

$$\lim_{n\to\infty}\frac{e^{\frac{1}{n}\ln a}-1}{\frac{1}{n}\ln a}=1. \quad \lim_{n\to\infty}(\sqrt[n]{a}-1)\sqrt[n]{(2n-1)!!}=\frac{2}{e}\cdot 1\cdot\ln a=\frac{2\ln a}{e}.$$

Theorem 5. If $(x_n)_{n\geq 1}$, $x_n=\sum_{k=1}^{n}\frac{1}{k}$ şi $a>0$, then $\lim_{n\to\infty}e^{2x_n}(\sqrt[n^4]{a}-1)\sqrt[n]{n!\cdot(2n-1)!!}=2e^{2(\gamma-1)}\ln a$.

Proof. $e^{2x_n}(\sqrt[n^4]{a}-1)\sqrt[n]{n!\cdot(2n-1)!!}=\frac{e^{2x_n}}{n^2}\cdot n^4\cdot(\sqrt[n^4]{a}-1)\cdot\frac{\sqrt[n]{n!}}{n}\cdot\frac{\sqrt[n]{(2n-1)!!}}{n}=$

$$=\frac{e^{2x_n}}{e^{2\ln n}}\cdot n^4\cdot\left(e^{\frac{\ln a}{n^4}}-1\right)\cdot\frac{\sqrt[n]{n!}}{n}\cdot\frac{\sqrt[n]{(2n-1)!!}}{n}==e^{2(x_n-\ln n)}\cdot\frac{e^{\frac{\ln a}{n^4}}-1}{\frac{\ln a}{n^4}}\cdot\frac{\sqrt[n]{n!}}{n}\cdot\frac{\sqrt[n]{(2n-1)!!}}{n}\cdot\ln a.$$

$\lim_{n\to\infty}e^{2(x_n-\ln n)}=e^{2\gamma}$, where γ is Euler-Mascheroni constant; $\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{e}$;

$$\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!}{n^n}}=\lim_{n\to\infty}\frac{(2n+1)!!}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!}=\lim_{n\to\infty}\frac{2n+1}{n+1}\left(\frac{n}{n+1}\right)^{n+1}=\frac{2}{e};$$

$\lim_{n\to\infty}\frac{e^{\frac{1}{n^4}\ln a}-1}{\frac{1}{n^4}\ln a}=1$. Hence, $\lim_{n\to\infty}e^{2x_n}(\sqrt[n^4]{a}-1)\sqrt[n]{n!\cdot(2n-1)!!}=e^{2\gamma}\cdot 1\cdot\frac{1}{e}\cdot\frac{2}{e}\cdot\ln a=2e^{2(\gamma-1)}\ln a$.

Theorem 6. If $a>0$, then $\lim_{x\to\infty}(a^{\frac{1}{x}}-1)(\Gamma(x+1))^{\frac{1}{x}}=\frac{\ln a}{e}$.

Proof. $f(x)=(a^{\frac{1}{x}}-1)(\Gamma(x+1))^{\frac{1}{x}}=x(a^{\frac{1}{x}}-1)\cdot\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x}=x(e^{\frac{\ln a}{x}}-1)\cdot\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x}=$

$$=\frac{e^{\frac{\ln a}{x}}-1}{\frac{\ln a}{x}}\cdot\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x}\cdot\ln a;$$

$$\lim_{x\to\infty}\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x}=\lim_{n\to\infty}\frac{(\Gamma(n+1))^{\frac{1}{n}}}{n}=\lim_{n\to\infty}\frac{(n!)^{\frac{1}{n}}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}}=\lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\frac{n^n}{n!}=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=\frac{1}{e}$$

Hence, $\lim_{x\to\infty}(a^{\frac{1}{x}}-1)(\Gamma(x+1))^{\frac{1}{x}}=\lim_{x\to\infty}f(x)=1\cdot\frac{1}{e}\cdot\ln a=\frac{\ln a}{e}$.

Theorem 7. $\lim_{x\to\infty} x\sin\frac{1}{x^2}(\Gamma(x+1))^{\frac{1}{x}}=\frac{1}{e}$.

Prrof. $f(x)=x\sin\frac{1}{x^2}(\Gamma(x+1))^{\frac{1}{x}}=x^2\sin\frac{1}{x^2}\cdot\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x}$; $\lim_{x\to\infty} x^2\sin\frac{1}{x^2}=1$;

$$\lim_{x\to\infty}\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x}=\lim_{n\to\infty}\frac{(\Gamma(n+1))^{\frac{1}{n}}}{n}=\lim_{n\to\infty}\frac{(n!)^{\frac{1}{n}}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}}=\lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\frac{n^n}{n!}=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=\frac{1}{e}$$

So, $\lim_{x\to\infty} x\sin\frac{1}{x^2}(\Gamma(x+1))^{\frac{1}{x}}=1\cdot\frac{1}{e}=\frac{1}{e}$.

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THE SPECIAL TRIANGLE WITH SIDES $\sqrt{a(b+c)},\sqrt{b(c+a)},\sqrt{c(a+b)}$

By Marius Drăgan, Neculai Stanciu-Romania

Abstract. This paper presents some geometric and algebraic inequalities starting from a certain triangle.

Keywords: geometric identities, geometric inequalities, algebraic inequalities.

MSC 2010: 51M16, 26D05.

INTRODUCTION: It is well-known that if u,v,w **are the sides of a triangle, then** $\sqrt{u},\sqrt{v},\sqrt{w}$ **are the sides of a triangle. If we take** $u=a(b+c),v=b(c+a),w=c(a+b)$**, where** a,b,c **are the sides of a triangle, then we obtain that** $\sqrt{a(b+c)},\sqrt{b(c+a)},\sqrt{c(a+b)}$ **are the sides of a triangle. If we denote** $a'=\sqrt{a(b+c)},b'=\sqrt{b(c+a)},c'=\sqrt{c(a+b)}$**,** $x=ab+bc+ca$ **and we compute area** S'**, of triangle with sides** a',b',c'**, we obtain** $16S'^2=2\sum(a'b')^2-\sum a'^4=2[ab(c^2+x)+bc(a^2+x)+ca(b^2+x)]-a^2(b^2+c^2+2bc)-$
$-b^2(c^2+a^2+2ac)-c^2(a^2+b^2+2ab)=2abc^2+2a^2bc+2ab^2c+2x(ab+bc+ca)-2a^2b^2-$
$-2b^2c^2-2c^2a^2-2ab^2c-2a^2bc-2abc^2=2[(ab+bc+ca)^2-a^2b^2-b^2c^2-c^2a^2]=$
$=4abc(a+b+c)=16Rrp\cdot 2p\Rightarrow S'=\sqrt{2Rrp^2}\Rightarrow S'=p\sqrt{2Rr}$ **.**

Also we obtain that:

$$p'=\frac{1}{2}\sum\sqrt{a(b+c)}\ ;\ r'=\frac{S'}{p'}=\frac{2p\sqrt{2}\sqrt{Rr}}{\sum\sqrt{a(b+c)}}\ ;$$

$$R'=\frac{a'b'c'}{4S'}=\frac{\sqrt{abc\prod(a+b)}}{4p\sqrt{2Rr}}=\frac{\sqrt{4Rrp\prod(a+b)}}{4p\sqrt{2Rr}}=\frac{\sqrt{p\prod(a+b)}}{2\sqrt{2}p}=\sqrt{\frac{\prod(a+b)}{8p}};$$

$$\cos A'=\frac{b'^2+c'^2-a'^2}{2b'c'}=\frac{b(a+c)+c(a+b)-a(b+c)}{2\sqrt{bc(a+b)(a+c)}}=\sqrt{\frac{bc}{(a+b)(a+c)}};$$

$$\sin A'=\sqrt{1-\frac{bc}{(a+b)(a+c)}}=\sqrt{\frac{a(a+b+c)}{(a+b)(a+c)}}=\sqrt{\frac{2ap}{(a+b)(a+c)}};$$

$$tgA'=\sqrt{\frac{2ap}{bc}}=a\cdot\sqrt{\frac{2p}{abc}}=a\cdot\sqrt{\frac{2p}{4Rrp}}=\frac{a}{\sqrt{2Rr}};$$

$$m'_a=\sqrt{\frac{2(b'^2+c'^2)-a'^2}{4}}=\sqrt{\frac{2[b(a+c)+c(a+b)]-a(b+c)}{4}}=\sqrt{\frac{ab+ac+4bc}{4}}=$$

$$=\frac{1}{2}\sqrt{ab+ac+4bc};\ h'_a=\frac{2S'}{a'}=\frac{2\sqrt{2}p\sqrt{Rr}}{a'}=\frac{2\sqrt{2}p\sqrt{Rr}}{\sqrt{a(b+c)}}.$$

In the next we obtain some results related to triangle with sides a',b',c' **.**

MAIN RESULTS

Proposition 1. A refinement of *Ionescu-Weitzenböck* inequality.

In any triangle ABC **is true the inequality:**

$$ab+bc+ca\ge 4\sqrt{3}\sqrt{\frac{R}{2r}}S.$$

Proof. By *Ionescu-Weitzenböck* inequality we have successively that

$$a'^2+b'^2+c'^2\ge 4\sqrt{3}S'\Leftrightarrow$$

$$\Leftrightarrow a(b+c)+b(c+a)+c(a+b)\ge 4\sqrt{3}p\sqrt{2Rr}\Leftrightarrow ab+bc+ca\ge 4\sqrt{3}S\sqrt{\frac{R}{2r}}\overset{Euler}{\ge}4\sqrt{3}S$$ **, q.e.d.**

Proposition 2. In any triangle ABC **is true the inequality:**

$$\sum\sqrt{ab(a+c)(b+c)}\ge 2\sqrt{6}\sqrt{Rr}\,p+p^2+r^2+4Rr$$

Proof. By *Hadwiger-Finsler* inequality we obtain successively

$$a'^2+b'^2+c'^2\ge 4\sqrt{3}S'+\sum(a'-b')^2\Leftrightarrow$$

$$\Leftrightarrow \sum a'b' \geq 2\sqrt{3}S' + \frac{1}{2}\sum a'^2 \Leftrightarrow \sum \sqrt{ab(a+c)(b+c)} \geq 2\sqrt{6}\sqrt{Rr}\,p + \frac{1}{2}\sum a(b+c) =$$

$$= 2\sqrt{6}\sqrt{Rr}\,p + \sum ab = 2\sqrt{6}\sqrt{Rr}\,p + p^2 + r^2 + 4Rr$$ **, q.e.d.**

Proposition 3. In any triangle ABC is true the inequality:

$$\sum \sqrt{ab(a+c)(b+c)} \geq 4\sqrt{6}\sqrt{Rr}\,p$$

Proof. This inequality results from *Gordon* inequality $a'b' + b'c' + c'a' \geq 4\sqrt{3}S' \Leftrightarrow$

$$\Leftrightarrow \sum \sqrt{ab(a+c)(b+c)} \geq 4\sqrt{6}\sqrt{Rr}\,p$$ **, q.e.d.**

Proposition 4. In any triangle ABC is true the inequality:

$$\sum \sqrt{ab(a+c)(b+c)} \leq \frac{2\sqrt{6}}{3}\sqrt{Rr}\,p + \frac{5}{3}\sum ab$$

Proof. By reverse *Hadwiger-Finsler* inequality we deduce successively

$$\sum a'^2 \leq 4\sqrt{3}S' + 3\sum (a'-b')^2 \Leftrightarrow$$

$$\Leftrightarrow 6\sum a'b' \leq 4\sqrt{3}S' + 5\sum a'^2 \Leftrightarrow \sum a'b' \leq \frac{2\sqrt{3}}{3}S' + \frac{5}{6}\sum a'^2$$

$$\Leftrightarrow \sum \sqrt{ab(a+c)(b+c)} \leq \frac{2\sqrt{3}}{3}p\sqrt{2Rr} + \frac{5}{3}\sum ab$$ **, q.e.d.**

Proposition 5. In any triangle ABC is true the inequality:

$$\sqrt{a(b+c)} + \sqrt{b(c+a)} + \sqrt{c(a+b)} \geq \sqrt[4]{864Rrp^2}$$

Proof. From *Mitrinović* inequality, i.e. $p' \geq 3\sqrt{3}r'$ we obtain

$$\frac{1}{2}\sum \sqrt{a(b+c)} \geq 3\sqrt{3}\cdot\frac{2p\sqrt{2}\sqrt{Rr}}{\sum a(b+c)} \Leftrightarrow \left(\sum \sqrt{a(b+c)}\right)^2 \geq 12p\sqrt{6}\sqrt{Rr} \Leftrightarrow$$

$$\Leftrightarrow \sum \sqrt{a(b+c)} \geq \sqrt[4]{864Rrp^2}$$ **, q.e.d.**

Proposition 6. In any triangle ABC is true the inequality:

$$\sum \sqrt{a(b+c)} \geq \sqrt[3]{54}\cdot\sqrt[6]{Rrp\prod(a+b)} \geq 2^{\frac{7}{6}}\cdot 3\cdot\sqrt[3]{Rrp}$$

Proof. From $p'^2 \geq \frac{27}{2}R'r'$ we have successively:

$$\frac{1}{4}(\sum\sqrt{a(b+c)})^2 \geq \frac{27}{2}\sqrt{\frac{\prod(a+b)}{8p}}\frac{2\sqrt{2}\sqrt{Rr}\,p}{\sum\sqrt{a(b+c)}} \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{4}(\sum\sqrt{a(b+c)})^3 \geq \frac{27}{2}\sqrt{Rrp\prod(a+b)} \Leftrightarrow (\sum\sqrt{a(b+c)})^3 \geq 54\sqrt{Rrp\prod(a+b)},$$

where by *Césaro* inequality, i.e. $\prod(a+b) \geq 8abc = 32Rrp$ we obtain the desired inequality.

Proposition 7. In any triangle ABC is true the inequality:

$$\sum\sqrt{a(b+c)} \geq \sqrt[3]{54}\cdot\sqrt[6]{Rrp\prod(a+b)} \geq \sqrt[4]{864Rrp^2} \geq 2^{\frac{7}{6}}\cdot 3\cdot\sqrt[3]{Rrp}$$

Proof. The first two inequalities from the above yields immediately from

$p'^2 \geq \frac{27}{2}R'r' \geq 27r'^2$, and the last is equivalent to

$$(\sqrt[4]{864Rrp^2})^{12} \geq \left(2^{\frac{7}{6}}\cdot 3\cdot\sqrt[3]{Rrp}\right)^{12} \Leftrightarrow 2^{15}\cdot 3^9\cdot p^2 \geq 2^{14}\cdot 3^{12}\cdot Rr \Leftrightarrow p^2 \geq \frac{27Rr}{2},\ \text{true.}$$

Proposition 8. In any triangle ABC is true the inequality:

$$8p\left(\sum\sqrt{a(b+c)}\right)^2 \leq 27\prod(a+b)$$

Proof. From $p' \leq \frac{3\sqrt{3}}{2}R'$ we obtain $\frac{1}{2}\sum a(b+c) \leq \frac{3\sqrt{3}}{2}\sqrt{\frac{\prod(a+b)}{8p}} \Leftrightarrow$

$$\Leftrightarrow \left(\sum\sqrt{a(b+c)}\right)^2 \leq \frac{27}{8}\frac{\prod(a+b)}{p},\ \text{q.e.d.}$$

Proposition 9. In any triangle ABC is true the inequality:

If $x,y,z>0$, such that $x+y+z=1$, then

$$\sum\sqrt{1-x^2} \geq \sqrt[3]{54}\cdot\sqrt[6]{\frac{\prod(1-x^2)}{4}} \geq \sqrt[4]{216\prod(1-x)} \geq 2^{\frac{7}{6}}\cdot 3\cdot\sqrt[3]{\frac{\prod(1-x)}{4}}.$$

Proof. Using *Ravi* substitutions $a = y + z, b = x + z, c = x + y, p = x + y + z, r = \sqrt{\frac{xyz}{x + y + z}}$

$R = \frac{\prod (x + y)}{4\sqrt{xyz(x + y + z)}}$**, from 7. we obtain**

$$\sum \sqrt{(y+z)(2x+y+z)} \geq \sqrt[3]{54} \cdot \sqrt[6]{\frac{\prod (x+y) \prod (2x+y+z)}{4}} \geq$$

$$\geq \sqrt[4]{216(x+y+z)\prod (x+y)} \geq 2^{\frac{7}{6}} \cdot 3 \cdot \sqrt[3]{\frac{\prod (x+y)}{4}},$$

and by $x + y + z = 1$**, q.e.d.**

Proposition 10. In any triangle ABC **is true the inequality:**

If $x, y, z > 0$**, such that** $x + y + z = 1$**, then** $8\left(\sum \sqrt{1 - x^2}\right)^2 \leq 27 \prod (1 + x)$**.**

Proof. By *Ravi* substitutions and 8. yields $8p\left(\sum \sqrt{a(b + c)}\right)^2 \leq 27 \prod (a + b) \Leftrightarrow$

$$\Leftrightarrow 8(x+y+z)\left(\sum \sqrt{(y+z)(2x+y+z)}\right)^2 \leq 27 \prod (2x+y+z) \text{ , q.e.d.}$$

Proposition 11. In any triangle ABC **is true the inequality:**

$$\left(\sum \frac{ab + ac + 2bc}{\sqrt{ab + ac + 4bc}}\right)^2 \leq \frac{9}{2} \frac{\prod (a+b)}{p}$$

Proof. From *Tereshin,*inequality i.e. $\sum \frac{b'^2 + c'^2}{m'_a} \leq 12R'$ yields that

$$\sum \frac{b(a+c) + c(a+b)}{\frac{1}{2}\sqrt{ab + ac + 4bc}} \leq 12\sqrt{\frac{\prod (a+b)}{8p}} \Leftrightarrow \sum \frac{ab + ac + 2bc}{\sqrt{ab + ac + 4bc}} \leq \frac{3\sqrt{2}}{2}\sqrt{\frac{\prod (a+b)}{p}} \text{, q.e.d.}$$

Proposition 12. In any triangle ABC **is true the inequality:**

$$16p\sum ab \leq 9\prod (a+b)$$

Proof. We have successively

$$\sum a'\sin A' \le \frac{9}{2}R' \Leftrightarrow \sum\sqrt{a(b+c)}\sqrt{\frac{2ap}{(a+b)(a+c)}} \le \frac{9}{2}\sqrt{\frac{\prod(a+b)}{8p}} \Leftrightarrow$$

$$\Leftrightarrow 8p\sum\frac{a\sqrt{b+c}}{\sqrt{(a+b)(a+c)}} \le 9\sqrt{\prod(a+b)} \Leftrightarrow 8p\sum a(b+c) \le 9\prod(a+b) \Leftrightarrow$$

$$\Leftrightarrow 16p\sum ab \le 9\prod(a+b)$$, **q.e.d.**

Proposition 13. In any triangle ABC is true the inequality:

$$\sum\frac{1}{\sqrt{ab+bc+4bc}} \ge \sqrt{\frac{8p}{\prod(a+b)}}$$

Proof. From $\frac{1}{m'_a}+\frac{1}{m'_b}+\frac{1}{m'_c} \ge \frac{2}{R'}$ we have $\sum\frac{1}{\sqrt{ab+bc+4bc}} \ge \frac{1}{\sqrt{\frac{\prod(a+b)}{8p}}}$, q.e.d.

Proposition 14. In any triangle ABC is true the inequality:

$$\sum\sqrt{ab+ac+4bc} \le 9\sqrt{\frac{\prod(a+b)}{8p}}$$

Proof. From $m'_a+m'_b+m'_c \le \frac{9}{2}R'$, we deduce $\sum\sqrt{ab+ac+4bc} \le 9\sqrt{\frac{\prod(a+b)}{8p}}$, q.e.d.

Proposition 15. In any triangle ABC is true the inequality:

$$\sum\frac{bc}{(a+b)(a+c)} \le 6\cdot\frac{abc}{\prod(b+c)}$$

Proof. Yields immediately from $\cos^2 A'+\cos^2 B'+\cos^2 C' \ge 6\cos A'\cos B'\cos C'$.

Proposition 16. In any triangle ABC is true the inequality: $\sum\sqrt{a(b+c)} \ge \sqrt{12\sqrt{6}}\sqrt{Rrp^2}$

Proof. From $(\sin A'+\sin B'+\sin C')^2 \ge 6\sqrt{3}\sin A'\sin B'\sin C'$ we obtain successively

$$\left(\sum\sqrt{\frac{2ap}{(a+b)(b+c)}}\right)^2 \ge 6\sqrt{3}\cdot\sqrt{\frac{8abcp^3}{\prod(a+b)^2}} \Leftrightarrow \frac{2p\left(\sum\sqrt{a(b+c)}\right)^2}{\prod(a+b)} \ge \frac{6\sqrt{3}p\sqrt{8\cdot 4Rrp^2}}{\prod(a+b)} \Leftrightarrow$$

q.e.d.

Reference: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

AN AMAZING CHAIN OF INTEGRALS

By Pham Duc Nam-Vietnam

$\boldsymbol{PROBLEM\ (1)\_PROVE}$:

$$\boldsymbol{I=\int_0^1\frac{log^2(1+x)\,log(1+x^2)}{1+x}dx}$$
$$\boldsymbol{=\frac{5}{2}Li_4\left(\frac{1}{2}\right)-\frac{35}{16}\zeta(3)\,log\left(\frac{1}{2}\right)-\frac{7\pi^2}{64}log^2(2)+\frac{41}{96}log^4(2)-\frac{209\pi^4}{7680}}$$

Naren Bhandari-Nepal

Solution:

$$*\,I=\int_0^1\frac{log^2(1+x)\,log(1+x^2)}{1+x}dx$$
$$=\int_0^1 log^2(1+x)\,log(1+x^2)\,d(log(1+x)),\begin{cases}u=log(1+x^2)\\ dv=log^2(1+x)\,d(log(1+x))\end{cases}$$
$$\Rightarrow\begin{cases}du=\dfrac{2x}{1+x^2}\\ v=\dfrac{1}{3}log^3(1+x)\end{cases}$$
$$\Rightarrow I=\frac{1}{3}log^3(1+x)\,log(1+x^2)\Big|_0^1-\frac{2}{3}\int_0^1\frac{x\,log^3(1+x)}{1+x^2}dx=\frac{1}{3}log^4(2)-\frac{2}{3}J$$
$$*\,J=\int_0^1\frac{x\,log^3(1+x)}{1+x^2}dx,let\colon x\to\frac{1}{1+x}\Rightarrow J$$
$$=\int_{\frac{1}{2}}^1\frac{(x-1)\,log^3(x)}{x(2x^2-2x+1)}dx$$
$$=\int_{\frac{1}{2}}^1\frac{(x-1)\,log^3(x)}{x((1-x)^2+x^2)}dx=\int_{\frac{1}{2}}^1 log^3(x)\left(\frac{2x-1}{(1-x)^2+x^2}-\frac{1}{x}\right)dx$$

Use: $\dfrac{2x-1}{(1-x)^2+x^2}=-\Re\left(\dfrac{1+i}{1-(1+i)x}\right)\Rightarrow J$

$$=\int_{\frac{1}{2}}^1 log^3(x)\left(-\Re\left(\frac{1+i}{1-(1+i)x}\right)-\frac{1}{x}\right)dx$$
$$=\frac{1}{4}log^4(2)-\Re\int_{\frac{1}{2}}^1 log^3(x)\frac{1+i}{1-(1+i)x}dx$$
$$=\frac{1}{4}log^4(2)-\Re\int_0^1 log^3(x)\frac{1+i}{1-(1+i)x}dx+\Re\int_0^{\frac{1}{2}}log^3(x)\frac{1+i}{1-(1+i)x}dx$$
$$=\frac{1}{4}log^4(2)+6\Re Li_4(1+i)+\Re\int_0^{\frac{1}{2}}log^3(x)\frac{1+i}{1-(1+i)x}dx\xrightarrow{x=\frac{x}{2}}$$
$$=\frac{1}{4}log^4(2)+6\Re Li_4(1+i)$$
$$+\Re\int_0^1\frac{1}{2}(log(x)-log(2))^3\frac{1+i}{1-\frac{1+i}{2}x}dx$$ .Expand out and integrate yields:

$$J=\frac{1}{4}log^4(2)+6\Re Li_4(1+i)-6\Re Li_4\left(\frac{1+i}{2}\right)-6\,log(2)\,\Re Li_3\left(\frac{1+i}{2}\right)-3\,log^2(2)\,\Re Li_2\left(\frac{1+i}{2}\right)$$
$$-\frac{1}{2}log^4(2)$$
$$=6\Re\left(Li_4(1+i)-Li_4\left(\frac{1+i}{2}\right)\right)-6\,log(2)\,\Re Li_3\left(\frac{1+i}{2}\right)$$
$$-3\,log^2(2)\,\Re Li_2\left(\frac{1+i}{2}\right)-\frac{1}{4}log^4(2)$$

$*$ Use: $\Re Li_3\left(\frac{1+i}{2}\right)=\frac{35}{64}\zeta(3)-\frac{5\pi^2}{192}log(2)+\frac{1}{48}log^3(2)\,,\Re Li_2\left(\frac{1+i}{2}\right)$
$$=\frac{5\pi^2}{96}-\frac{1}{8}log^2(2)\,,\Re\left(Li_4(1+i)-Li_4\left(\frac{1+i}{2}\right)\right)$$
$$=-\frac{5}{8}Li_4\left(\frac{1}{2}\right)+\frac{209\pi^4}{30720}+\frac{7\pi^2}{256}log^2(2)-\frac{3}{128}log^4(2)$$
$$\Rightarrow J=6\left(-\frac{5}{8}Li_4\left(\frac{1}{2}\right)+\frac{209\pi^4}{30720}+\frac{7\pi^2}{256}log^2(2)-\frac{3}{128}log^4(2)\right)$$
$$-6\,log(2)\left(\frac{35}{64}\zeta(3)-\frac{5\pi^2}{192}log(2)+\frac{1}{48}log^3(2)\right)-3\,log^2(2)\left(\frac{5\pi^2}{96}-\frac{1}{8}log^2(2)\right)$$
$$-\frac{1}{4}log^4(2)$$
$$=-\frac{15}{4}Li_4\left(\frac{1}{2}\right)+\frac{209\pi^4}{5120}+\frac{21\pi^2}{128}log^2(2)-\frac{9}{64}log^4(2)-\frac{105}{32}log(2)\,\zeta(3)$$
$$\Rightarrow I=\frac{1}{3}log^4(2)-\frac{2}{3}J$$
$$=\frac{1}{3}log^4(2)$$
$$-\frac{2}{3}\left(-\frac{15}{4}Li_4\left(\frac{1}{2}\right)+\frac{209\pi^4}{5120}+\frac{21\pi^2}{128}log^2(2)-\frac{9}{64}log^4(2)-\frac{105}{32}log(2)\,\zeta(3)\right)$$
$$=\frac{5}{2}Li_4\left(\frac{1}{2}\right)+\frac{35}{16}log(2)\,\zeta(3)-\frac{7\pi^2}{64}log^2(2)+\frac{41}{96}log^4(2)-\frac{209\pi^4}{7680}$$
$$=\frac{5}{2}L\quad_4\left(\frac{1}{2}\right)-\frac{35}{16}\zeta(3)\,log\left(\frac{1}{2}\right)-\frac{7\pi^2}{64}log^2(2)+\frac{41}{96}log^4(2)-\frac{209\pi^4}{7680}\,(Q.E.D)$$

$\boldsymbol{PROBLEM(3)\_PROVE}$:
$$\boldsymbol{I=\int_0^1\frac{x\,log^2(x)\,log(1-x)}{1+x^2}dx=-\frac{5}{4}Li_4\left(\frac{1}{2}\right)+\frac{13\pi^4}{3840}-\frac{5}{96}log^4(2)+\frac{5\pi^2}{96}log^2(2)}$$

Naren Bhandari-Nepal

Solution: $I=\int_0^1\frac{x\,log^2(x)\,log(1-x)}{1+x^2}dx=\frac{1}{2}\int_0^1 log^2(x)\,log(1-x)\,d(log(x^2+1))\xrightarrow{IBP}=$
$\underbrace{\frac{1}{2}log(x^2+1)\,log^2(x)\,log(1-x)\Big|_0^1}_{=0}-\int_0^1\frac{log(x)\,log(1-x)\,log(x^2+1)}{x}dx+\frac{1}{2}\int_0^1\frac{log^2(x)\,log(x^2+1)}{1-x}dx=$
$\frac{1}{2}J-K$

$*$ Use： Cornel Ioan Valean's generalized alternating harmonic series:

$$I(m)=\frac{(-1)^m}{(m-1)!}\int_0^1\frac{log^{m-1}(x)\,log\left(\frac{1+x^2}{2}\right)}{1-x}dx$$
$$=m\zeta(m+1)-2^{-m}(1-2^{-m+1})\,log(2)\,\zeta(m)$$
$$-\sum_{k=0}^{m-1}\beta(k+1)\beta(m-k)-\sum_{k=1}^{m-2}2^{-m-1}\left(1-2^{-k}\right)\left(1-2^{-m+k+1}\right)\zeta(k+1)\zeta(m-k)$$

Put : $m=3\Rightarrow I(3)=-\frac{1}{2}\int_0^1\frac{log^2(x)\,log\left(\frac{1+x^2}{2}\right)}{1-x}dx$

$$=3\zeta(4)-\frac{3}{32}log(2)\,\zeta(3)-\beta(1)\beta(3)-\beta(2)\beta(2)-\beta(3)\beta(1)-\frac{1}{64}\zeta(2)\zeta(2)$$

$$=3\zeta(4)-\frac{3}{32}log(2)\,\zeta(3)-\frac{\pi}{2}.\frac{\pi^3}{32}-G^2-\frac{1}{64}\zeta^2(2)\Rightarrow\int_0^1\frac{log^2(x)\,log\left(\frac{1+x^2}{2}\right)}{1-x}dx$$
$$=-6\zeta(4)+\frac{3}{16}log(2)\,\zeta(3)+\frac{\pi^4}{32}+2G^2+\frac{1}{32}\zeta^2(2)$$

$$\Leftrightarrow\int_0^1\frac{log^2(x)\,log(x^2+1)}{1-x}dx-\int_0^1\frac{log^2(x)\,log(2)}{1-x}dx$$
$$=-6\zeta(4)+\frac{3}{16}log(2)\,\zeta(3)+\frac{\pi^4}{32}+2G^2+\frac{1}{32}\zeta^2(2)$$

$$\Leftrightarrow\int_0^1\frac{log^2(x)\,log(x^2+1)}{1-x}dx-2\,log(2)\,\zeta(3)$$
$$=-6\zeta(4)+\frac{3}{16}log(2)\,\zeta(3)+\frac{\pi^4}{32}+2G^2+\frac{1}{32}\zeta^2(2)\Rightarrow J$$
$$=-6\zeta(4)+\frac{3}{16}log(2)\,\zeta(3)+\frac{\pi^4}{32}+2G^2+\frac{1}{32}\zeta^2(2)+2\,log(2)\,\zeta(3)$$
$$=-\frac{\pi^4}{15}+\frac{3}{16}log(2)\,\zeta(3)+\frac{\pi^4}{32}+2G^2+\frac{\pi^4}{1152}+2\,log(2)\,\zeta(3)=-\frac{199\pi^4}{5760}+2G^2+\frac{35}{16}log(2)\,\zeta(3)$$

$$*\,K=\int_0^1\frac{log(x)\,log(1-x)\,log(x^2+1)}{x}dx=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\int_0^1x^{2k-1}\,log(x)\,log(1-x)\,dx$$

From : $\int_0^1x^{k-1}\,log(1-x)\,dx=-\frac{H_k}{k}\xrightarrow{Derivative\ w.r.t\ k}\int_0^1x^{k-1}\,log(x)\,log(1-x)\,dx=\frac{H_k}{k^2}$

$$+\frac{{H_k}^{(2)}}{k}-\frac{\zeta(2)}{k}$$

$$\Rightarrow K=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\left(\frac{H_{2k}}{4k^2}+\frac{{H_{2k}}^{(2)}}{2k}-\frac{\zeta(2)}{2k}\right)$$
$$=\frac{1}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}H_{2k}}{k^3}+\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}{H_{2k}}^{(2)}}{k^2}-\frac{\zeta(2)}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}$$

Use : $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}=\frac{\pi^2}{12},\sum_{k=1}^{\infty}\frac{(-1)^{k-1}H_{2k}}{k^3}$

$$=\frac{195}{32}\zeta(4)+\frac{5}{4}log^2(2)\,\zeta(2)-\frac{35}{8}log(2)\,\zeta(3)-\frac{5}{24}log^4(2)-5Li_4\left(\frac{1}{2}\right)(*),\sum_{k=1}^{\infty}\frac{(-1)^{k-1}H_{2k}^{(2)}}{k^2}$$

$$=2G^2-\frac{353}{64}\zeta(4)-\frac{5}{4}log^2(2)\,\zeta(2)+\frac{35}{8}log(2)\,\zeta(3)+\frac{5}{24}log^4(2)+5Li_4\left(\frac{1}{2}\right)(**)$$

$$\Rightarrow K=\frac{1}{4}\left(\frac{195}{32}\zeta(4)+\frac{5}{4}log^2(2)\,\zeta(2)-\frac{35}{8}log(2)\,\zeta(3)-\frac{5}{24}log^4(2)-5Li_4\left(\frac{1}{2}\right)\right)$$
$$+\frac{1}{2}\left(2G^2-\frac{353}{64}\zeta(4)-\frac{5}{4}log^2(2)\,\zeta(2)+\frac{35}{8}log(2)\,\zeta(3)+\frac{5}{24}log^4(2)+5Li_4\left(\frac{1}{2}\right)\right)-\frac{\zeta(2)}{2}.\frac{\pi^2}{12}$$

$$=G^2-\frac{5}{16}log^2(2)\,\zeta(2)+\frac{35}{32}log(2)\,\zeta(3)-\frac{119}{64}\zeta(4)+\frac{5}{96}log^4(2)+\frac{5}{4}Li_4\left(\frac{1}{2}\right)$$
$$=G^2-\frac{5\pi^2}{96}log^2(2)+\frac{5}{96}log^4(2)+\frac{5}{4}Li_4\left(\frac{1}{2}\right)-\frac{119\pi^4}{5760}+\frac{35}{32}log(2)\,\zeta(3)$$

$$*\Rightarrow I=\frac{1}{2}J-K=\frac{1}{2}\left(-\frac{199\pi^4}{5760}+2G^2+\frac{35}{16}log(2)\,\zeta(3)\right)$$
$$-\left(G^2-\frac{5\pi^2}{96}log^2(2)+\frac{5}{96}log^4(2)+\frac{5}{4}Li_4\left(\frac{1}{2}\right)-\frac{119\pi^4}{5760}+\frac{35}{32}log(2)\,\zeta(3)\right)$$
$$=-\frac{5}{4}Li_4\left(\frac{1}{2}\right)+\frac{13\pi^4}{3840}-\frac{5}{96}log^4(2)+\frac{5\pi^2}{96}log^2(2)\,(Q.E.D)$$

$NOTE$: (∗), (∗
∗) From the book: (Almost) Impossible Integrals, Sums, and Series by Cornel Ioan Valean

$$\boldsymbol{PROBLEM(4)\_PROVE: I=\int_0^1\frac{x\,log(x)\,log^2(1-x)}{1+x^2}dx}$$
$$\boldsymbol{=-\frac{15}{8}Li_4\left(\frac{1}{2}\right)+\frac{167\pi^4}{23040}-\frac{5}{64}log^4(2)+\frac{\pi^2}{32}log^2(2)}$$

Naren Bhandari-Nepal

Solution: ∗ Power series of: $\frac{x}{x^2+1}=\sum_{n=1}^{\infty}(-1)^{n-1}x^{2n-1}$,and: $\int_0^1 x^{n-1}\,log^2(1-x)\,dx=\frac{H_n^2+H_n^{(2)}}{n}$

⇒ Derivative with respect to n

$$\Rightarrow\int_0^1 x^{n-1}\,log(x)\,log^2(1-x)\,dx=\frac{2\zeta(3)}{n}+\frac{2\zeta(2)H_n}{n}-\frac{H_n^{(2)}}{n^2}-\frac{H_n^2}{n^2}-\frac{2H_nH_n^{(2)}}{n}-\frac{2H_n^{(3)}}{n}$$

$$\Rightarrow I=\sum_{n=1}^{\infty}(-1)^{n-1}\int_0^1 x^{2n-1}\log(x)\log^2(1-x)\,dx$$
$$=\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{2\zeta(3)}{2n}+\frac{2\zeta(2)H_{2n}}{2n}-\frac{H_{2n}^{(2)}}{4n^2}-\frac{H_{2n}^2}{4n^2}-\frac{2H_{2n}H_{2n}^{(2)}}{2n}-\frac{2H_{2n}^{(3)}}{2n}\right)$$
$$=\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{\zeta(3)}{n}+\frac{\zeta(2)H_{2n}}{n}-\frac{H_{2n}^{(2)}}{4n^2}-\frac{H_{2n}^2}{4n^2}-\frac{H_{2n}H_{2n}^{(2)}}{n}-\frac{H_{2n}^{(3)}}{n}\right)$$

$*\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\zeta(3)}{n}=\zeta(3)\log(2)$

$*\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\zeta(2)H_{2n}}{n}=\zeta(2)\left(\frac{5}{48}\pi^2-\frac{1}{4}\log^2(2)\right)$

$*\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_{2n}^{(3)}}{n}$,use: Cornel Ioan Valean's generalized alternating harmonic series:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_{2n}^{(m)}}{n}=\frac{(-1)^m}{(m-1)!}\int_0^1\frac{\log^{m-1}(x)\log\left(\frac{x^2+1}{2}\right)}{1-x}dx$$
$$=m\zeta(m+1)-2^{-m}(1-2^{-m+1})\log(2)\zeta(m)$$
$$-\sum_{k=0}^{m-1}\beta(k+1)\beta(m-k)-\sum_{k=1}^{m-1}2^{-m-1}(1-2^{-k})(1-2^{-m+k+1})\zeta(k+1)\zeta(m-k)$$

Let: $m=3\Rightarrow\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_{2n}^{(3)}}{n}=\frac{199\pi^4}{11520}-\frac{3}{32}\log(2)\zeta(3)-G^2$

$*$ From : generating function of: $\sum_{n=1}^{\infty}\frac{H_n^2}{n^2}x^n$
$$=-\frac{1}{3}\log^3(1-x)\log(x)-\log^2(1-x)Li_2(1-x)+\frac{1}{2}Li_2^2(x)$$
$$+2\log(1-x)Li_3(1-x)+Li_4(x)-2Li_4(1-x)+2\zeta(4),\text{ then:}$$
$$\sum_{n=1}^{\infty}\frac{H_{2n}^2}{n^2}(-1)^{n-1}=2G^2-\frac{5}{48}\log^4(2)+\log^2(2)\zeta(2)-\frac{35}{16}\log(2)\zeta(3)+\frac{231}{32}\zeta(4)-\pi G\log(2)$$
$$-2\pi\Im\left(Li_3\left(\frac{1+i}{2}\right)\right)-\frac{5}{2}Li_4\left(\frac{1}{2}\right)$$

$*$ From : generating function of: $\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n}x^n=\frac{1}{6}\log^3(1-x)\log(x)-\frac{1}{24}\log^4(1-x)$
$$+\frac{1}{2}\log^2(1-x)Li_2(1-x)-\log(1-x)Li_3(1-x)+Li_4(1-x)-Li_4\left(\frac{x}{x-1}\right)$$
$$-\zeta(4),\text{ then:}$$
$$\sum_{n=1}^{\infty}\frac{H_{2n}H_{2n}^{(2)}}{n}(-1)^{n-1}=\frac{5}{96}\log^4(2)-\frac{3}{8}\log^2(2)\zeta(2)+\frac{35}{64}\log(2)\zeta(3)-\frac{137}{128}\zeta(4)+\frac{1}{4}\pi G\log(2)$$
$$+\frac{\pi}{2}\Im\left(Li_3\left(\frac{1+i}{2}\right)\right)+\frac{5}{4}Li_4\left(\frac{1}{2}\right)$$

$$*\sum_{n=1}^{\infty}\frac{H_{2n}^{(2)}}{n^2}(-1)^{n-1}=2G^2-\frac{353}{64}\zeta(4)-\frac{5}{4}log^2(2)\,\zeta(2)+\frac{35}{8}\log(2)\zeta(3)+\frac{5}{24}log^4(2)+5Li_4\left(\frac{1}{2}\right)(*)$$

$*$ Combine these results: I

$$=\zeta(3)\,log(2)+\frac{5}{48}\zeta(2)\pi^2-\frac{1}{4}\zeta(2)log^2(2)-\frac{1}{4}\left(2G^2-\frac{353}{64}\zeta(4)-\frac{5}{4}log^2(2)\,\zeta(2)+\frac{35}{8}\log(2)\,\zeta(3)+\frac{5}{24}log^4(2)+5Li_4\left(\frac{1}{2}\right)\right)$$
$$-\frac{1}{4}\left(2G^2-\frac{5}{48}log^4(2)+log^2(2)\,\zeta(2)-\frac{35}{16}log(2)\,\zeta(3)+\frac{231}{32}\zeta(4)-\pi G\,log(2)-2\pi\Im\left(Li_3\left(\frac{1+i}{2}\right)\right)-\frac{5}{2}Li_4\left(\frac{1}{2}\right)\right)$$
$$-\left(\frac{5}{96}log^4(2)-\frac{3}{8}log^2(2)\,\zeta(2)+\frac{35}{64}log(2)\,\zeta(3)-\frac{137}{128}\zeta(4)+\frac{1}{4}\pi G\,log(2)+\frac{\pi}{2}\Im\left(Li_3\left(\frac{1+i}{2}\right)\right)+\frac{5}{4}Li_4\left(\frac{1}{2}\right)\right)-\left(\frac{199\pi^4}{11520}-\frac{3}{32}log(2)\,\zeta(3)-G^2\right)$$
$$=-\frac{15}{8}Li_4\left(\frac{1}{2}\right)+\frac{167\pi^4}{23040}-\frac{5}{64}log^4(2)+\frac{\pi^2}{32}log^2(2)\,(Q.E.D)$$

$(*)$: $https://math.stackexchange.com/questions/3803424/how-to-find-sum-n-1-infty-frac-1nh-2nn3-and-sum-n-1-infty-f/3803762\#3803762$

PROBLEM(2)_PROVE:

$$\boldsymbol{I=\int_0^1\frac{x\,log(x)\,log(1-x)}{1+x^2}log\left(\frac{x}{1-x}\right)dx=\frac{5}{8}Li_4\left(\frac{1}{2}\right)-\frac{89\pi^4}{23040}+\frac{5}{192}log^4(2)+\frac{\pi^2}{48}log^2(2)}$$

Naren Bhandari

Solution: From : problem(3), problem(4) :

$$P_3=\int_0^1\frac{x\,log^2(x)\,log(1-x)}{1+x^2}dx=-\frac{5}{4}Li_4\left(\frac{1}{2}\right)+\frac{13\pi^4}{3840}-\frac{5}{96}log^4(2)+\frac{5\pi^2}{96}log^2(2)$$
$$P_4=\int_0^1\frac{x\,log(x)\,log^2(1-x)}{1+x^2}dx=-\frac{15}{8}Li_4\left(\frac{1}{2}\right)+\frac{167\pi^4}{23040}-\frac{5}{64}log^4(2)+\frac{\pi^2}{32}log^2(2)$$
$$I=P_3-P_4=-\frac{5}{4}Li_4\left(\frac{1}{2}\right)+\frac{13\pi^4}{3840}-\frac{5}{96}log^4(2)+\frac{5\pi^2}{96}log^2(2)-\left(-\frac{15}{8}Li_4\left(\frac{1}{2}\right)+\frac{167\pi^4}{23040}-\frac{5}{64}log^4(2)+\frac{\pi^2}{32}log^2(2)\right)$$
$$=\frac{5}{8}Li_4\left(\frac{1}{2}\right)-\frac{89\pi^4}{23040}+\frac{5}{192}log^4(2)+\frac{\pi^2}{48}log^2(2)\,(Q.E.D)$$

PROVING THE REFLECTIVE PROPERTY OF AN ELLIPSE

By Benny Le Van[1]-Vietnam

The reflective property of an ellipse states that any rays passing through a focus shall hit the boundary and reflect to the remaining focus. This property has been applied in multiple sites around the world[2]. In this article, we shall prove this geometric property**.** SOLUTION:

Given ellipse (E) as illustrated in Figure I with the following equation:

$$(\boldsymbol{E}):\frac{\boldsymbol{x}^2}{\boldsymbol{a}^2}+\frac{\boldsymbol{y}^2}{\boldsymbol{b}^2}=\mathbf{1} \qquad \textbf{(1)}$$

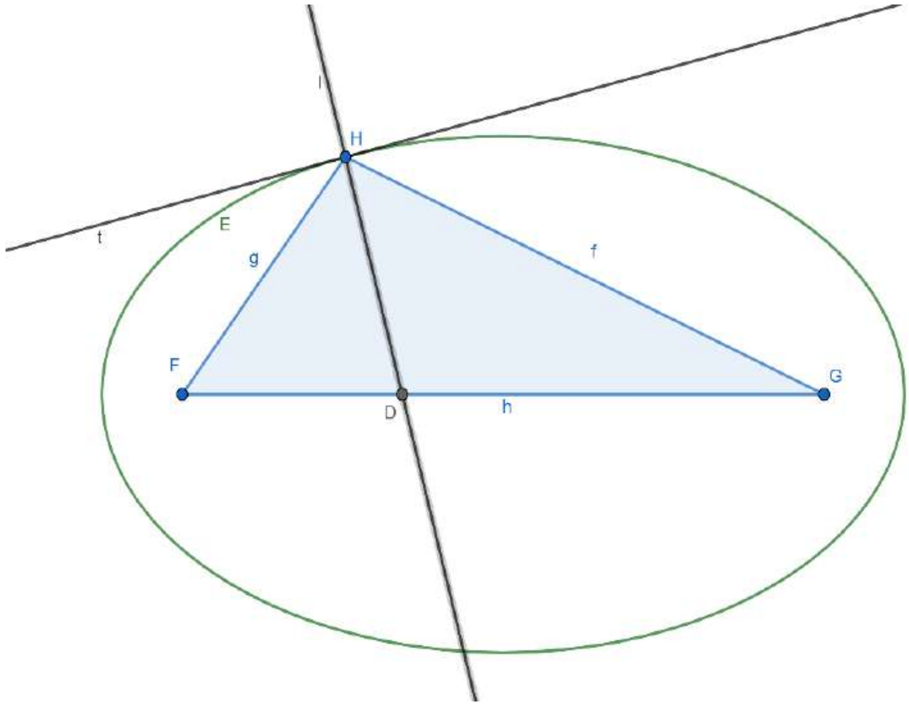

Figure I. The illustrated ellipse, constructed via GeoGebra.

In (1), a and b are semi-axes of (E) such that $0 < b < a$, and the focus is accordingly c. The foci are, respectively, $F(-c, 0)$ and $G(c, 0)$. Given point $H(j, k)$ which belongs to (E). If $k = 0$, the problem is straightforwardly proven. We consider $k \neq 0$. We are supposed to prove that HD is the bisector of triangle HFG.

The tangent line (t) of (E) at H and its correspondingly perpendicular line (l) are therefore:

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[1] https://orcid.org/0000-0002-6428-8731.

[2] See: Dawson, S. (2021). The Reflective Property of an Ellipse. Retrieved February 10, 2023, from https://personal.math.ubc.ca/~cass/courses/m309-01a/dawson/index.html#:~:text=The%20Reflective%20property%20of%20an%20ellipse%20is%20simply%20this%3A%20when,pass%20through%20the%20other%20focus.

$$(t):\frac{jx}{a^2}+\frac{ky}{b^2}=1 \quad (l):\begin{cases} x=j+\frac{ju}{a^2} \\ y=k+\frac{ku}{b^2} \\ (u\epsilon\mathbb{R}) \end{cases}$$

Let $D(d,0)$ be the intersection of (l) and FG. Since $y_D=0$, we get:

$$k+\frac{ku}{b^2}=0 \Leftrightarrow u=-b^2$$

Accordingly,

$$d=j\left(1-\frac{b^2}{a^2}\right)=\frac{jc^2}{a^2}$$

We calculate DF and DG as follows:

$$DF=\frac{jc^2}{a^2}+c=\frac{c}{a^2}(a^2+jc) \quad (2)$$

$$DG=c-\frac{jc^2}{a^2}=\frac{c}{a^2}(a^2-jc) \quad (3)$$

From (2) and (3), we get:

$$\frac{DF}{DG}=\frac{a^2+jc}{a^2-jc} \quad (4)$$

In (4), it is noticeable that there exist $\varphi\epsilon[0,2\pi]$ such that $j=a\cos\varphi$. This is to ensure that $a^2\geq|jc|$, resulting in $DF>0$ and $DG>0$.

We calculate HF as follow:

$$HF^2=(j+c)^2+k^2=j^2+2jc+c^2+k^2 \quad (5)$$

As point H belongs to (E), we get the following relation:

$$\frac{j^2}{a^2}+\frac{k^2}{b^2}=1\Rightarrow\frac{k^2}{b^2}=\frac{a^2-j^2}{a^2}\Rightarrow k^2=\frac{b^2}{a^2}(a^2-j^2)=\frac{(a^2-c^2)(a^2-j^2)}{a^2} \quad (6)$$

Replacing (6) into (5), we obtain:

$$HF^2=j^2+2jc+c^2+\frac{(a^2-c^2)(a^2-j^2)}{a^2}$$
$$=\frac{a^2j^2+2a^2jc+a^2c^2+a^4-a^2(j^2+c^2)+j^2c^2}{a^2}=\frac{a^4+2a^2jc+j^2c^2}{a^2}$$

Therefore,

$$HF=\frac{a^2+jc}{a} \quad (7)$$

As point H belongs to (E), we apply the definition of an ellipse:

$$HF+HG=2a \quad (8)$$

From (7) and (8), we get:

$$HG = 2a - HF = 2a - \frac{a^2 + jc}{a} = \frac{a^2 - jc}{a} \tag{9}$$

From (7) and (9), we get:

$$\frac{HF}{HG} = \frac{a^2 + jc}{a^2 - jc} \tag{10}$$

From (4) and (10), we obtain:

$$\frac{DF}{DG} = \frac{HF}{HG}$$

Henceforth, HD is the bisector of triangle HFG, which consequences in QED.

ABOUT AN INEQUALITY BY GEORGE APOSTOLOPOULOS-V

By Marin Chirciu – Romania

1) In ΔABC the following relationship holds:

$$\frac{4r}{R^2} \le \frac{h_a}{r_b r_c} + \frac{h_b}{r_c r_a} + \frac{h_c}{r_a r_b} \le \frac{R}{2r^2}$$

Proposed by George Apostolopoulos – Greece

Solution: We prove the following lemma:

Lemma.

2) In ΔABC the following relationship holds:

$$\frac{h_a}{r_b r_c} + \frac{h_b}{r_c r_a} + \frac{h_c}{r_a r_b} = \frac{1}{2R}\left[1 + \left(\frac{4R + r}{s}\right)^2\right]$$

Proof: Using the formulas $h_a = \frac{2S}{a}$ and $r_a = \frac{S}{s-a}$ we obtain:

$\sum \frac{h_a}{r_b r_c} = \sum \frac{\frac{2S}{a}}{\frac{S}{s-b} \cdot \frac{S}{s-c}} = \frac{2}{S} \sum \frac{(s-b)(s-c)}{a} = \frac{2}{rs} \cdot \frac{r[s^2 + (4R+r)^2]}{4Rs} = \frac{1}{2R}\left[1 + \left(\frac{4R+r}{s}\right)^2\right]$, which follows from

$$\sum \frac{(s-b)(s-c)}{a} = \frac{r[s^2 + (4R+r)^2]}{4Rs}$$

Let's get back to the main problem. LHS inequality. Using the Lemma the inequality can be written:

$$\frac{1}{2R}\left[1 + \left(\frac{4R + r}{s}\right)^2\right] \ge \frac{4r}{R^2} \Leftrightarrow s^2(R - 8r) + R(4R + r)^2 \ge 0$$

We distinguish the following cases:

Case 1). If $(R-8r)\geq 0$, the inequality is obvious.

Case 2). If $(R-8r)<0$, the inequality can be rewritten $R(4R+r)^2\geq s^2(8r-R)$, which follows from Blundon – Gerretsen's inequality $s^2\leq\frac{R(4R+r)^2}{2(2R-r)}$

It remains to prove that $R(4R+r)^2\geq\frac{R(4R+r)^2}{2(2R-r)}(8r-R)\Leftrightarrow R\geq 2r$ (Euler's inequality)

Equality holds if and only if the triangle is equilateral.

RHS inequality. Using the Lemma the inequality can be written:

$\frac{1}{2R}\left[1+\left(\frac{4R+r}{s}\right)^2\right]\geq\frac{R}{2r^2}\Leftrightarrow s^2(R^2-r^2)\geq r^2(4R+r)^2$, which follows from Gerretsen's inequality $s^2\geq 16Rr-5r^2\geq\frac{r(4R+r)^2}{R+r}$.

It remains to prove that $\frac{r(4R+r)^2}{R+r}(R^2-r^2)\geq r^2(4R+r)^2\Leftrightarrow R\geq 2r$ (Euler's inequality)

Equality holds if and only if the triangle is equilateral.

Remark: Let's interchange h_a with r_a.

3) In ΔABC the following relationship holds:

$$\frac{1}{r}\leq\frac{r_a}{h_bh_c}+\frac{r_b}{h_ch_a}+\frac{r_c}{h_ah_b}\leq\frac{R}{2r^2}$$

Proposed by Marin Chirciu – Romania

Solution: We prove the following lemma: **Lemma.**

4) In ΔABC the following relationship holds:

$$\frac{r_a}{h_bh_c}+\frac{r_b}{h_ch_a}+\frac{r_c}{h_ah_b}=\frac{1}{4r}\left[1+\left(\frac{4R+r}{s}\right)^2\right]$$

Proof: Using the formulas $h_a=\frac{2S}{a}$ and $r_a=\frac{S}{s-a}$ we obtain:

$$\sum\frac{r_a}{h_bh_c}=\sum\frac{\frac{S}{s-a}}{\frac{2S}{b}\cdot\frac{2S}{c}}=\frac{1}{4S}\sum\frac{bc}{s-a}=\frac{1}{4rs}\cdot\frac{s^2+(4R+r)^2}{s}=\frac{1}{4r}\left[1+\left(\frac{4R+r}{s}\right)^2\right]$$

which follows from $\sum\frac{bc}{s-a}=\frac{s^2+(4R+r)^2}{s}$

Let's get back to the main problem. LHS inequality. Using the Lemma the inequality can be written:

$$\frac{1}{4r}\left[1+\left(\frac{4R+r}{s}\right)^2\right] \geq \frac{1}{r} \Leftrightarrow (4R+r)^2 \geq 3s^2 \text{ (Doucet's inequality)}$$

Equality holds if and only if the triangle is equilateral. RHS inequality:

Using the Lemma the inequality can be written:

$\frac{1}{4r}\left[1+\left(\frac{4R+r}{s}\right)^2\right] \leq \frac{R}{2r^2} \Leftrightarrow s^2(2R-r) \geq r(4R+r)^2$, which follows from Gerretsen's inequality $s^2 \geq 16Rr - 5r^2 \geq \frac{r(4R+r)^2}{R+r}$.

It remains to prove that $\frac{r(4R+r)^2}{R+r}(2R-r) \geq r(4R+r)^2 \Leftrightarrow R \geq 2r$ (Euler's inequality)

Equality holds if and only if the triangle is equilateral.

Remark: Between the sums $\frac{h_a}{r_b r_c}+\frac{h_b}{r_c r_a}+\frac{h_c}{r_a r_b}$ and $\frac{r_a}{h_b h_c}+\frac{r_b}{h_c h_a}+\frac{r_c}{h_a h_b}$ the following relationship holds:

5) In ΔABC the following relationship holds:

$$\frac{h_a}{r_b r_c}+\frac{h_b}{r_c r_a}+\frac{h_c}{r_a r_b} \leq \frac{r_a}{h_b h_c}+\frac{r_b}{h_c h_a}+\frac{r_c}{h_a h_b}$$

Proposed by Marin Chirciu – Romania

Using the following Lemmas, the inequality holds:

$$\frac{1}{2R}\left[1+\left(\frac{4R+r}{s}\right)^2\right] \leq \frac{1}{4r}\left[1+\left(\frac{4R+r}{s}\right)^2\right] \Leftrightarrow R \geq 2r \text{ (Euler's inequality)}$$

Equality holds if and only if the triangle is equilateral.

Remark. The following sequence of inequalities can be written:

6) In ΔABC the following relationship holds:

$$\frac{4r}{R^2} \leq \frac{h_a}{r_b r_c}+\frac{h_b}{r_c r_a}+\frac{h_c}{r_a r_b} \leq \frac{r_a}{h_b h_c}+\frac{r_b}{h_c h_a}+\frac{r_c}{h_a h_b} \leq \frac{R}{2r^2}$$

Solution See inequalities 1) and 3). Equality holds if and only if the triangle is equilateral.

Reference: Romanian Mathematical Magazine-www.ssmrmh.ro

ABOUT AN INEQUALITY BY HOANG LE NHAT TUNG-II

By Marin Chirciu-Romania

If $x, y, z > 0$ such that $x + y + z = 3$ then:

$$3\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}+1\right)\geq 4(xy+yz+zx)$$

Hence, find the minimum value of expression

$$P=\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}+\frac{3}{8}\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}\right)$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution: Step 1. We prove that $3\sqrt[3]{x}+2x^2\geq 5x$; (1)

Using AM-GM inequality, we get:

$$\sqrt[3]{x}+\sqrt[3]{x}+\sqrt[3]{x}+x^2+x^2\geq 5\sqrt[5]{\sqrt[3]{x}\cdot\sqrt[3]{x}\cdot\sqrt[3]{x}\cdot x^2\cdot x^2}=5x\Leftrightarrow$$

$3\sum\sqrt[3]{x}+2\sum x^2\geq 15$ equality for $\sqrt[3]{x}=x^2\Leftrightarrow x=1$.

Step 2. We prove that $3\sum\sqrt[3]{x}+2\sum x^2\geq 15$; (2)

Summing relations (1), we get: $3\sum\sqrt[3]{x}+2\sum x^2\geq 5\sum x=5\cdot 3=15$, equality for $x=y=z=1$.

Step 3. We prove that: $3\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}+1\right)\geq 4(xy+yz+zx)$; (3)

Using hypothesis condition, we have:

$\sum x^2=(x+y+z)^2-2(xy+yz+zx)=9-2\sum yz$ and with condition (2) it follows:

$$3\sum\sqrt[3]{x}+2\left(9-2\sum yz\right)\geq 15\Leftrightarrow 3\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}\right)+3\geq 4(xy+yz+zx)\Leftrightarrow$$

$3\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}+1\right)\geq 4(xy+yz+zx)$, with equality for $x=y=z=1$.

Let's solve the proposed problem:

We show that:

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\geq\frac{9}{2(xy+yz+zx)};\ (4)$$

Using Bergstrom Inequality, we have:

$$\sum\frac{x}{y+z}=\sum\frac{x^2}{xy+xz}\geq\frac{(\sum x)^2}{\sum(xy+zx)}=\frac{9}{2\sum yz}$$

Equality holds if $\frac{x}{y+z}=\frac{y}{z+x}=\frac{z}{x+y}\Leftrightarrow x=y=z.$

Step 5. We show that:

$$P=\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}+\frac{3}{8}\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}\right)\geq\frac{21}{8};\ \ (5)$$

Inequality (3) it can be rewritten as:

$$3\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}\right)\geq 4(xy+yz+zx)-3;\ \ (6)$$

From (5),(6) we get:

$$P=\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}+\frac{3}{8}\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}\right)\geq\frac{9}{2\sum yz}+\frac{1}{8}\left(4\sum yz-3\right)=$$

$$=\frac{9}{2\sum yz}+\frac{\sum yz}{2}-\frac{3}{8}\overset{AGM}{\geq}2\sqrt{\frac{9}{2\sum yz}\cdot\frac{\sum yz}{2}}-\frac{3}{8}=2\cdot\frac{3}{2}-\frac{3}{8}=\frac{21}{8},$$

Equality holds if $x=y=z=1$.

So, $P\geq\frac{21}{8}$, with equality for $x=y=z=1$ then minimum of expression P is $\frac{21}{8}$ which is attained for $(x,y,z)=(1,1,1)$.

2. If $x,y,z>0$ such that $x+y+z=3$ then

$$\mathbf{2\left(\sqrt[4]{x}+\sqrt[4]{y}+\sqrt[4]{z}\right)+3\geq 3(xy+yz+zx)}$$

Hence, find the minimum value of expression

$$\mathbf{P=\frac{x}{y+nz}+\frac{y}{z+nx}+\frac{z}{x+ny}+\frac{2}{3(n+1)}\left(\sqrt[4]{x}+\sqrt[4]{y}+\sqrt[4]{z}\right);n\geq 0}$$

Proposed by Marin Chirciu-Romania

Solution by proposer:

Step 1. We prove that: $4\sqrt[4]{x}+3x^2\geq 7x;\ \ (1)$. Using AM-GM inequality, we get:

$$\sqrt[4]{x}+\sqrt[4]{x}+\sqrt[4]{x}+\sqrt[4]{x}+x^2+x^2+x^2\geq 7\sqrt[7]{\sqrt[4]{x}\cdot\sqrt[4]{x}\cdot\sqrt[4]{x}\cdot\sqrt[4]{x}\cdot x^2\cdot x^2\cdot x^2}=7x\Leftrightarrow$$

$4\sqrt[4]{x}+3x^2\geq 7x$, equality if $\sqrt[4]{x}=x^2\Leftrightarrow x=1$.

Step 2. We prove that $4\sum\sqrt[4]{x}+3\sum x^2\geq 21;(2)$

Summing inequalities (1), we get $4\sum\sqrt[4]{x}+3\sum x^2\geq 7\sum x=7\cdot 3=21$, Equality if $x = y = z = 1$.

Step 3. We prove that:

$$2\left(\sqrt[4]{x} + \sqrt[4]{y} + \sqrt[4]{z}\right) + 3 \geq 3(xy + yz + zx); (3)$$

Using hypothesis condition, we have:

$\sum x^2 = (x + y + z)^2 - 2(xy + yz + zx) = 9 - 2\sum yz$ and with inequality (2), it follows:

$$4\sum \sqrt[4]{x} + 3\left(9 - 2\sum yz\right) \geq 21 \Leftrightarrow 2\left(\sqrt[4]{x} + \sqrt[4]{y} + \sqrt[4]{z}\right) + 3 \geq 3(xy + yz + zx)$$

Equality holds if $x = y = z = 1$.Let's solve the proposed problem.

Step 4. We show that:

$$\frac{x}{y + nz} + \frac{y}{z + nx} + \frac{z}{x + ny} \geq \frac{9}{(n + 1)(xy + yz + zx)}; (4)$$

Using Bergstrom inequality, we get:

$$\sum \frac{x}{y + nz} = \sum \frac{x^2}{xy + nxz} \geq \frac{(\sum x)^2}{\sum(xy + nxz)} = \frac{9}{(n + 1)\sum yz}$$

Equality holds if $\frac{x}{y+nz} = \frac{y}{z+nx} = \frac{z}{x+ny} \Leftrightarrow x = y = z$.

Step 5. We show that:

$$P = \frac{x}{y + nz} + \frac{y}{z + nx} + \frac{z}{x + ny} + \frac{2}{3(n + 1)}\left(\sqrt[4]{x} + \sqrt[4]{y} + \sqrt[4]{z}\right) \geq \frac{5}{n + 1}; (5)$$

Inequality (3) it can rewritten as:

$$2\left(\sqrt[4]{x} + \sqrt[4]{y} + \sqrt[4]{z}\right) \geq 3(xy + yz + zx) - 3; (6)$$

From (5),(6) we get:

$$P = \frac{x}{y + nz} + \frac{y}{z + nx} + \frac{z}{x + ny} + \frac{2}{3(n + 1)}\left(\sqrt[4]{x} + \sqrt[4]{y} + \sqrt[4]{z}\right) \geq$$

$$\geq \frac{9}{(n + 1)\sum yz} + \frac{1}{3(n + 1)}\left(3\sum yz - 3\right) = \frac{9}{(n + 1)\sum yz} + \frac{1}{n + 1}\sum yz - \frac{1}{n + 1} \overset{AGM}{\geq}$$

$$\geq 2\sqrt{\frac{9}{(n + 1)\sum yz} \cdot \frac{1}{n + 1}\sum yz} - \frac{1}{n + 1} = 2 \cdot \frac{3}{n + 1} - \frac{2}{n + 1} = \frac{5}{n + 1}.$$

Equality holds if: $\frac{9}{(n+1)\sum yz} = \frac{1}{n+1}\sum yz \Leftrightarrow (\sum yz)^2 = 9 \Leftrightarrow \sum yz = 3$.

Equality in (5) if $x = y = z = 1$.

From $P \geq \frac{5}{n+1}$, equality if $x = y = z = 1$ it follows that minimum of expression P is $\frac{5}{n+1}$ which is attained for $(x, y, z) = (1,1,1)$.

Note. See problem UP.310 from 21-RMM Sumer Edition 2021 proposed by Hoang Le Nhat Tung-Hanoi-Vietnam.Remark. The problem it can be developed.

3. If $x, y, z > 0$ such that $x + y + z = 3, n \in \mathbb{N}, n \geq 2$ then

$$n\left(\sqrt[n]{x} + \sqrt[n]{y} + \sqrt[n]{z}\right) + 3(n-2) \geq 2(n-1)(xy + yz + zx)$$

Hence, find the minimum value of expression

$$P = \frac{x}{y + \lambda z} + \frac{y}{z + \lambda x} + \frac{z}{x + \lambda y} + \frac{n}{2(n-1)(\lambda+1)}\left(\sqrt[n]{x} + \sqrt[n]{y} + \sqrt[n]{z}\right); \lambda \geq 0$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Step 1. We show that: $n\sqrt[n]{x} + (n-1)x^2 \geq (2n-1)x$; (1)

Using AM-GM inequality, we get:

$$n\sqrt[n]{x} + (n-1)x^2 = \sqrt[n]{x} + \sqrt[n]{x} + \cdots + \sqrt[n]{x} + x^2 + x^2 + \cdots + x^2 \geq$$

$$\geq (2n-1)\sqrt[2n-1]{\sqrt[n]{x} \cdot \sqrt[n]{x} \cdot \ldots \cdot \sqrt[n]{x} \cdot x^2 \cdot x^2 \cdot \ldots \cdot x^2} = (2n-1)x \Leftrightarrow$$

$n\sqrt[n]{x} + (n-1)x^2 \geq (2n-1)x$, equality for $\sqrt[n]{x} = x^2 \Leftrightarrow x = 1$.

Step 2. We prove that: $n\sum\sqrt[n]{x} + (n-1)\sum x^2 \geq 3(2n-1)$; (2)

Summing inequalities (1) we get:

$$n\sum\sqrt[n]{x} + (n-1)\sum x^2 \geq (2n-1)\sum x = (2n-1) \cdot 3 = 3(2n-1).$$

Equality for $x = y = z = 1$.

Step 3.We show that:

$$n\left(\sqrt[n]{x} + \sqrt[n]{y} + \sqrt[n]{z}\right) + 3(n-2) \geq 2(n-1)(xy + yz + zx); \quad (3)$$

Using Hypothesis condition, we have:

$\sum x^2 = (x+y+z)^2 - 2(xy + yz + zx) = 9 - 2\sum yz$ and with inequality (2), it follows:

$$n\sum\sqrt[n]{x} + (n-1)\sum x^2 \geq 3(2n-1) \Leftrightarrow$$

$$n\sum\sqrt[n]{x} + (n-1)\left(9 - 2\sum yz\right) \geq 3(2n-1) \Leftrightarrow$$

$n\sum\sqrt[n]{x}+3(n-2)\geq 2(n-1)\sum yz$, equality for $x=y=z=1$.

Let's solve the proposed prolem.

Step 4. We show that:

$$\frac{x}{y+\lambda z}+\frac{y}{z+\lambda x}+\frac{z}{x+\lambda y}\geq\frac{9}{(\lambda+1)(xy+yz+zx)};\quad (4)$$

Using Bergstrom inequality, we get:

$$\sum\frac{x}{y+\lambda z}=\sum\frac{x^2}{xy+\lambda xz}\geq\frac{(\sum x)^2}{\sum(xy+\lambda xz)}\geq\frac{9}{(\lambda+1)\sum yz}$$

Equality for $\frac{x}{y+\lambda z}=\frac{y}{z+\lambda x}=\frac{z}{x+\lambda y}\Leftrightarrow x=y=z$.

Step 5. We show that:

$$P=\frac{x}{y+\lambda z}+\frac{y}{z+\lambda x}+\frac{z}{x+\lambda y}+\frac{n}{2(n-1)(\lambda+1)}\left(\sqrt[n]{x}+\sqrt[n]{y}+\sqrt[n]{z}\right)\geq\frac{3(3n-2)}{2(n-1)(\lambda+1)};(5)$$

Inequality (3) it can rewritten as:

$$n\left(\sqrt[n]{x}+\sqrt[n]{y}+\sqrt[n]{z}\right)\geq 2(n-1)(xy+yz+zx)-3(n-2);(6)$$

Using (5),(6) we get:

$$P=\frac{x}{y+\lambda z}+\frac{y}{z+\lambda x}+\frac{z}{x+\lambda y}+\frac{n}{2(n-1)(\lambda+1)}\left(\sqrt[n]{x}+\sqrt[n]{y}+\sqrt[n]{z}\right)\geq$$

$$\geq\frac{9}{(\lambda+1)\sum yz}+\frac{1}{2(n-1)(\lambda+1)}\left[2(n-1)\sum yz-3(n-2)\right]=$$

$$=\frac{9}{(\lambda+1)\sum yz}+\frac{1}{\lambda+1}\sum yz-\frac{3(n-2)}{2(n-1)(\lambda+1)}\overset{AGM}{\geq}$$

$$2\sqrt{\frac{9}{(\lambda+1)\sum yz}\cdot\frac{1}{\lambda+1}\sum yz}-\frac{3(n-2)}{2(n-1)(\lambda+1)}=$$

$$=2\cdot\frac{3}{\lambda+1}-\frac{3(n-2)}{2(n-1)(\lambda+1)}=\frac{3(3n-2)}{2(n-1)(\lambda+1)}$$

Equality holds in AGM if $\frac{9}{(\lambda+1)\sum yz}=\frac{1}{\lambda+1}\sum yz\Leftrightarrow(\sum yz)^2=9\Leftrightarrow\sum yz=3$.

Equality holds in (5) if $x=y=z=1$.

From $P\geq\frac{3(3n-2)}{2(n-1)(\lambda+1)}$, with equality for $x=y=z=1$ it follows that minimum of expression P is $\frac{3(3n-2)}{2(n-1)(\lambda+1)}$ attained for $(x,y,z)=(1,1,1)$.

Note: For $n = 3, \lambda = 2$ Problem UP.310. from 21-RMM-Summer Edition 2021 proposed by Hoang Le Nhat Tung-Hanoi-Vietnam.

Reference: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

ABOUT AN INEQUALITY BY MARIAN URSĂRESCU-VI

By Marin Chirciu – Romania

1) Prove that in any acute-angled triangle the following inequality holds:

$$\frac{h_a^2}{w_a^2}+\frac{h_b^2}{w_b^2}+\frac{h_c^2}{w_c^2}\ge\frac{12r^2}{R^2}$$

Proposed by Marian Ursărescu – Romania

Solution We prove the following lemma:

2) In ΔABC the following relationship holds:

$$\frac{h_a^2}{w_a^2}+\frac{h_b^2}{w_b^2}+\frac{h_c^2}{w_c^2}\ge\frac{(\sum bc)^2}{4R^2s^2}$$

Proof. Using the following formulas: $h_a=\frac{2S}{a}$ and $w_a=\frac{2bc}{b+c}\cos\frac{A}{2}$, we obtain:

$$\frac{h_a^2}{w_a^2}=\frac{\left(\frac{2S}{a}\right)^2}{\left(\frac{2bc}{b+c}\cos\frac{A}{2}\right)^2}=\frac{S^2}{a^2b^2c^2}\cdot\frac{bc(b+c)^2}{s(s-a)}\ge\frac{S^2}{16R^2S^2}\cdot\frac{bc\cdot 4bc}{s(s-a)}=\frac{b^2c^2}{4R^2s(s-a)}$$

$$\text{It follows }\sum\frac{h_a^2}{w_a^2}\ge\frac{1}{4R^2}\sum\frac{b^2c^2}{s(s-a)}\overset{\text{Bergstrom}}{\ge}\frac{1}{4R^2}\cdot\frac{(\sum bc)^2}{\sum s(s-a)}=\frac{(\sum bc)^2}{4R^2s^2}$$

Let's get back to the main problem:

Using the Lemma it suffices to prove that:

$$\frac{(\sum bc)^2}{4R^2s^2}\ge\frac{12r^2}{R^2}\Leftrightarrow\frac{(s^2+r^2+4Rr)^2}{4R^2s^2}\ge\frac{12r^2}{R^2}\Leftrightarrow s^2(s^2+8Rr-46r^2)+r^2(4R+r)^2\ge 0$$

We distinguish the following cases:

Case 1). If $(s^2+8Rr-46r^2)\ge 0$, the inequality is obvious.

Case 2). If $(s^2+8Rr-46r^2)<0$, the inequality can be rewritten:

$r^2(4R+r)^2\ge s^2(46r^2-8Rr-s^2)$, which follows from Blundon-Gerretsen's inequality:

$$16Rr-5r^2\le s^2\le\frac{R(4R+R)^2}{2(2R-r)}. \text{ It remains to prove that:}$$

$$r^2(4R+r)^2\ge\frac{R(4R+r)^2}{2(2R-r)}(46r^2-8Rr-16Rr+5r^2)\Leftrightarrow 24R^2-47Rr-2r^2\ge 0\Leftrightarrow$$

$$\Leftrightarrow (R-2r)(4R+r)\ge 0, \text{ obviously from Euler's inequality } R\ge 2r.$$

Equality holds if and only if the triangle is equilateral.

Remark. From the above proof, the condition of acute-angled triangle it is not necessary.

Remark. Inequality can be strengthened:

3) In ΔABC the following inequality holds:

$$\frac{h_a^2}{w_a^2}+\frac{h_b^2}{w_b^2}+\frac{h_c^2}{w_c^2}\ge\frac{6r}{R}$$

Proposed by Marin Chirciu – Romania

Solution Using Lemma, it suffices to prove that:

$$\frac{(\sum bc)^2}{4R^2s^2}\ge\frac{6r}{R}\Leftrightarrow\frac{(s^2+r^2+4Rr)^2}{4R^2s^2}\ge\frac{6r}{R}\Leftrightarrow s^2(s^2+2r^2-16Rr)+r^2(4R+r)^2\ge 0$$

We distinguish the following cases:

Case 1). If $(p^2+2r^2-16Rr)\ge 0$, the inequality is obvious.

Case 2). If $(s^2+2r^2-16Rr)<0$, the inequality can be rewritten:

$r^2(4R+r)^2\ge s^2(16Rr-2r^2-s^2)$, which follows from Blundon-Gerretsen's inequality:

$$16Rr-5r^2\le s^2\le\frac{R(4R+r)^2}{2(2R-r)}. \text{ It remains to prove that:}$$

$$r^2(4R+r)^2\ge\frac{R(4R+R)^2}{2(2R-r)}(16Rr-2r^2-16Rr+5r^2)\Leftrightarrow R\ge 2r \text{ (Euler)}$$

Equality holds if and only if the triangle is equilateral.

Remark. Inequality 3) is stronger than inequality 1)

4) In ΔABC the following relationship holds:

$$\frac{h_a^2}{w_a^2}+\frac{h_b^2}{w_b^2}+\frac{h_c^2}{w_c^2}\ge\frac{6r}{R}\ge\frac{12r^2}{R^2}$$

Solution: See inequality 3) and $\frac{6r}{R}\ge\frac{12r^2}{R^2}\Leftrightarrow R\ge 2r$ (Euler)

Equality holds if and only if the triangle is equilateral.

Remark: If we replace h_a with r_a we propose:

5) In ΔABC the following relationship holds:

$$\frac{r_a^2}{w_a^2}+\frac{r_b^2}{w_b^2}+\frac{r_c^2}{w_c^2}\ge\frac{3R}{2r}$$

Proposed by Marin Chirciu – Romania

Solution We prove the following lemma:

Lemma.

6) In ΔABC the following relationship holds:

$$\frac{r_a^2}{w_a^2}+\frac{r_b^2}{w_b^2}+\frac{r_c^2}{w_c^2}\ge\sum\frac{r^2s}{(s-a)^3}$$

Proof. Using the following formulas: $r_a=\frac{S}{s-a}$ and $w_a=\frac{2bc}{b+c}\cos\frac{A}{2}$, we obtain:

$$\frac{r_a^2}{w_a^2}=\frac{\left(\frac{S}{s-a}\right)^2}{\left(\frac{2bc}{b+c}\cos\frac{A}{2}\right)^2}=\frac{S^2}{4s}\cdot\frac{(b+c)^2}{bc(s-a)^3}\ge\frac{r^2s^2}{4s}\cdot\frac{4bc}{bc(s-a)^3}=\frac{r^2s}{(s-a)^3}$$

It follows $\sum\frac{r_a^2}{w_a^2}\ge\sum\frac{r^2s}{(s-a)^3}$.

Let's get back to the main problem.

Using Lemma it suffices to prove that: $\sum\frac{r^2s}{(s-a)^3}\ge\frac{3R}{2r}$

Using the identity in triangle: $\sum\frac{1}{(s-a)^3}=\frac{(4R+r)^3-12s^2R}{r^3s^3}$ the inequality holds:

$\sum\frac{r^2s}{(s-a)^3}\ge\frac{3R}{2r}$ we write $r^2s\cdot\frac{(4R+r)^3-12s^2R}{r^3s^3}\ge\frac{3R}{2r}\Leftrightarrow 2(4R+r)^3-24s^2R\ge 3s^2R\Leftrightarrow$

$\Leftrightarrow 2(4R+r)^3\ge 27s^2R$, it follows from Blundon-Gerretsen's inequality.

$s^2\le\frac{R(4R+r)^2}{2(2R-r)}$. It remains to prove that:

$$2(4R+r)^3\ge 27R\cdot\frac{R(4R+r)^2}{2(2R-r)}\Leftrightarrow 5R^2-8Rr-4r^2\ge 0\Leftrightarrow$$

$\Leftrightarrow (R-2r)(5R+2r)\ge 0$, obviously from Euler's inequality $R\ge 2r$.

Equality holds if and only if the triangle is equilateral.

Reference: Romanian Mathematical Magazine-www.ssmrmh.ro

ABOUT AN INEQUALITY BY NGUYEN VAN CANH-XIII

By Marin Chirciu – Romania

1) In ΔABC the following relationship holds:

$$\frac{3}{4(2R^2+r^2)} \le \frac{1}{a^2+2b^2}+\frac{1}{b^2+2c^2}+\frac{1}{c^2+2a^2} \le \frac{1}{12r^2}$$

Proposed by Nguyen Van Canh – Vietnam

Solution: LHS inequality: Using Bergstrom's inequality we have:

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \ge \frac{(1+1+1)^2}{a^2+b^2+c^2} = \frac{9}{a^2+2b^2}, \text{ wherefrom } \sum\frac{1}{a^2+2b^2} \le \sum\frac{1}{9}\left(\frac{1}{a^2}+\frac{2}{b^2}\right) = \frac{1}{3}\sum\frac{1}{a^2}.$$

$$E = \frac{1}{a^2+2b^2}+\frac{1}{b^2+2c^2}+\frac{1}{c^2+2a^2} \le \frac{1}{3}\sum\frac{1}{a^2} \overset{(1)}{\le} \frac{1}{3}\cdot\frac{1}{4r^2} = \frac{1}{12r^2} = RHS, \text{ where (1)} \Leftrightarrow$$

$$\Leftrightarrow \sum\frac{1}{a^2} \le \frac{1}{4r^2} \text{ (J. Steining, 1963)}$$

Equality holds if and only if the triangle is equilateral. RHS inequality: Using Bergstrom's inequality we have:

$$E = \frac{1}{a^2+2b^2}+\frac{1}{b^2+2c^2}+\frac{1}{c^2+2a^2} \ge \frac{(1+1+1)^2}{\sum(a^2+2b^2)} = \frac{9}{3\sum a^2} = \frac{3}{\sum a^2} \overset{(1)}{\ge} \frac{3}{4(2R^2+r^2)} = \text{RHS}$$

$$\text{where (1)} \Leftrightarrow \frac{3}{\sum a^2} \ge \frac{3}{4(2R^2+r^2)} \Leftrightarrow \sum a^2 \le 4(2R^2+r^2) \Leftrightarrow$$

$$\Leftrightarrow 2(s^2-r^2-4Rr) \le 4(2R^2+r^2) \Leftrightarrow s^2 \le 4R^2+4Rr+3r^2, \text{ (Gerretsen's inequality)}$$

Equality holds if and only if the triangle is equilateral.

Remark: The problem can be developed.

2) In ΔABC the following relationship holds:

$$\frac{9}{4(n+1)(2R^2+r^2)} \le \frac{1}{a^2+nb^2}+\frac{1}{b^2+nc^2}+\frac{1}{c^2+na^2} \le \frac{1}{4(n+1)r^2}, n \in \mathbb{N}$$

Marin Chirciu-Romania

Solution: RHS inequality.Using Bergstrom's inequality we have:

$$\frac{1}{a^2}+\frac{1}{b^2}+\cdots+\frac{1}{b^2} \ge \frac{(1+1+\cdots+1)^2}{a^2+b^2+\cdots+b^2} = \frac{(n+1)^2}{a^2+nb^2}$$

wherefrom $\sum\frac{1}{a^2+nb^2} \le \sum\frac{1}{(n+1)^2}\left(\frac{1}{a^2}+\frac{n}{b^2}\right) = \frac{1}{n+1}\sum\frac{1}{a^2}$. We obtain:

$E=\frac{1}{a^2+nb^2}+\frac{1}{b^2+nc^2}+\frac{1}{c^2+na^2}\leq\frac{1}{n+1}\sum\frac{1}{a^2}\overset{(1)}{\leq}\frac{1}{n+1}\cdot\frac{1}{4r^2}=\frac{1}{4(n+1)r^2}=RHS$, where (1) $\Leftrightarrow$

$\Leftrightarrow\sum\frac{1}{a^2}\leq\frac{1}{4r^2}$, (J. Steining, 1963). Equality holds if and only if the triangle is equilateral.LHS inequality. Using Bergstrom's inequality we have:

$$E=\frac{1}{a^2+nb^2}+\frac{1}{b^2+nc^2}+\frac{1}{c^2+na^2}\geq\frac{(1+1+1)^2}{\sum(n^2+nb^2)}=\frac{9}{(n+1)\sum a^2}=$$

$$=\frac{9}{(n+1)\sum a^2}\overset{(2)}{\geq}\frac{9}{4(n+1)(2R^2+r^2)}=LHS$$

where (2) $\Leftrightarrow\frac{9}{(n+1)\sum a^2}\geq\frac{9}{4(n+1)(2R^2+r^2)}\Leftrightarrow\sum a^2\leq 4(2R^2+r^2)\Leftrightarrow$

$\Leftrightarrow 2(s^2-r^2-4Rr)\leq 4(2R^2+r^2)\Leftrightarrow s^2\leq 4R^2+4Rr+3r^2$, (Gerretsen's inequality)

Equality holds if and only if the triangle is equilateral.

Note: For $n=2$ we obtain the proposed problem by Nguyen Van Canh, Vietnam in RMM 12/2020.

Reference: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

SPECIAL TRIANGLES

By Bogdan Fuștei-Romania

We consider triangle ABC and X a point in the plane of triangle.

AX,BX,CX intersect BC,AC și CX in points X_a,X_b, X_c.

AX_a,BX_b,CX_c are cevians of point X.

AX_{a8}, BX_{b8}, $CX_{c8}\leftrightarrow n_a$, n_b, n_c are cevians of Nagel in triangle ABC.

Then we have the next theorem:

n_a , n_b, n_c are cevians of Nagel in triangle ABC ,

can be lengths of sides of a triangle(Cevians as Sides of Triangles-Zvonko Cerin).

We show that: $p^2=n_a^2+2r_ah_a$ (and analogs);

$r_ah_a=\frac{2r_ar_br_c}{r_b+r_c}$ (and analogs) ;

$h_a=\frac{2r_br_c}{r_b+r_c}$ (and analogs);

$p^2-n_a^2$=$2r_ah_a \to (p+n_a)(p-n_a)$= $2r_ah_a \to\ p-n_a=\frac{2r_ah_a}{p+n_a} \to p=n_a+\frac{4r_ar_br_c}{(n_a+p)(r_b+r_c)}$

p+p>p-true, cevians of Nagel can be lenghts of sides of a triangle then we have:

$n_a+n_b>n_c$(and analogs).From all what we present we have the next conclusion:

$\frac{1}{(n_a+p)(r_b+r_c)}+\frac{1}{(n_b+p)(r_a+r_c)}>\frac{1}{(n_c+p)(r_a+r_b)}$ (and analogs),so

1) $\frac{1}{(n_a+p)(r_b+r_c)},\frac{1}{(n_b+p)(r_a+r_c)},\frac{1}{(n_c+p)(r_a+r_b)}$ **can be lenghts of sides of a triangle.**

LEMA:If x,y,z are lengths sides of a triangle, then
$\sqrt{x},\sqrt{y},\sqrt{z}$ are lengths sides of acute triangle.

Now we will use the next relation relația $r_b+r_c=4R\cos^2\frac{A}{2}$ (and analogs), and we obtain next results:

2) $\frac{1}{\sqrt{n_a+p}\cos\frac{A}{2}},\frac{1}{\sqrt{n_b+p}\cos\frac{B}{2}},\frac{1}{\sqrt{n_c+p}\cos\frac{C}{2}}$ **can be lenghts of sides of acute triangle.**

$\frac{2r_ah_a}{p+n_a}=p-n_a$ (and analogs) $\to\frac{2r_ah_a}{p-n_a}=p+n_a$ (and analogs).

n_a , n_b, n_c are cevians of Nagel in triangle ABC , can be lengths of sides of a triangle,

then we obtain:

3) $\frac{r_ah_a}{p-n_a},\frac{r_bh_b}{p-n_b},\frac{r_ch_c}{p-n_c}$ **can be lenghts of sides of a triangle.**

4) $\frac{r_ah_a}{p+n_a},\frac{r_bh_b}{p+n_b},\frac{r_ch_c}{p+n_c}$ **can be lenghts of sides of a triangle.**

We use $r_ah_a=\frac{2r_ar_br_c}{r_b+r_c}$ (and analogs) ,and we obtain:

5) $\frac{1}{(p-n_a)(r_b+r_c)},\frac{1}{(p-n_b)(r_a+r_c)},\frac{1}{(p-n_c)(r_a+r_b)}$ **can be lenghts of sides of a triangle.**

$r_b+r_c=4R\cos^2\frac{A}{2}$ (and analog) and we use Lema we obtain next results:

6) $\frac{1}{\sqrt{p-n_a}\cos\frac{A}{2}}$, $\frac{1}{\sqrt{p-n_b}\cos\frac{B}{2}}$, $\frac{1}{\sqrt{p-n_c}\cos\frac{C}{2}}$ **can be lenghts of sides of acute triangle .**

We shown that $\frac{n_a}{h_a}=\frac{\sqrt{(b-c)^2+4r^2}}{2r}$ (and analogs), but also we shown that $\frac{n_a}{h_a},\frac{n_b}{h_b},\frac{n_c}{h_c}$ can be lenghts of sides of a triangle, so we will have the next results :

7) $\sqrt{(\mathbf{b}-\mathbf{c})^2+4\mathbf{r}^2}$,$\sqrt{(\mathbf{a}-\mathbf{b})^2+4\mathbf{r}^2}$, $\sqrt{(\mathbf{c}-\mathbf{a})^2+4\mathbf{r}^2}$ **can be lenghts of sides of a** triangle.

But AN_a=$\sqrt{(b-c)^2+4r^2}$(and analogs), N_a-point of Nagel ,so we obtain:

8) $\mathbf{AN_a}$, $\mathbf{BN_a}$, $\mathbf{CN_a}$ **can be lenghts of sides of a triangle.**

We know that AN_a=$\frac{an_a}{p}$ (and analogs)(RMM 33) and AG_e=$\frac{g_a(r_b+r_c)}{4R+r}$ (and analogs)(RMM 32) also we will use the next theorem:

(Murray Klamkin's Duality Principle for Triangle Inequalities):If P a point in Int(ΔABC), let PA=x,PB=y,PC=z,AB=c,AC=b,BC=a.Then ax,by,cz can be the lenghts of the sides of a triangle.

Using this theorem we obtain the next results :

9)$\mathbf{a^2n_a}$,$\mathbf{b^2n_b}$, $\mathbf{c^2n_c}$ **can be lenghts of sides of a triangle.**

10)$\mathbf{a}\sqrt{\mathbf{n_a}}$, $\mathbf{b}\sqrt{\mathbf{n_b}}$,$\mathbf{c}\sqrt{\mathbf{n_c}}$ **can be lenghts of sides of acute triangle**.

11)$\mathbf{ag_a(r_b+r_c)}$, $\mathbf{bg_b(r_a+r_c)}$, $\mathbf{cg_c(r_a+r_b)}$ **can be lenghts of sides of a triangle.**

12)$\sqrt{\mathbf{ag_a}}\cos\frac{A}{2}$,$\sqrt{\mathbf{bg_b}}\cos\frac{B}{2}$,$\sqrt{\mathbf{cg_c}}\cos\frac{C}{2}$
can be lenghts of sides of acute triangle.

(we used relation $r_b+r_c=4R\cos^2\frac{A}{2}$(and analogs)).

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NEW TRIANGLE INEQUALITIES WITH BROCARD'S ANGLE

By Bogdan Fuștei-Romania, Mohamed Amine Ben Ajiba-Morocco

ABSTRACT : In this paper are created new inequalities in triangle using Brocard's angle.

Panaitopol's Inequality : In any triangle ABC, we have

$$\frac{m_a}{h_a} \le \frac{R}{2r} \ \text{(and analogs)}, \qquad (1)$$

with equality if and only if the triangle ABC is equilateral.

Proof :

Considering the origin of the complex plane at the circumcenter of triangle ABC and

let z_1, z_2, z_3 be the coordinates of points A, B, C, respectively.

Using the formulas $h_a = \frac{2S}{a}$ and

$S = pr$, the desired inequality (1) can be rewritten as follows,

$am_a \le Rp$. We have :

$$2am_a = 2|z_2 - z_3|.\left|z_1 - \frac{z_2 + z_3}{2}\right| = |(z_2 - z_3)(2z_1 - z_2 - z_3)|$$

$$= |z_1(z_2 - z_3) + z_3(z_3 - z_1) + z_2(z_1 - z_2)|$$
$$\le |z_1|.|z_2 - z_3| + |z_3|.|z_3 - z_1| + |z_2|.|z_1 - z_2|$$

$= Ra + Rb + Rc = 2Rp$,

This completes the proof of (1). Equality holds if and only if the triangle ABC is equilateral.

Now, in triangle ABC, we have the following relations (see, Bogdan Fuștei – *About Nagel's and Gergonnes's cevian* – www.ssmrmh.ro),

$$2r_b r_c = h_a(r_b + r_c) \quad \text{(and analogs)},$$

$$4m_a^2 = 4r_b r_c + (b - c)^2 = 2h_a(r_b + r_c) + (b - c)^2 \quad \text{(and analogs)}$$

On the other hand, using the formulas $h_a = \frac{2S}{a}$ and $r_a = \frac{S}{p - a}$ (and analogs), we have

$$r_b + r_c = \frac{S}{p - b} + \frac{S}{p - c} = \frac{S(2p - b - c)}{(p - b)(p - c)} = \frac{4S.a}{(a - b + c)(a + b - c)} = \frac{4S.a}{a^2 - (b - c)^2},$$

$$r_b + r_c - 2h_a = \frac{4S.a}{a^2-(b-c)^2} - 2h_a = \frac{2h_a.a^2}{a^2-(b-c)^2} - 2h_a = \frac{2h_a(b-c)^2}{a^2-(b-c)^2}.$$

Using these relations and identities, we have

$$(r_b + r_c)^2 - 4{m_a}^2 = (r_b + r_c)(r_b + r_c - 2h_a) - (b-c)^2$$
$$= \frac{2h_a(r_b + r_c)(b-c)^2}{a^2-(b-c)^2} - (b-c)^2.$$

and since we have, $2h_a(r_b + r_c) = 4r_b r_c = 4p(p-a) = (b+c)^2 - a^2$, and,

$$\cos A = \frac{b^2+c^2-a^2}{2bc}, \quad rr_a = (p-b)(p-c), \text{ then}$$

$$(r_b + r_c)^2 - 4{m_a}^2 = \left(\frac{(b+c)^2-a^2}{a^2-(b-c)^2} - 1\right)(b-c)^2 = \frac{2(b^2+c^2-a^2)(b-c)^2}{4(p-b)(p-c)}$$

$$= \frac{2.2bc\cos A\,(b-c)^2}{4rr_a} = \frac{bc\cos A\,(b-c)^2}{rr_a}.$$

$$\Rightarrow (r_b + r_c)^2 - 4{m_a}^2 = \frac{bc\cos A\,(b-c)^2}{rr_a} \text{ (and analogs).}$$

Using the relation $\sin^2\frac{A}{2} = \frac{(p-b)(p-c)}{bc} = \frac{rr_a}{bc}$, we get

$$\boldsymbol{(r_b + r_c)^2 - 4{m_a}^2 = \frac{\cos A\,(b-c)^2}{\sin^2\frac{A}{2}}} \text{ (and analogs)} \quad (2)$$

Using this identity and the formulas $a = 4R\,sin\frac{A}{2}cos\frac{A}{2}$ and $r_b + r_c = 4R\cos^2\frac{A}{2}$, we get

$$r_b + r_c - \frac{4{m_a}^2}{r_b + r_c} = \frac{\cos A\,(b-c)^2}{4R\cos^2\frac{A}{2}.\sin^2\frac{A}{2}} = \frac{4R\cos A\,(b-c)^2}{a^2},$$

then we have the following new identity

$$\boldsymbol{r_b + r_c - \frac{4{m_a}^2}{r_b + r_c} = 4R\cos A\left(\frac{b-c}{a}\right)^2} \text{ (and analogs)} \quad (3)$$

Now, by the arithmetic − geometric means inequality, we have

$$\boldsymbol{r_b + r_c + \frac{4{m_a}^2}{r_b + r_c} \geq 2\sqrt{(r_b + r_c).\frac{4{m_a}^2}{r_b + r_c}} = 4m_a} \quad (4)$$

From the results (3) and (4), we have

$$\boldsymbol{2(r_b + r_c) \geq 4m_a + 4R\cos A\left(\frac{b-c}{a}\right)^2.}$$

Which gives us the following inequality, in any triangle ABC, we have

$$\boldsymbol{\frac{r_b + r_c}{2} \geq m_a + R\cos A\left(\frac{b-c}{a}\right)^2} \quad (\textbf{and analogs}).$$

Again, from the results (2) and (3), we have

$$\boldsymbol{4R\cos A\left(\frac{b-c}{a}\right)^2 + \frac{8{m_a}^2}{r_b + r_c} \geq 4m_a,}$$

or,

$$\boldsymbol{R\cos A\left(\frac{b-c}{a}\right)^2 + \frac{2{m_a}^2}{r_b + r_c} \geq m_a} \qquad (5)$$

On the other hand, using the relations $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ and

$\cos^2\dfrac{A}{2} = \dfrac{1 + \cos A}{2}$, we have

$$4{m_a}^2 = 2(b^2 + c^2) - a^2 = (b-c)^2 + 2bc + (b^2 + c^2 - a^2)$$
$$= (b-c)^2 + 2bc + 2bc\cos A$$

$$= (b-c)^2 + 2bc(1 + \cos A) = (b-c)^2 + 4bc\cos^2\frac{A}{2}.$$

and by the formulas $r_b + r_c = 4R\cos^2\dfrac{A}{2}$ and $h_a = \dfrac{bc}{2R}$, we obtain

$$\frac{2{m_a}^2}{r_b + r_c} = \frac{4bc\cos^2\frac{A}{2} + (b-c)^2}{2.4R\cos^2\frac{A}{2}} = h_a + \frac{(b-c)^2}{8R\cos^2\frac{A}{2}}.$$

Note that $a = 4R\,sin\dfrac{A}{2}cos\dfrac{A}{2}$, we obtain the following identity

$$\boldsymbol{\frac{2{m_a}^2}{r_b + r_c} = h_a + 2R\sin^2\frac{A}{2}\left(\frac{b-c}{a}\right)^2} \quad (\textbf{and analogs})$$

Replacing this identity in (5), we get

$$\boldsymbol{h_a + R\left(\cos A + 2\sin^2\frac{A}{2}\right)\left(\frac{b-c}{a}\right)^2 \geq m_a.}$$

Using the relation

$2\sin^2\frac{A}{2}=1-\cos A$, we obtain the following inequality, in any triangle ABC

$$\boldsymbol{m_a \le h_a + R\left(\frac{b-c}{a}\right)^2} \ \textbf{(and analogs)} \qquad (6)$$

By the results (3) and (4), the equality in (6) holds if

$$r_b + r_c = \frac{4{m_a}^2}{r_b + r_c} \Leftrightarrow \cos A\left(\frac{b-c}{a}\right)^2 = 0, \text{i.e. } b = c \text{ or } A = \frac{\pi}{2}.$$

From this result, we have

$$\sqrt{\frac{\boldsymbol{m_a - h_a}}{\boldsymbol{R}}} \le \frac{|\boldsymbol{b-c}|}{\boldsymbol{a}},$$

Then we have, in any triangle ABC, the follwing inequality

$$\boldsymbol{a\sqrt{\frac{m_a - h_a}{R}} \le |b-c|} \ \textbf{(and analogs)} \qquad (7)$$

Adding this inequality with similar ones and using the identity

$$|a-b| + |b-c| + |c-a| = 2\big(max(a,b,c) - min(a,b,c)\big),$$

we obtain the following inequality, in any triangle ABC

$$\boldsymbol{\frac{1}{2}\left(a\sqrt{\frac{m_a - h_a}{R}} + b\sqrt{\frac{m_b - h_b}{R}} + c\sqrt{\frac{m_c - h_c}{R}}\right) \le max(a,b,c) - min(a,b,c).} \qquad (8)$$

From the inequality (6), we have

$$\frac{m_a}{h_a} \le 1 + \frac{R(b-c)^2}{a^2 h_a},$$

and by the formulas $R = \frac{abc}{4S}$ and $ah_a = 2S$, we obtain

$$\boldsymbol{\frac{m_a}{h_a} \le 1 + \frac{bc(b-c)^2}{8S^2}} \ \textbf{(and analogs)} \qquad (9)$$

Since m_a, m_b, m_c can be the sides of triangle with area

$S_m = \frac{3S}{4}$, median $\overline{m_a} = \frac{3a}{4}$ (and analogs) altitude

$\overline{h_a} = \frac{2S_m}{m_a} = \frac{3S}{2m_a}$ (and analogs), then by using the inequality (9) in $\Delta m_a m_b m_c$,

we obtain

$$\boldsymbol{\frac{m_a}{h_a} \leq 1 + \frac{2m_b m_c (m_b - m_c)^2}{9S^2}} \text{ (and analogs)} \qquad (10)$$

Now, in any triangle ABC, we have the following relation

$$4{m_a}^2 = {n_a}^2 + {g_a}^2 + 2r_b r_c \quad \text{(and analogs)}$$

Using this relation and the identity (2), we have

$${r_b}^2 + {r_c}^2 = (4{m_a}^2 - 2r_b r_c) + [(r_b + r_c)^2 - 4{m_a}^2] = {n_a}^2 + {g_a}^2 + \frac{\cos A\,(b-c)^2}{\sin^2 \frac{A}{2}}.$$

Then we obtain the following identity

$${r_b}^2 + {r_c}^2 = {n_a}^2 + {g_a}^2 + \frac{\cos A\,(b-c)^2}{\sin^2 \frac{A}{2}} \quad \text{(and analogs)}$$

Which gives us the following inequality, in any non– obtuse triangle ABC, we have

$$\boldsymbol{{r_b}^2 + {r_c}^2 \geq {n_a}^2 + {g_a}^2} \text{ (and analogs)} \qquad (11)$$

with equality if $b = c$ or $A = \frac{\pi}{2}$.

In this part, we will prove the following inequality, in any triangle ABC, we have

$$\boldsymbol{\frac{m_b}{h_c} + \frac{m_c}{h_b} \leq \frac{R}{r}} \quad \text{(and analogs)} \qquad (12)$$

Using the result (6) and the formulas $\mathrm{R} = \frac{\mathrm{abc}}{4\mathrm{S}}, h_a = \frac{2S}{a}$ (and analogs), we have

$$\frac{m_b}{h_c} + \frac{m_c}{h_b} \leq \left(\frac{h_b}{h_c} + \frac{R(c-a)^2}{h_c b^2}\right) + \left(\frac{h_c}{h_b} + \frac{R(a-b)^2}{h_b c^2}\right)$$
$$= \left(\frac{c}{b} + \frac{c^2 a(c-a)^2}{8bS^2}\right) + \left(\frac{b}{c} + \frac{b^2 a(a-b)^2}{8cS^2}\right).$$

Then we have,

$$\frac{m_b}{h_c} + \frac{m_c}{h_b} \leq \frac{8S^2(b^2 + c^2) + c^3 a(c-a)^2 + b^3 a(a-b)}{8bcS^2}$$

So to prove (12) it suffices to prove that

$$8S^2(b^2+c^2)+c^3a(c-a)^2+ab^3(a-b)\le 8bcS^2.\frac{R}{r}$$

and by the formulas $4RS=abc, S=pr$ and the following identity

$$16S^2=2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4),$$

the last inequality is equivalent to

$$[2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)](b^2+c^2)+2c^3a(c-a)^2+2ab^3(a-b)$$
$$\le 2ab^2c^2(a+b+c).$$

which, after expanding and simplifying, equivalent to

$$(b^2+c^2)a^4-2(b^3+c^3)a^3+2(b^4-b^2c^2+c^4)a^2-2(b+c)(b^2+bc+c^2)(b-c)^2a$$
$$+(b^2+c^2)(b^2-c^2)^2\ge 0$$

or,

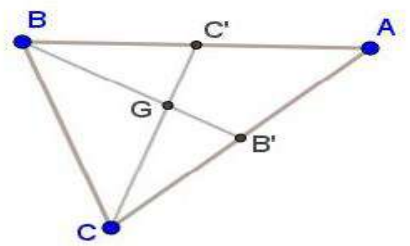
$$(b^2+c^2)\left(a^2-\frac{(b^3+c^3)a}{b^2+c^2}\right)^2+\frac{(b^2+bc+c^2)^2}{b^2+c^2}.\left(a-\frac{(b^2+c^2)(b+c)}{b^2+bc+c^2}\right)^2(b-c)^2\ge 0,$$

which is true and the proof of (12) is complete. Equality holds if $\mathrm{b}=\mathrm{c}$.

Now, we will prove the following inequality, in any triangle ABC, we have

$$\mathbf{2\frac{m_a}{h_a}\le\frac{m_b}{h_c}+\frac{m_c}{h_b}}\quad \textbf{(and analogs)}\qquad \mathbf{(13)}$$

Let B', C' be the midpoints of AC, AB, and let G be the centroid of triangle ABC.



By Ptolemy's inequality in the quadrilateral $AB'GC'$, we have

$$B'C'.GA\le AC'.GB'+AB'.GC',$$

or more explicitly,

$$\frac{a}{2}.\frac{2m_a}{3}\le\frac{c}{2}.\frac{m_b}{3}+\frac{b}{2}.\frac{m_c}{3}$$

which is equialent to

$$2\frac{m_a}{h_a} \le \frac{m_b}{h_c} + \frac{m_c}{h_b}.$$

From the results (12) and (13), we obtain the following refinement

of Panaitopol's inequality :

$$\mathbf{2\frac{m_a}{h_a} \le \frac{m_b}{h_c} + \frac{m_c}{h_b} \le \frac{R}{r}} \quad (\textbf{and analogs}) \qquad (\mathbf{14})$$

Now, using the formula

$h_a = \dfrac{bc}{2R}$ (and analogs), we obtain the equivalent expression of (9) :

$$\mathbf{\frac{m_b}{b} + \frac{m_c}{c} \le \frac{a}{2r}} \quad (\textbf{and analogs}) \qquad (\mathbf{15})$$

We have, in any triangle ABC the following relation

$$p^2 = {n_a}^2 + 2r_a h_a \quad \text{(and analogs)}$$

This relation can be rewritten as follows

$$\frac{p + n_a}{h_a} = \frac{2r_a}{p - n_a} \quad \text{(and analogs)}$$

By this relation and the formulas $h_a = \dfrac{2S}{a}, S = pr$, we have

$$\frac{a}{2r} = \frac{p}{h_a} = \frac{p + n_a}{h_a} - \frac{n_a}{h_a} = \frac{2r_a}{p - n_a} - \frac{n_a}{h_a} \quad \text{(and analogs)}$$

From the result (15), we obtain

$$\mathbf{\frac{m_b}{b} + \frac{m_c}{c} + \frac{n_a}{h_a} \le \frac{2r_a}{p - n_a}} \quad (\textbf{and analogs}) \qquad (\mathbf{16})$$

or,

$$\mathbf{\left(\frac{m_b}{b} + \frac{m_c}{c} + \frac{n_a}{h_a}\right)^{-1} \ge \frac{p - n_a}{2r_a}} \quad (\textbf{and analogs})$$

Adding this inequality with similar ones and using the identity

$\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}$, we obtain the following inequality

$$\sum_{cyc}\left(\frac{m_b}{b}+\frac{m_c}{c}+\frac{n_a}{h_a}\right)^{-1}\geq\frac{p}{2r}-\frac{1}{2}\left(\frac{n_a}{r_a}+\frac{n_b}{r_b}+\frac{n_c}{r_c}\right)\qquad(17)$$

From the inequality (16), we have

$$\frac{1}{2}(p-n_a)\left(\frac{m_b}{b}+\frac{m_c}{c}+\frac{n_a}{h_a}\right)\leq r_a\ \text{(and analogs)}$$

Adding this inequality with similar ones and using the identity

$r_a+r_b+r_c=4R+r$, we obtain the following inequality

$$\frac{1}{2}\sum_{cyc}(p-n_a)\left(\frac{m_b}{b}+\frac{m_c}{c}+\frac{n_a}{h_a}\right)\leq 4R+r\qquad(18)$$

We have the following identity

$$\frac{{g_a}^2}{h_a}+\frac{{g_b}^2}{h_b}+\frac{{g_c}^2}{h_c}=2R+5r.$$

Then we obtain the following inequality

$$\frac{1}{2}\sum_{cyc}(p-n_a)\left(\frac{m_b}{b}+\frac{m_c}{c}+\frac{n_a}{h_a}\right)\leq\sum_{cyc}\frac{{g_a}^2}{h_a}+2(R-2r)\qquad(19)$$

Again, by the relation $p^2={n_a}^2+2r_ah_a$ (and analogs), we have

$$\frac{p-n_a}{h_a}=\frac{2r_a}{p+n_a}\ \text{(and analogs)}$$

By this relation and the formulas $h_a=\frac{2S}{a}, S=pr$, we have

$$\frac{a}{2r}=\frac{p}{h_a}=\frac{p-n_a}{h_a}+\frac{n_a}{h_a}=\frac{2r_a}{p+n_a}+\frac{n_a}{h_a}\ \text{(and analogs)}$$

From the result (15), we obtain

$$\frac{m_b}{b}+\frac{m_c}{c}-\frac{n_a}{h_a}\leq\frac{2r_a}{p+n_a}\ \text{(and analogs)}\qquad(20)$$

or,

$$\frac{1}{2}(p+n_a)\left(\frac{m_b}{b}+\frac{m_c}{c}-\frac{n_a}{h_a}\right)\leq r_a\ \text{(and analogs)}$$

Adding this inequality with similar ones and using the identity

$r_a + r_b + r_c = 4R + r$, we obtain the following inequality

$$\frac{1}{2}\sum_{cyc}(p+n_a)\left(\frac{m_b}{b}+\frac{m_c}{c}-\frac{n_a}{h_a}\right) \le 4R + r \tag{21}$$

Using the following identity

$$\frac{{g_a}^2}{h_a}+\frac{{g_b}^2}{h_b}+\frac{{g_c}^2}{h_c}=2R+5r.$$

we obtain the following inequality

$$\frac{1}{2}\sum_{cyc}(p+n_a)\left(\frac{m_b}{b}+\frac{m_c}{c}-\frac{n_a}{h_a}\right) \le \sum_{cyc}\frac{{g_a}^2}{h_a}+2(R-2r) \tag{22}$$

In this part, we will prove the following inequality, in any non– obtuse triangle ABC, we have

$$\frac{2{m_a}^2}{r_b+r_c} \le \frac{b^2+c^2}{4R} \quad (\text{and analogs}) \tag{23}$$

By the formulas $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$ and

$\cos^2\dfrac{A}{2} = \dfrac{1+\cos A}{2}$, we have the two relations

$$4{m_a}^2 = 2(b^2+c^2)-a^2 = b^2+c^2+(b^2+c^2-a^2) = b^2+c^2+2bc\cos A$$

$$r_b + r_c = 4R\cos^2\frac{A}{2} = 2R(1+\cos A).$$

Using these relations, the inequality (23) is succesively equivalent to

$$4{m_a}^2 \le \frac{r_b+r_c}{2R}(b^2+c^2) \Leftrightarrow b^2+c^2+2bc\cos A \le (1+\cos A)(b^2+c^2) \Leftrightarrow 0 \le (b-c)^2\cos A,$$

which is true because $\cos A \ge 0$. Equality in (23) holds if $b = c$ or $A = \frac{\pi}{2}$.

Using the identity (3), we obtain the following inequality, in any non– obtuse triangle ABC, we have

$$r_b + r_c \le \frac{b^2+c^2}{2R} + 4R\cos A\left(\frac{b-c}{a}\right)^2 \quad (\text{and analogs}) \tag{24}$$

By the inequality (23) and the formulas $h_a = \frac{bc}{2R}, s_a = \frac{2bcm_a}{b^2+c^2}$, we obtain the following inequality, in any non– obtuse triangle ABC holds

$$\frac{r_b + r_c}{2} \geq \frac{m_a s_a}{h_a} \geq m_a \text{ (and analogs)} \qquad (25)$$

In this part, we will prove the following inequality chains, in any triangle ABC, if ω − Brocard′s angle, we have

$$\frac{R}{r} \geq \frac{m_b}{h_c} + \frac{m_c}{h_b} \geq \frac{1}{\sin\omega} \geq max\left\{2\frac{m_b+m_c}{h_b+h_c}, \frac{m_b}{h_b} + \frac{m_c}{h_c}\right\} \text{ (and analogs)} \qquad (26)$$

$$\frac{R}{r} \geq \frac{m_b}{h_c} + \frac{m_c}{h_b} \geq max\left\{\frac{1}{\sin\omega}, 2\frac{m_a}{h_a}\right\} \geq \frac{b}{c} + \frac{c}{b} \geq \frac{c+a}{a+b} + \frac{a+b}{c+a} \text{ (and analogs)} \qquad (27)$$

$$\frac{R}{r} \geq \frac{m_b}{h_c} + \frac{m_c}{h_b} \geq max\left\{\frac{1}{\sin\omega}, 2\frac{m_a}{h_a}\right\} \geq \frac{m_b}{m_c} + \frac{m_c}{m_b} \geq \frac{m_a+m_b}{m_b+m_c} + \frac{m_b+m_c}{m_a+m_b} \text{ (and analogs)} \qquad (28)$$

Lemma 1. In triangle ABC, ω − Brocard′s angle, we have

$$\frac{m_b}{h_c} + \frac{m_c}{h_b} \geq \frac{1}{\sin\omega} \text{ (and analogs)} \qquad (29)$$

Proof. Using the known median formulae we have

$$4cm_b = \sqrt{4c^2(2c^2+2a^2-b^2)}$$

$$= \sqrt{(3c^2+a^2-b^2)^2 + 2(a^2b^2+b^2c^2+c^2a^2) - (a^4+b^4+c^4)},$$

by the identity $16S^2 = 2(a^2b^2+b^2c^2+c^2a^2) - (a^4+b^4+c^4)$, we get

$$\frac{m_b}{h_c} = \frac{cm_b}{2S} = \frac{\sqrt{(3c^2+a^2-b^2)^2+(4S)^2}}{8S}.$$

Similarly, we get

$$\frac{m_c}{h_b} = \frac{\sqrt{(3b^2+a^2-c^2)^2+(4S)^2}}{8S}.$$

By the triangle inequality, we have

$$\sqrt{x^2+y^2} + \sqrt{z^2+t^2} \geq \sqrt{(x+z)^2+(y+t)^2},$$

for all real numbers x, y, z, t, with equality if $xt = yz$.

Using this inequality, the formula $\sin\omega = \frac{2S}{\sqrt{a^2b^2+b^2c^2+c^2a^2}}$ and the identity

$16S^2 = 2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)$, we obtain

$$\frac{m_b}{h_c} + \frac{m_c}{h_b} = \frac{\sqrt{(3c^2 + a^2 - b^2)^2 + (4S)^2} + \sqrt{(3b^2 + a^2 - c^2)^2 + (4S)^2}}{8S}$$

$$\geq \frac{\sqrt{(2a^2 + 2b^2 + 2c^2)^2 + (8S)^2}}{8S} = \frac{\sqrt{16(a^2b^2 + b^2c^2 + c^2a^2)}}{8S} = \frac{1}{\sin\omega},$$

which completes the proof of (29).

The equality in (29) holds if $(3c^2 + a^2 - b^2).4S = (3b^2 + a^2 - c^2).4S$, i.e. $b = c$.

Lemma 2. In triangle ABC, we have

$$\boldsymbol{4m_bm_c \leq 2a^2 + bc} \qquad (\mathbf{30})$$

Proof. Using the known median formulae we have

$$(4m_bm_c)^2 = (2c^2 + 2a^2 - b^2)(2b^2 + 2a^2 - c^2)$$
$$= 4a^4 + 2a^2(b^2 + c^2) - (2b^4 - 5b^2c^2 + 2c^4)$$

$$= (2a^2 + bc)^2 + 2a^2(b^2 + c^2 - 2bc) - 2(b^4 - 2b^2c^2 + c^4)$$

$$= (2a^2 + bc)^2 - 2[(b + c)^2 - a^2](b - c)^2 \leq (2a^2 + bc)^2,$$

the last inequality is true by $b + c > a$. Equality holds if b = c.

Lemma 3. In triangle ABC, $\boldsymbol{\omega}$ − Brocard's angle, we have

$$\frac{\mathbf{1}}{\boldsymbol{\sin\omega}} \geq \mathbf{2}\frac{\boldsymbol{m_b + m_c}}{\boldsymbol{h_b + h_c}} \quad (\textbf{and analogs}) \qquad (\mathbf{31})$$

Proof. By the formulas $h_a = \frac{2S}{a}$ (and analogs) and $\sin\omega = \frac{2S}{\sqrt{a^2b^2 + b^2c^2 + c^2a^2}}$, the

inequality (31) can be rewritten as follows

$$\sqrt{a^2b^2 + b^2c^2 + c^2a^2} \geq \frac{2bc(m_b + m_c)}{b + c}.$$

Using the known median formulae and the inequality (30), we have

$$\big(2bc(m_b + m_c)\big)^2 = (bc)^2(4{m_b}^2 + 4{m_c}^2 + 2.4m_bm_c)$$

$$\leq (bc)^2[(2c^2 + 2a^2 - b^2) + (2b^2 + 2a^2 - c^2) + 2(2a^2 + bc)] = (bc)^2[8a^2 + (b + c)^2]$$

By the AM − GM inequality, we have $8(bc)^2 = 4bc.2bc \leq (b + c)^2(b^2 + c^2)$,

$$\left(\frac{2bc(m_b+m_c)}{b+c}\right)^2 \le \frac{(b+c)^2(b^2+c^2)a^2+(bc)^2(b+c)^2}{(b+c)^2} = a^2b^2+b^2c^2+c^2a^2,$$

which completes the proof of (31). Equality holds if $b = c$.

Lemma 4. In triangle ABC, ω − Brocard's angle, we have

$$\frac{1}{\sin\omega} \ge \frac{m_b}{h_b}+\frac{m_c}{h_c} \quad \text{(and analogs)} \tag{32}$$

Proof. By the formulas $h_a = \frac{2S}{a}$ (and analogs) and $\sin\omega = \frac{2S}{\sqrt{a^2b^2+b^2c^2+c^2a^2}}$, the inequality (32) can be rewritten as follows

$$\sqrt{a^2b^2+b^2c^2+c^2a^2} \ge bm_b + cm_c.$$

By the following inequality $(x+y)^2 \le 2(x^2+y^2)$,

for all real numbers x, y, and using the known median formulae, we have

$$(bm_b+cm_c)^2 \le 2(b^2{m_b}^2+c^2{m_c}^2) = \frac{b^2(2c^2+2a^2-b^2)+c^2(2b^2+2a^2-c^2)}{2}$$

$$= (a^2b^2+b^2c^2+c^2a^2) - \frac{(b^2-c^2)^2}{2} \le a^2b^2+b^2c^2+c^2a^2,$$

which completes the proof of (32). Equality holds if $b = c$.

From the inequalities (12), (29), (31) and (32) yields the desired inequality chain (26).

Lemma 5. In triangle ABC, ω − Brocard's angle, we have

$$\frac{1}{\sin\omega} \ge \frac{b}{c}+\frac{c}{b} \tag{33}$$

Proof. By the formula $\sin\omega = \frac{2S}{\sqrt{a^2b^2+b^2c^2+c^2a^2}}$, the inequality (33) can be rewritten as

$$bc\sqrt{a^2b^2+b^2c^2+c^2a^2} \ge 2S(b^2+c^2).$$

Using the identity $16S^2 = 2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)$, we have

$$4\big(2S(b^2+c^2)\big)^2 = [2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)](2b^2c^2+b^4+c^4)$$

$$= 4b^2c^2(a^2b^2+b^2c^2+c^2a^2) - a^4(b^2+c^2)^2 + 2(b^4+c^4)(a^2b^2+c^2a^2) - (b^4+c^4)^2$$

$$= 4b^2c^2(a^2b^2+b^2c^2+c^2a^2) - [a^2(b^2+c^2)-(b^4+c^4)]^2$$

$\le 4b^2c^2(a^2b^2+b^2c^2+c^2a^2)$, which completes the proof of (33).

By Tereshin's inequality, we have

$$m_a \geq \frac{b^2+c^2}{4R} \quad \text{(and analogs)}$$

and by the formula $h_a = \dfrac{bc}{2R}$, we obtain

$$\mathbf{2\frac{m_a}{h_a} \geq \frac{b^2+c^2}{bc} = \frac{b}{c}+\frac{c}{b} \quad \text{(and analogs)}} \qquad (34)$$

Lemma 6. If a, b, c be positive real numbers, then we have

$$\mathbf{\frac{b}{c}+\frac{c}{b} \geq \frac{c+a}{a+b}+\frac{a+b}{c+a}.} \qquad (35)$$

Proof. The desired inequality is successively equivalent to

$$\frac{b}{c}-\frac{a+b}{c+a} \geq \frac{c+a}{a+b}-\frac{c}{b} \Leftrightarrow \frac{a(b-c)}{c(c+a)} \geq \frac{a(b-c)}{b(a+b)} \Leftrightarrow \frac{a(b-c)[b(a+b)-c(c+a)]}{bc(c+a)(a+b)} \geq 0$$

$$\Leftrightarrow \frac{a(a+b+c)(b-c)^2}{bc(c+a)(a+b)} \geq 0,$$

which is true and the proof of (35) is complete. Equality holds if $\mathrm{b} = \mathrm{c}$.

From the inequalities (12), (13), (29), (33), (34) and (35) yields the desired inequality chain (27).

Since m_a, m_b, m_c can be the sides of triangle with area $S_m = \dfrac{3S}{4}$,

$$\text{median } \overline{m_a} = \frac{3a}{4} \text{ (and analogs) altitude}$$

$$\overline{h_a} = \frac{2S_m}{m_a} = \frac{3S}{2m_a} \quad \text{(and analogs), and by the formula}$$

$$\sin\omega = \frac{2S}{\sqrt{a^2b^2+b^2c^2+c^2a^2}},$$

and the identity $m_a{}^2m_b{}^2 + m_b{}^2m_c{}^2 + m_c{}^2m_a{}^2 = \dfrac{9}{16}(a^2b^2+b^2c^2+c^2a^2)$, then we have

$$\frac{\overline{m_a}}{\overline{h_a}} = \frac{m_a}{h_a} \text{ and } \sin\omega_m = \frac{2S_m}{\sqrt{m_a{}^2m_b{}^2+m_b{}^2m_c{}^2+m_c{}^2m_a{}^2}} = \frac{2S}{\sqrt{a^2b^2+b^2c^2+c^2a^2}} = \sin\omega.$$

Applying the inequalities (33), (34) and (35) in $\Delta m_a m_b m_c$ and using the previous results, we obtain the following inequalities

$$\frac{1}{\sin\omega} \geq \frac{m_b}{m_c} + \frac{m_c}{m_b} \quad \text{(and analogs)} \tag{36}$$

$$2\frac{m_a}{h_a} \geq \frac{m_b}{m_c} + \frac{m_c}{m_b} \quad \text{(and analogs)} \tag{37}$$

$$\frac{m_b}{m_c} + \frac{m_c}{m_b} \geq \frac{m_a + m_b}{m_b + m_c} + \frac{m_b + m_c}{m_a + m_b} \quad \text{(and analogs)} \tag{38}$$

From the inequalities $(9), (10), (26), (33), (34)$ and (35) yields the desired inequality chain (28).

Reference : ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

PROPOSED PROBLEMS

PROBLEMS FOR JUNIORS

J.2341 If $a, b, c, d, e, t, u > 0$, then:

$$(a^2 + t)(b^2 + t)(c^2 + t)(d^2 + u)(e^2 + u) \geq \frac{3}{4}t^2u(a + b + c)^2(d + e)^2$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2342 In ΔABC the following relationship holds:

$$\frac{m_a + m_b}{h_c^3} + \frac{m_b + m_c}{h_a^3} + \frac{m_c + m_a}{h_b^3} \geq \frac{2\sqrt{3}}{F}$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți-Romania

J.2343 If $x, y, z > 0$, then in ΔABC the following relationship holds:

$$\frac{x + y}{z} \cdot ab + \frac{y + z}{x} \cdot bc + \frac{z + x}{y} \cdot ca \geq 8\sqrt{3}F$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți -Romania

J.2344 If $x, y > 0$, then in ΔABC the following relationship holds:

$$\frac{a^3}{xr+yh_a}+\frac{b^3}{xr+yh_b}+\frac{c^3}{xr+yh_c}\ge\frac{24}{x+3y}\cdot F$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți -Romania

J.2345 If $m \ge 0, x, y, z > 0$, then:

$$(x^{2m+2}+1)(y^{2m+2}+1)(z^{2m+2}+1)\ge\frac{3^{m+1}}{2^{5m+2}}(x+y+z)^{2m+2}$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți -Romania

J.2346 If $m \ge 0, x, y > 0$, then:

$$(x^{2m+2}+1)(y^{2m+2}+1)\ge\frac{(x+y)^{2m+2}}{4^m}$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2347 If $x, y > 0$, then in ΔABC holds:

$$(x^2+y^2)\left(r_a^2+r_b^2+r_c^2\right)\ge 6xy\sqrt{3}F+(xr_a-yr_b)^2+(xr_b-yr_c)^2+(xr_c-yr_a)^2$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2348 If $x, y > 0$, then in ΔABC holds:

$$x^2(a^2+b^2+c^2)+y^2\left(r_a^2+r_b^2+r_c^2\right)\ge 12xyF+(xa-yr_a)^2+(xb-yr_b)^2+(xc-yr_c)^2$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2349 If $M \in Int(\Delta ABC), d_a = d(M, BC), d_b = d(M, CA), d_c = d(M, AB)$, then:

$$\frac{a^{m+2}\cdot x^{m+1}}{d_a^m}+\frac{b^{m+2}\cdot y^{m+1}}{d_b^m}+\frac{c^{m+2}\cdot z^{m+1}}{d_c^m}\ge 2^{m+2}(xy+yz+zx)^{\frac{m+1}{2}}\cdot F$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2350 If $x, y, z > 0$, then in ΔABC holds: $\frac{a^3\cdot x^2}{h_a}+\frac{b^3\cdot y^2}{h_b}+\frac{c^3\cdot z^2}{h_c}\ge\frac{8}{3}(xy+yz+zx)\cdot F$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2351 If $m \ge 0, x, y, z > 0$ and $T, U \in Int(\Delta ABC), t_a = d(T, BC), t_b = d(T, CA), t_c = d(T, AB)$,

$u_a = d(U, BC), u_b = d(U, CA), u_c = d(U, AB)$, then:

$$\frac{a^{m+2}\cdot x^{m+1}}{(yt_a+zu_a)^m}+\frac{b^{m+2}\cdot y^{m+1}}{zt_b+xu_b}+\frac{c^{m+2}\cdot z^{m+1}}{(xt_c+yu_c)^m}\ge\frac{4(xy+yz+zx)^{\frac{m+1}{2}}}{(x+y+z)^m}\cdot F$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2352 In ΔABC the following relationship holds:

$$(a^2b^2+1)(b^2c^2+1)(c^2a^2+1)\ge 36F^2$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți -Romania

J.2353 If $x,y,t>0$, then in ΔABC holds:

$$\left(\left(xm_a^2+ym_b^2\right)^2+t^2\right)\left(\left(xm_b^2+ym_c^2\right)^2+t^2\right)\left((xm_c^2+ym_a^2)^2+t^2\right)\ge\frac{81}{4}t^4(x+y)^2\cdot F^2$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți -Romania

J.2354 If $t,x,y,z>0$, then in ΔABC holds:

$$((x^2+a^4)^2+t^2)((y^2+b^4)^2+t^2)((z^2+c^4)^2+t^2)\ge 48(xy+yz+zx)t^4F^2$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2355 If $x,y,z>0$, then in ΔABC holds:

$$(a^4+x^2)(b^4+y^2)(c^4+y^2)\ge 36\cdot\sqrt[3]{(xyz)^4}\cdot F^2$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2356 If $m\ge 0$ and $M\in Int(\Delta ABC), d_a=d(M,BC), d_b=d(M,CA), d_c=d(M,AB)$, then:

$$\frac{a^{m+1}\cdot b}{(xr+yd_b)^m}+\frac{b^{m+1}\cdot c}{(xr+yd_c)^m}+\frac{c^{m+1}\cdot a}{(xr+yd_a)^m}\ge\frac{2^{m+2}\cdot\left(\sqrt{3}\right)^{m+1}}{(x+y)^m}\cdot F$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2357 If $x,y>0$ and $T,U\in Int(\Delta ABC), t_a=d(T,BC), t_b=d(T,CA), t_c=d(T,AB)$,

$u_a=d(U,BC), u_b=d(U,CA), u_c=d(U,AB)$, then:

$$\frac{a^3b}{(xt_b+yu_c)^2}+\frac{b^3c}{(xt_c+yu_c)^2}+\frac{c^3a}{(xt_a+yu_a)^2}\ge\frac{48\sqrt{3}}{(x+y)^2}\cdot F$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2358 If $x,y>0$ and $U,V\in Int(\Delta ABC), u_a=d(U,BC), u_b=d(U,CA), u_c=d(U,AB)$,

$v_a=d(V,BC), v_b=d(V,CA), v_c=d(V,AB)$, then:

$$\frac{a^3}{xu_a+yv_a}+\frac{b^3}{xu_b+yv_b}+\frac{c^3}{xu_c+yv_c}\ge\frac{24}{x+y}\cdot F$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2359 If $m\ge 0, x,y>0$ and $M\in Int(\Delta ABC), d_a=d(M,BC), d_b=d(M,CA), d_c=d(M,AB)$, then:

$$\frac{a^{m+2}}{(xr+yd_a)^m}+\frac{b^{m+2}}{(xr+yd_b)^m}+\frac{c^{m+2}}{(xr+yd_c)^m}\geq\frac{2^{m+2}\left(\sqrt{3}\right)^{m+1}}{(x+y)^m}\cdot F$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți -Romania

J.2360 If $m\geq 0, x, y>0$, then in ΔABC holds:

$$\frac{a^{m+2}}{(xr+yh_a)^m}+\frac{b^{m+2}}{(xr+yh_b)^m}+\frac{c^{m+2}}{(xr+yh_c)^m}\geq\frac{2^{m+2}\left(\sqrt{3}\right)^{m+1}}{(x+3y)^m}\cdot F$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți -Romania

J.2361 If $t>0$, then in ΔABC holds:

$$(m_a^4+t^2)\left(m_b^4+t^2\right)(mc^4+t^2)\geq\frac{81}{4}t^4F^2$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2362 If $a, b, c, t>0$ then:

$$(a^2+t)(b^2+t)(c^2+t)\geq\frac{9}{4}t^2(ab+bc+ca)$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2363 If $x, y>0$, then in ΔABC holds:

$$((xr_a+yr_c)^2+r^2)((xr_b+yr_c)^2+r^2)((xr_c+yr_a)^2+r^2)\geq\frac{9}{4}(x+y)^2r^2F^2$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2364 In ΔABC the following relationship holds:

$$((r_a+r_b)^2+R^2)((r_b+r_c)^2+R^2)((r_c+r_a)^2+R^2)\geq 16\sqrt{3}\cdot F^3$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2365 Let $a, b, c>0$ such that $a+b+c=3$. Prove that,

$$\frac{1}{a^2+a}+\frac{1}{b^2+b}+\frac{1}{c^2+c}\geq\frac{3}{4}(a^2+b^2+c^2-1).$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

J.2366 *Inspired by a problem of Sir Daniel Sitaru.*

In acute ΔABC the following relationship holds:

$$8m_am_bm_c\geq(w_a+w_b)(w_b+w_c)(w_c+w_a)\geq 8s_as_bs_c.$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

J.2367 Find the largest positive constant k such that the following inequality

$$(a+b+c)\left(a\sqrt{a^2+bc}+b\sqrt{b^2+ca}+c\sqrt{c^2+ab}\right)\geq k\big(ab(a+b)+bc(b+c)+ca(c+a)\big),$$

holds for all nonnegative real numbers a,b,c.

Proposed by Mohamed Amine Ben Ajiba-Morocco

J.2368 In ΔABC the following relationship holds:

$$\sin^8 A\,.\cos A+\sin^8 B\,.\cos B+\sin^8 C\,.\cos C\leq\frac{243}{512}$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

J.2369 Let $a,b,c>0$ such that $abc=1$. Prove that:

$$\frac{1}{a^3+a^2+a}+\frac{1}{b^3+b^2+b}+\frac{1}{c^3+c^2+c}+\frac{8}{a+b+c}\geq\frac{11}{3}.$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

J.2370 In ΔABC the following relationship holds:

$$1+\frac{4R}{r}\geq\sqrt{\cot^2\frac{A}{2}+\cot^2\frac{B}{2}+3}+\sqrt{\cot^2\frac{B}{2}+\cot^2\frac{C}{2}+3}+\sqrt{\cot^2\frac{C}{2}+\cot^2\frac{A}{2}+3}\geq\frac{5\sqrt{3}s}{6r}+\frac{3}{2}.$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

J.2371 If in acute $\Delta ABC, r_br_c=max(r_br_c,\ r_cr_a,\ r_ar_b)$ then:

$$\frac{R+r}{R-r}\geq max\left(\frac{n_a^2}{r_br_c}+\frac{2r_ah_a}{r_br_c},\quad\frac{n_b^2}{r_br_c}+\frac{2h_b}{r_c},\quad\frac{n_c^2}{r_br_c}+\frac{2h_c}{r_b}\right)$$

Proposed by Bogdan Fuștei-Romania

J.2372 In acute ΔABC the following relationship holds:

$$4(m_a+m_b+m_c)^2\geq 27min(ab,bc,ca)$$

Proposed by Daniel Sitaru-Romania

J.2373 In ΔABC the following relationship holds:

$$\frac{a}{2r}\geq\frac{m_a}{h_a}+\frac{2r_a}{s+n_a}+\frac{\sqrt{g_a}\left(\sqrt{n_a}-\sqrt{g_a}\right)}{h_a}$$

Proposed by Bogdan Fuștei-Romania

J.2374 In ΔABC the following relationship holds:

$$\sum_{cyc} h_a\,(r_b + r_c - 2m_a) \geq 0$$

Proposed by Bogdan Fuștei-Romania

J.2375 Find $x, y, z, t \in \mathbb{R}$ such that:

$$\sum_{cyc} \frac{x}{x^2+1} + \frac{1}{2}\sum_{cyc} \frac{1}{x^2+1} = 1 + \sqrt{5}$$

Proposed by Daniel Sitaru, Claudia Nănuți -Romania

J.2376 Solve for real numbers:

$$sinx\sqrt{1 - sin^2x} = 1 + cosy\sqrt{1 - cos^2y}$$

Proposed by Daniel Sitaru, Claudia Nănuți -Romania

J.2377 In acute $\Delta ABC, a = min(a, b, c)$. Prove that:

$$a\left(\frac{b}{a^2+c^2-b^2} + \frac{c}{a^2+b^2-c^2}\right) \geq \frac{2}{cos(B-C)}$$

Proposed by Daniel Sitaru-Romania

J.2378 If $\lambda > 0$ then in ΔABC holds:

$$\frac{r_a + \lambda r}{r_b + r_c} + \frac{r_b + \lambda r}{r_c + r_a} + \frac{r_c + \lambda r}{r_a + r_b} \leq \frac{R}{r} + \frac{1-\lambda}{2}$$

Proposed by Mehmet Șahin-Turkiye

J.2379 If $a = min(a, b, c)$ in acute ΔABC then: $\frac{2m_a}{r_a} \geq \frac{n_b}{h_c} + \frac{n_c}{h_b}$

Proposed by Bogdan Fuștei-Romania

J.2380 If $a = min(a, b, c), I$ —incenter in acute ΔABC then:

$$\frac{1}{r}\sum_{cyc} AI \geq \sqrt{2\left(\frac{n_b}{h_c} + \frac{n_c}{h_b}\right)} + \sqrt{\frac{2(n_b + h_b)}{r_b}} + \sqrt{\frac{2(n_c + h_c)}{r_c}}$$

Proposed by Bogdan Fuștei-Romania

J.2381 TRUE OR FALSE: If I —incenter in ΔABC then holds:

$$\frac{IA^2}{BC} + \frac{IB^2}{CA} + \frac{IC^2}{AB} \leq R\sqrt{3}$$

Proposed by George Apostolopoulos-Greece

J.2382 If in ΔABC holds $r+2R=s$ then

$$\sum_{cyc}\frac{m_a^2}{\sin^2\frac{A}{2}}\cdot\sum_{cyc}\frac{m_a^2}{\cos^2\frac{A}{2}}\geq 324R^2r^2+\prod_{cyc}(a-2m_a)$$

Proposed by Daniel Sitaru-Romania

J.2383 In ΔABC the following relationship holds:

$$2(m_a-R)\geq(2R+r-r_a)\cos(B-C)$$

Proposed by Bogdan Fuștei-Romania

J.2384 In ΔABC the following relationship holds:

$$2+\frac{1}{\sin^2\omega}\geq\frac{2}{3}\left(\sum_{cyc}\frac{m_a}{h_a}\right)^2+\max\left\{\left(\frac{m_a}{h_a}-\frac{m_b}{h_b}\right)^2,\left(\frac{m_b}{h_b}-\frac{m_c}{h_c}\right)^2,\left(\frac{m_c}{h_c}-\frac{m_a}{h_a}\right)^2\right\}$$

Proposed by Bogdan Fuștei-Romania

J.2385 In ΔABC the following relationship holds:

$$\frac{a}{b}\sqrt{\cot\frac{A}{2}}+\frac{b}{c}\sqrt{\cot\frac{B}{2}}+\frac{c}{a}\sqrt{\cot\frac{C}{2}}\leq 3\sqrt[4]{3}\cdot\frac{R}{2r}$$

Proposed by George Apostolopoulos-Greece

J.2386 In ΔABC the following relationship holds:

$$\left(\sum_{cyc}a^2m_b\right)\left(\sum_{cyc}a^2m_c\right)\geq\left(\sum_{cyc}m_am_b\right)\left(\sum_{cyc}a^2b^2\right)$$

Proposed by Daniel Sitaru-Romania

J.2387 In ΔABC the following relationship holds:

$$\frac{m_a^6h_b^2}{m_b^4h_a^3}+\frac{m_b^6h_c^2}{m_c^4h_b^3}+\frac{m_c^6h_a^2}{m_a^4h_c^3}\geq\frac{9R}{2}$$

Proposed by Daniel Sitaru, Claudia Nănuți -Romania

J.2388 In ΔABC the following relationship holds:

$$\left(\frac{m_a^2}{bc\cdot\cos\frac{B}{2}}\right)^2+\left(\frac{m_b^2}{ca\cdot\cos\frac{C}{2}}\right)^2+\left(\frac{m_c^2}{ab\cdot\cos\frac{A}{2}}\right)^2\geq 2+\frac{r}{2R}$$

Proposed by Daniel Sitaru, Claudia Nănuți -Romania

J.2389 AD, BE, CF —internal bisectors in ΔABC. Prove that:

$$\frac{EF}{BE+CF}+\frac{FD}{CF+AD}+\frac{DE}{AD+BE}\le\frac{\sqrt{3}R}{4r}$$

Proposed by George Apostolopoulos-Greece

J.2390 If $a,b,c,d\in[1,2]$ then:

$$(a+b+c+d+e)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\right)\le 28$$

When equality holds?

Proposed by Daniel Sitaru, Claudia Nănuți -Romania

J.2391

$$\sum_{cyc}\sqrt{a}\left(\sqrt{a+|b-c|}-\sqrt{a-|b-c|}\right)\ge 2(\max(a,b,c)-\min(a,b,c))$$

Proposed by Hesen Memmedov-Azerbaijan

J.2392 If $1\le x\le 2\le y\le 3\le z$ then:

$$\frac{2}{z}+\frac{2z}{3}\ge\frac{x}{2}+\frac{1}{x}+\frac{y}{6}+\frac{1}{y}$$

Proposed by Daniel Sitaru, Claudia Nănuți -Romania

J.2393 In any acute or right triangle ABC holds:

$$\frac{2m_a}{r_a}\ge\frac{n_b}{h_c}+\frac{n_c}{h_b}$$

Proposed by Bogdan Fuștei-Romania

J.2394 In ΔABC holds:

$$\frac{n_a}{h_a}+\frac{n_b}{h_b}+\frac{n_c}{h_c}+1\le\frac{2R}{r}$$

Proposed by Bogdan Fuștei-Romania

J.2395 In ΔABC the following relationship holds:

$$\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\left(\sqrt{\frac{2(a+c)-b}{2(a+b)-c}}+\sqrt{\frac{2(a+b)-c}{2(a+c)-b}}\right)\le\sqrt{\frac{2R}{r}}$$

Proposed by Bogdan Fuștei-Romania

J.2396 If I −incenter then in ΔABC holds:

$$2\sqrt{3}r \le \sqrt[3]{a \cdot AI^2 + b \cdot BI^2 + c \cdot CI^2} \le \sqrt{3}R$$

Proposed by George Apostolopoulos-Greece

J.2397 In acute ΔABC holds:

$$\frac{a^2+\lambda bc}{b^2+c^2}+\frac{b^2+\lambda ca}{c^2+a^2}+\frac{c^2+\lambda ab}{a^2+b^2} \ge \frac{6r^2}{R^2}\cdot(1+\lambda),\quad \lambda>0$$

Proposed by Mehmet Șahin-Turkiye

J.2398 In ΔABC holds:

$$2\sum_{cyc}\left(\frac{m_b}{b}+\frac{m_c}{c}-\frac{n_a}{h_a}\right)^{-1} \ge \frac{s}{r}+\sum_{cyc}\frac{n_a}{r_a}$$

Proposed by Bogdan Fuștei-Romania

J.2399 In acute ΔABC holds:

$$\frac{b^2+c^2}{a^2}cosA+\frac{c^2+a^2}{b^2}cosB+\frac{a^2+b^2}{c^2}cosC \ge 24\left(\frac{r}{R}\right)^2-3$$

Proposed by Mehmet Șahin-Turkiye

J.2400 Solve for real numbers: $(2^x+3^x)\cdot\sqrt{6^{1-x}}=5$

Proposed by Daniel Sitaru-Romania

J.2401 If $a \in \mathbb{R}_+, m \in [1,\infty)$ and $b,c,d,x,y,z \in \mathbb{R}_+^*, X=x+y+z, cX \ge d\max\{x,y,z\}$, then:

$$\frac{aX+bx}{(cX-dx)^m}+\frac{aX+by}{(cX-dy)^m}+\frac{aX+bz}{(cX-dz)^m} \ge \frac{3^m(3a+b)}{(3c-d)^m X^{m-1}}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

J.2402 If $m \ge 0$ then in ΔABC holds:

$$\frac{a^2\sin^{2m}A}{(\sin B\sin C)^m}+\frac{b^2\sin^{2m}B}{(\sin C\sin A)^m}+\frac{c^2\sin^{2m}C}{(\sin A\sin B)^m} \ge 36r^2$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

J.2403 If $m \in [0,\infty), x,y,z,t \in (0,\infty)$, then in any ΔABC, holds:

$$\sum_{cyc}\frac{\left(xm_a^2+ym_b^2\right)^{m+1}}{(zh_a^2+th_b^2)^m} \ge \frac{(x+y)^{m+1}}{(z+t)^m}3\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

J.2404 Prove that the number $\overline{11}_x \cdot \overline{12}_x \cdot \overline{13}_x \cdot \overline{14}_x \cdot \overline{15}_x \cdot \overline{17}_x \cdot \overline{18}_x \cdot \overline{19}_x$ cannot be perfect cube for any base of numeration x. ***Proposed by Neculai Stanciu-Romania***

J.2405 Prove this cryptarithm: $ACDEA \times BCDEB \leq ACDEB \times BCDEA$

Proposed by Neculai Stanciu-Romania

J.2406 Prove that in acute triangle ABC holds:

$$\sum_{cyc}\left[\left(\frac{\sin A \sin B}{\sin C}\right)^2 + \left(\frac{\cos A \cos B}{\cos C}\right)^2\right] \geq 3$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

J.2407 Solve for real numbers:

$$\begin{cases} \dfrac{2^{y-x+1}+1}{4^{y-x}+1} = 5^{y-x+1} \\ y^5 + 2x - 1 = \log\left(\dfrac{1}{y^2-2x}\right) \end{cases}$$

Proposed by Neculai Stanciu-Romania

J.2408 If $x, y, z > 0$, then prove that:

$$\sum_{cyc}\frac{x}{\lambda x + y + z} \leq \frac{3}{\lambda + 2}, (\forall)\lambda \geq 1$$

Proposed by Neculai Stanciu, George Florin Șerban-Romania

J.2409 Determine all triplets $(x, y, z) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ which satisfy:

$$\frac{x(y+1)}{x-1} = \frac{y(z+1)}{y-1} + \frac{z(x+1)}{z-1} = 6$$

Proposed by Neculai Stanciu-Romania

J.2410 For $x \in [-1,1]$, prove that:

$$\left|\frac{\left(x+\sqrt{x^2-1}\right)^{n+1} + \left(x-\sqrt{x^2-1}\right)^{n+1} + \left(x+\sqrt{x^2-1}\right)^{n-1} + \left(x-\sqrt{x^2-1}\right)^{n-1}}{\left(x+\sqrt{x^2-1}\right)^{n} + \left(x-\sqrt{x^2-1}\right)^{n}}\right| \leq 2$$

Proposed by Neculai Stanciu-Romania

J.2411 If $ABCDA'B'C'D'$ is a rectangular parallelepiped with $AB = a, BC = b, AA' = c$ and O is the center of face $ABCD$ and O' is the center of face $BCC'B'$, then compute the distance from the lines $B'O$ and BO'.

Proposed by Neculai Stanciu-Romania

J.2412 Prove that 5 divide $n(4n^2+1)(6n^2+1)$, for any natural number n.

Proposed by Neculai Stanciu-Romania

J.2413 If $M\in Int(\Delta ABC), d_a=d(M,BC), d_b=d(M,CA), d_c=d(M,AB), F_a=[MBC]$, $F_b=[MCA], F_c=[MAB]$, then:

$$\frac{a^5}{d_a\cdot F_a}+\frac{b^5}{d_b\cdot F_b}+\frac{c^5}{d_c\cdot F_c}\ge 96\sqrt{3}\cdot F$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți -Romania

J.2414 If $m\ge 0$ and $x,y,z>0$ then in ΔABC holds:

$$\left(a^{m+1}+b^{m+1}+c^{m+1}\right)\left(\frac{1}{\left(ax+y\sqrt{bc}\right)^m}+\frac{1}{\left(bx+y\sqrt{ca}\right)^m}+\frac{1}{\left(cx+y\sqrt{ab}\right)^m}\right)\ge\frac{6\sqrt[4]{27}}{(x+y)^m}\cdot\sqrt{F}$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți -Romania

J.2415 If $x,y>0$ then in ΔABC holds:

$$\left(a^4+b^4+c^4\right)\left(\frac{1}{a^2x+bcy}+\frac{1}{b^2x+cay}+\frac{1}{c^2x+aby}\right)\ge\frac{12\sqrt{3}}{x+y}\cdot F$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

J.2416 If $m\ge 0$ and in $\Delta ABC, A_1\in BC, B_1\in CA, C_1\in AB$ such that $BA_1=mA_1C, CB_1=mB_1A$, $AC_1=mC_1B$, then:

$$\frac{a\cdot A_1B_1^2}{h_b}+\frac{b\cdot B_1C_1^2}{h_c}+\frac{c\cdot C_1A_1^2}{h_a}\ge\frac{8m}{(m+1)^2}\cdot F$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

J.2417 If $x\ge 0$ and in $\Delta ABC, A_1\in BC, B_1\in CA, C_1\in AB$ such that $BA_1=xA_1C, CB_1=xB_1A$, $AC_1=xC_1B$, then:

$$[A_1B_1C_1]=\frac{x^2-x+1}{(x+1)^2}\cdot F\ge\frac{x}{(x+1)^2}\cdot F$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

J.2418 If $t,u,x,y,z>0$ then:

$$\left(\left(\frac{x}{y+z}+\frac{y}{z+x}\right)^2+t^2\right)\left(\left(\frac{z}{x+y}\right)^2+u^2\right)\ge\left(\frac{uz}{y+z}+\frac{uy}{z+x}+\frac{tz}{x+y}\right)^2$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

J.2419 If $m, n, p > 0$ then in ΔABC holds:

$$\frac{w_a^x \cdot a^{x+y} \cdot b^z}{h_b^z} + \frac{w_b^x \cdot b^{x+y} \cdot c^z}{h_c^z} + \frac{w_a^x \cdot c^{x+y} \cdot a^z}{h_a^z} \ge 2^{x+2z} \cdot \left(\sqrt{F}\right)^{2x+y} \cdot \left(\sqrt[4]{3}\right)^{4-y-2z}$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți -Romania

J.2420 If $a, b, c > 0, a + b + c = 3$ then:

$$\sum_{cyc} \frac{1}{a^3 + b^3} + 3 \sum_{cyc} \frac{1}{ab(a+b)} \ge 6$$

Proposed by Daniel Sitaru, Claudia Nănuți -Romania

J.2421 If $0 \le a, b, c \le 2, a^2 + b^2 + c^2 = 6$ then find min$P$:

$$P = \sqrt{a^2 + b^2} + \sqrt{b^2 + c^2} + \sqrt{c^2 + a^2}$$

Proposed by Tran Quoc Thinh-Vietnam

J.2422 Solve for real numbers

$$\begin{cases} x^x = 2^{x+4} \\ z^2 + \frac{4z}{x} + log_y z = 91 \\ y^y = 3^{y+9} \end{cases}$$

Proposed by Daniel Sitaru-Romania

J.2423 In ΔABC the following relationship holds:

$$\frac{r_a}{b+c}\sqrt{sinA} + \frac{r_b}{c+a}\sqrt{sinB} + \frac{r_c}{a+b}\sqrt{sinC} \le \frac{9}{8}\sqrt{\frac{9R^2 - 24r^2}{2}}$$

Proposed by Mehmet Șahin-Turkiye

J.2424 Find all $(m, n, p) \in \mathbb{N}X\mathbb{N}X\mathbb{N}$ such that:

$$\begin{cases} m + n + p + mn + np + pm + mnp = 35 \\ (p+2) \cdot p^p = (p+2)^p \end{cases}$$

Proposed by Daniel Sitaru-Romania

J.2425 If $a, b, c > 0, abc = 2$ then: $\frac{a}{a^2+ab+5} + \frac{b}{b^2+2bc+3} + \frac{2c}{c^2+2ca+8} \le \frac{1}{2}$

Proposed by Tran Quoc Thinh-Vietnam

J.2426 If $a, b, c \ge 0, a + b + c = 2$ then: $a^2b^2 + b^2c^2 + c^2a^2 \le 1$

Proposed by Tran Quoc Thinh-Vietnam

J.2427 If $x, y, z > 0, x + y + z = 3$ then:

$$2\sum_{cyc} x\left(y + \frac{1}{y} + \frac{1}{y^3}\right) \geq 3\sum_{cyc} \frac{x}{y^2} + 9$$

Proposed by Daniel Sitaru-Romania

J.2428 If $a, b, c \geq 0, a + b + c = 2$ then: $a^2b^2 + b^2c^2 + c^2a^2 + \frac{11}{8}abc \leq 1$

Proposed by Tran Quoc Thinh-Vietnam

J.2429 In ΔABC the following relationship holds:

$$\frac{2}{3}\left(\frac{m_a}{h_a} + \frac{m_b}{h_b}\right) \cdot \frac{\sqrt{a^2 + b^2 + c^2}}{R} \leq \frac{h_a + h_b - h_c}{h_c} + \frac{2n_c}{\sqrt{4r^2 + (a - b)^2}}$$

Proposed by Bogdan Fuștei-Romania

J.2430 In ΔABC the following relationship holds:

$$(sinA + 2sinB)^4 + (sinB + 2sinC)^4 + (sinC + 2sinA)^4 \leq \frac{3^7}{2^3}\left(1 - \frac{r}{R}\right)$$

Proposed by Marian Ursărescu-Romania

J.2431 In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{1}{cot\left(\frac{\pi + B}{8}\right) + cot\left(\frac{\pi + C}{8}\right)} \leq \frac{\sqrt{3}}{2}$$

Proposed by Daniel Sitaru, Claudia Nănuți-Romania

J.2432 In $\Delta ABC, BA', CB', AC'$ −external bisectors, $CB' = CB, BA' = BA$,

$CB' = CB, BC' = a', CA' = b', AB' = c', R_1, R_2, R_3, R$ −circumradies of

$\Delta A'BC, \Delta B'CA, \Delta C'AB, \Delta ABC$. Prove that: $Ra'b'c' = R_1R_2R_3(a + b + c)$ and

$$\frac{a'}{R_3} + \frac{b'}{R_1} + \frac{c'}{R_2} \geq \frac{s}{2R}$$

Proposed by Mehmet Șahin-Turkiye

J.2433 In ΔABC the following relationship holds:

$$6r \cdot min(a, b, c) \leq 2Rs \leq 3R \cdot max(a, b, c)$$

Proposed by Daniel Sitaru, Claudia Nănuți-Romania

J.2434 If $a, b, c > 0, abc = 1$ then:

$$(a^3+1)(b^3+1)(c^3+1) \geq \left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right)$$

Proposed by Ilir Demiri-Azerbaijan

J.2435 In ΔABC the following relationship holds:

$$\frac{1}{3}min(h_a, h_b, h_c) \leq r \leq \frac{1}{3}max(h_a, h_b, h_c)$$

Proposed by Daniel Sitaru, Maria Carina Viespescu-Romania

J.2436 In ΔABC the following relationship holds:

$$2\sum_{cyc}\frac{h_a}{s-n_a} \geq \sum_{cyc}\frac{b+c-a}{a}\left(\frac{n_a}{h_a}+\frac{m_b}{b}+\frac{m_c}{c}\right)$$

Proposed by Bogdan Fuștei-Romania

J.2437 If $a, b, c, d, e, f > 0, a+d = b+e = c+f$ then:

$$a\left(\frac{1}{e}+\frac{1}{f}\right)+b\left(\frac{1}{d}+\frac{1}{f}\right)+c\left(\frac{1}{d}+\frac{1}{e}\right) \leq 2\left(\frac{a}{b}+\frac{b}{e}+\frac{c}{f}\right)$$

Proposed by Daniel Sitaru, Mihai Ionescu-Romania

J.2438 H –orthocenter in $\Delta ABC, A(2,2), B(6,2), C(4,2+2\sqrt{3}), M(8,8)$.

Find MH. If $\overrightarrow{MA} = \vec{x}, \overrightarrow{MB} = \vec{y}, \overrightarrow{MC} = \vec{z}$ then find $\overrightarrow{MH}$ in terms of $\vec{x}, \vec{y}, \vec{z}$.

Proposed by Daniel Sitaru, Claudia Nănuți -Romania

J.2439

$x, y > 0, x+y = 2, a = \left[2log_2\frac{2^{256}}{\binom{256}{128}}\right], [*] - GIF$. Prove that:

$$\sqrt{a+\frac{1}{x^2}}+\sqrt{a+\frac{1}{y^2}} \geq 6$$

Proposed by Daniel Sitaru, Claudia Nănuți -Romania

J.2440 In $\Delta ABC, I$ –incenter, the following relationship holds:

$$\sqrt{\frac{R}{r}}\cdot\sqrt{\frac{h_a}{s-n_a}}\cdot\left(\sqrt{\frac{n_a}{h_a}+\frac{m_b}{b}+\frac{m_c}{c}}\right)^{-1} \geq \frac{AI}{2r}$$

Proposed by Bogdan Fuștei-Romania

J.2441 If I −incenter in ΔABC then holds:

$$\frac{1}{r}\sum_{cyc} AI \geq \sum_{cyc}\sqrt{\frac{h_a}{r_a}}\cdot\sqrt{1+\sum_{cyc}\frac{n_a}{h_a}}$$

Proposed by Bogdan Fuștei-Romania

J.2442 I −incenter in $\Delta ABC, A(2,2), B(6,4), C(4,8), M(8,6)$. Find MI.

Proposed by Daniel Sitaru, Olivia Bercea-Romania

J.2443 Let $a,b,c>0$ such that $a+b+c=3$. Prove that:

$$\sqrt{8a^2+1}+\sqrt{8b^2+1}+\sqrt{8c^2+1}+3\left(\frac{a}{2a+1}+\frac{b}{2b+1}+\frac{c}{2c+1}\right)\leq 12$$

Proposed by Nguyen Van Canh-Vietnam

J.2444 Find a triplet $(a>0, b>0, c>0)$ such that:

$$|ax^2+bx+c|\leq 1, |ax+b|\leq 1, |bx+c|\leq 1, \forall x\in\mathbb{R}$$

Proposed by Nguyen Van Canh-Vietnam

J.2445 If $x,y,z>0, x^2+y^2+z^2\leq 3$ then: $\sum\frac{1}{7-x}\leq\frac{1}{2}$

Proposed by Marin Chirciu – Romania

J.2446 If $m,n\in\mathbb{R}_+^*$, then in any triangle ABC holds:

$$\frac{\tan\frac{A}{2}}{m+n\cdot\cot^2\frac{A}{2}}+\frac{\tan\frac{B}{2}}{m+n\cdot\cot^2\frac{B}{2}}+\frac{\tan\frac{C}{2}}{m+n\cdot\cot^2\frac{C}{2}}\geq\frac{(4R+r)^2r}{\left(np^2+mr(4R+r)\right)r}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

J.2447 If $x,y,z\in\mathbb{R}_+^*$, then in any triangle ABC holds:

$$\frac{\cot^3\frac{A}{2}}{x\tan\frac{A}{2}+y\tan\frac{B}{2}+z\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{\cot^3\frac{B}{2}}{x\tan\frac{B}{2}+y\tan\frac{C}{2}+z\tan\frac{C}{2}\tan\frac{A}{2}}+$$

$$+\frac{\cot^3\frac{C}{2}}{x\tan\frac{C}{2}+y\tan\frac{A}{2}+z\tan\frac{A}{2}\tan\frac{B}{2}}\geq\frac{p^4}{\left((4R+r)^2x+(y-2x)p^2+3zpr\right)r^2}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

J.2448 Let $\Delta ABC, \Delta A'B'C', \Delta A''B''C''$. Prove that:

$$\min\left\{\sum\sqrt{\frac{a}{b+c}},\sum\sqrt{\frac{a'}{b'+c'}},\sum\sqrt{\frac{a''}{b''+c''}}\right\}+\frac{RR'R''}{8rr'r''}\geq$$

$$\geq 1+\max\left\{\sum\sqrt{\frac{a}{a+b}},\sum\sqrt{\frac{a'}{a'+b'}},\sum\sqrt{\frac{a''}{b''+c''}}\right\}$$

Proposed by Nguyen Van Canh-Vietnam

J.2449 Compute without any software:

$$A=\sqrt{2023-\sqrt{2022+\sqrt{2022+\sqrt{2022+\sqrt{2022}}}}}$$

Proposed by Nguyen Van Canh-Vietnam

J.2450 In ΔABC:

$$\sum\frac{1}{w_a^{2n+1}(w_b+w_c)}\geq\frac{1}{18}\left(\frac{4}{3R^2}\right)^{n+1},n\in\mathbb{N}$$

Proposed by Marin Chirciu – Romania

J.2451 In ΔABC the following relationship holds:

$$\sum_{cyc}\left(\frac{n_a^2}{w_a^2}+\frac{4r_a}{r_b+r_c}\right)\geq 1+\frac{4R}{r}$$

Proposed by Bogdan Fuștei-Romania

J.2452 In ΔABC the following relationship holds:

$$\left(\sum_{cyc}\frac{n_a}{h_a}\right)^2\sum_{cyc}(s-n_b)(s-n_c)>4s^2$$

Proposed by Bogdan Fuștei-Romania

J.2453 In ΔABC the following relationship holds:

$$2\left(\frac{n_b}{h_b}+\frac{n_c}{h_c}\right)\geq\frac{cos\frac{A}{2}}{sin\frac{B}{2}sin\frac{C}{2}}$$

Proposed by Bogdan Fuștei-Romania

J.2454 In ΔABC the following relationship holds:

$$192\sqrt{3}R^3<rs^2\prod_{cyc}csc^3\left(\frac{A}{3}\right)$$

Proposed by Daniel Sitaru, Jipescu Ana-Romania

J.2455 In ΔABC the following relationship holds:

$$\sum_{cyc} sin\left(\frac{3A+5\pi}{18}\right) sin\left(\frac{3B+5\pi}{18}\right) \le \frac{9}{4}$$

Proposed by Daniel Sitaru, Carmen Terheci-Romania

J.2456 If $m,n \ge 0, m+n=1$ and $x,y,z>0$, then in ΔABC holds:

$$\frac{x\cdot a^m}{(y+z)h_a^n}+\frac{y\cdot b^m}{(z+x)h_b^n}+\frac{z\cdot c^m}{(x+y)h_c^n} \ge \frac{\sqrt[4]{27}}{2^n}\cdot\left(\sqrt{F}\right)^{1-2n}$$

Proposed by D.M.Bătinețu-Giurgiu, Sorina Tudor-Romania

J.2457 If $m,n \ge 0$, then in ΔABC holds:

$$\frac{m^3a^3+n^3b^3}{ab}+\frac{m^3b^3+n^3c^3}{bc}+\frac{m^3c^3+n^3a^3}{ca} \ge \frac{(m+n)^3s}{2}$$

Proposed by D.M.Bătinețu-Giurgiu, Corina Ionescu-Romania

J.2458 If $t,x,y,z>0$ then in ΔABC holds:

$$\frac{t^4+x^4}{(y+z)^2}\cdot a^8+\frac{t^4+y^4}{(z+x)^2}\cdot b^8+\frac{t^4+z^4}{(x+y)^2}\cdot c^8 \ge \frac{32t^2}{3}\cdot F^4$$

Proposed by D.M.Bătinețu-Giurgiu, Carmen Vlad-Romania

J.2459 If $m,t,u \ge 0, t+u=4(m+1)$ and $x,y,z>0$, then in ΔABC holds:

$$\frac{x^{m+1}a^t}{(y+z)^{m+1}\cdot h_a^u}+\frac{y^{m+1}b^t}{(z+x)^{m+1}\cdot h_b^u}+\frac{z^{m+1}c^t}{(x+y)^{m+1}\cdot h_c^u} \ge \frac{2^{3m-u+3}\left(\sqrt{F}\right)^{t+u}}{3^m}$$

Proposed by D.M.Bătinețu-Giurgiu, Daniela Barbu-Romania

J.2460 In ΔABC the following relationship holds:

$$\frac{6r^2s}{R} \le \sum_{cyc} h_a^2\cdot\tan\frac{A}{2} \le 3rs$$

Proposed by D.M.Bătinețu-Giurgiu, Gigi Zaharia-Romania

J.2461 In acute ΔABC holds:

$$\sum\frac{\sec^2 B+\sec^2 C}{\sec A} \ge 12$$

Proposed by Marin Chirciu – Romania

J.2462 Find all $n\in\mathbb{N}, p\in\mathbb{N}$ such that: $2n^3+n^2+1=p^2$.

Proposed by Kerimov Elsen-Azerbaijan

J.2463 Let $f(x)=ax^2+bx+c, g(x)=cx^2+ax+b\ (a,b,c\in\mathbb{R})$. Find all values of a,b,c such that $f\big(g(x)\big)=x^4-2x^3+4x^2-3x+3, \forall x\in\mathbb{R}$

Proposed by Nguyen Van Canh-Vietnam

J.2464 Let $a,b,c>0$. Prove that:

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\le 9+\frac{2024(a+b+c)^3}{a^2b^2+b^2c^2+c^2a^2}\cdot(|a-b|+|b-c|+|c-a|)$$

Proposed by Nguyen Van Canh-Vietnam

J.2465 In any ΔABC the following relationship holds:

$$\frac{m_a^2(m_a^2+m_bm_c)}{(m_b+m_c)^2}+\frac{m_b^2(m_b^2+m_cm_a)}{(m_c+m_a)^2}+\frac{m_c^2(m_c^2+m_am_b)}{(m_a+m_b)^2}\ge\frac{27r^2}{2}$$

Proposed by Zaza Mzhavandze – Georgia

J.2466 If $a,b,c>0, abc=2$ then:

$$\frac{a}{a^2+ab+5}+\frac{b}{b^2+2bc+3}+\frac{2c}{c^2+2ca+8}\le\frac{1}{2}$$

Proposed by Tran Quoc Thinh-Vietnam

J.2467 In $\Delta ABC, \Delta A'B'C'$ the following relationship holds:

$$\min\left\{\sum\sqrt[3]{\frac{w_a}{w_b+w_c}},\sum\sqrt[3]{\frac{m_a'}{m_b'+m_c'}}\right\}+\frac{R^2R'}{r^2r'}\ge$$

$$\ge 8+\max\left\{\sum\sqrt[3]{\frac{w_a'}{w_b'+w_c'}},\sum\sqrt[3]{\frac{m_a}{m_b+m_c}}\right\}$$

Proposed by Nguyen Van Canh-Vietnam

J.2468 In any ΔABC and $n\in\mathbb{N}$ the following relationship holds:

$$\frac{w_a^n(w_a^2+w_bw_c)}{(w_b+w_c)^2}+\frac{w_b^n(w_b^2+w_cw_a)}{(w_c+w_a)^2}+\frac{w_c^n(w_c^2+w_aw_b)}{(w_a+w_b)^2}\ge\frac{3^{n+1}\cdot r^n}{2}$$

Proposed by Zaza Mzhavanadze – Georgia

J.2469 In ΔABC the following relationship holds:

$$\frac{(4R+r)^2}{r^2s^5}\le\frac{\cot\frac{A}{2}}{h_a^5}+\frac{\cot\frac{B}{2}}{h_b^5}+\frac{\cot\frac{C}{2}}{h_c^5}\le\frac{R^4(4R+r)^2}{16r^6s^5}$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2470 If $x, y \geq 0$, then in ΔABC holds:

$$a^x b^y + b^x c^y + c^x a^y \geq 2^{x+y}\left(\sqrt[4]{3}\right)^{4-x-y}\left(\sqrt{F}\right)^{x+y}$$

Proposed by D.M.Bătinețu-Giurgiu, Mihaela Stăncele-Romania

J.2471 In ΔABC the following relationship holds:

$$\frac{m_a}{h_a} \cdot bc + \frac{m_b}{h_b} \cdot ca + \frac{m_c}{h_c} \cdot ab \geq 4\sqrt{3}F$$

Proposed by D.M.Bătinețu-Giurgiu, Cătălina Stan-Romania

J.2472 If $m \geq 0$ and $x, y, z > 0$, then in ΔABC holds:

$$\frac{x^{m+1}a^4}{(y+z)^{m+1}} + \frac{y^{m+1}b^4}{(z+x)^{m+1}} + \frac{z^{m+1}c^4}{(x+y)^{m+1}} \geq 2^{3-m} \cdot F^2$$

Proposed by D.M.Bătinețu-Giurgiu, Cristian Catană-Romania

J.2473 If $x, y, z > 0$, then in ΔABC holds:

$$\left(\frac{ax}{h_a} + \frac{by}{h_b} + \frac{cz}{h_c}\right)^2 \geq 4(xy + yz + zx)$$

Proposed by D.M.Bătinețu-Giurgiu, Sebastian Ilinca-Romania

J.2474 If $M \in Int(\Delta ABC), x = MA, y = MB, z = MC$ then:

$$\frac{x}{\sqrt{yz}} \cdot a + \frac{y}{\sqrt{zx}} \cdot b + \frac{z}{\sqrt{xy}} \cdot c \geq 2 \cdot \sqrt[4]{27} \cdot \sqrt{F}$$

Proposed by D.M.Bătinețu-Giurgiu, Simona Chiriță-Romania

J.2475 If $t > 0, u \geq 0$, then in ΔABC holds:

$$(ab)^{t(u+1)} + (bc)^{t(u+1)} + (ca)^{t(u+1)} \geq 4^{t(u+1)} \cdot 3^{1-t(u+1)}\left(\sqrt{3}\right)^{t(u+1)} \cdot F^{t(u+1)}$$

Proposed by D.M.Bătinețu-Giurgiu, Ionuț Ivănescu-Romania

J.2476 If $x, y, z > 0$, then in ΔABC holds:

$$\frac{ae^{x^2}}{(y+z)h_a} + \frac{be^{y^2}}{(z+x)h_b} + \frac{ce^{z^2}}{(x+y)h_c} > 2\sqrt{3}$$

Proposed by D.M.Bătinețu-Giurgiu, Rareș Tudorașcu-Romania

J.2477 If $x, y, z > 0$, then in ΔABC holds:

$$\frac{a^2e^{x^2}}{y+z} + \frac{b^2e^{y^2}}{z+x} + \frac{c^2e^{z^2}}{x+y} > 4\sqrt{3} \cdot F$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți -Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

PROBLEMS FOR SENIORS

S.2341 Let $(a_n)_{n\geq 1}, a_n \in (0,\infty)$ such that $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n\cdot\sqrt[n]{n!}} = a > 0$. Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}} - \frac{n^2}{\sqrt[n]{a_n}}\right)$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

S.2342 Solve for real numbers:

$$2e^x = 3 + (1+e^x)log\left(\frac{3+x}{2-x}\right)$$

Proposed by Khaled Abd Imouti-Syria

S.2343 If a,b,c,d —sides in a cyclic quadrilateral then:

$$\left(\prod_{cyc}(a+b+c-d)\right)^2 \leq 4(a^2+c^2)(b^2+d^2)\prod_{cyc}(a+b)$$

Proposed by Daniel Sitaru,Claudia Nănuți-Romania

S.2344 Solve for real numbers:

$$\begin{cases} x^2+y^2+z^2=3 \\ t^2+x+y+z+4t=-3 \\ 2xy+2yz+2zx=-3 \end{cases}$$

Proposed by Daniel Sitaru, Dan Nănuți-Romania

S.2345 If $x, y > 0$ then:

$$\left(1+\frac{x}{y}\right)^{y} \le e^{x} \le \left(1+\frac{x}{y}\right)^{x+y}$$

Proposed by Khaled Abd Imouti-Syria

S.2346 Find:

$$\Omega = \lim_{n\to\infty}\left(1+\sum_{k=1}^{n}\frac{k^3+k^2-2k-1}{\left((k+1)!\right)^2}\right)\cdot\sum_{k=1}^{n}\frac{k^2+k+1}{(k+2)!}$$

Proposed by Daniel Sitaru, Claudia Nănuți-Romania

S.2347 Find:

$$\Omega = \lim_{n\to\infty}\left(\sum_{k=2}^{n}\left[\frac{k+\sqrt{k}}{k}\right]\right)\left(\sum_{k=2}^{n}\left[\frac{k+\sqrt[3]{k}}{k}\right]\right)\left(\sum_{k=2}^{n}\left[\frac{k+\sqrt[5]{k}}{k}\right]\right)\left(\sum_{k=2}^{n}k^2\right)^{-1}$$

Proposed by Daniel Sitaru, Claudia Nănuți-Romania

S.2348 $a, b > 0, a \ne b, f_m(x) = a + (x-a)^m, g_m(x) = b + (x-b)^m$

$$f_m: (a,\infty) \to (a,\infty), g_m: (b,\infty) \to (b,\infty)$$

$G = \{f_m / m \in \mathbb{R}^*\}, H = \{g_m / m \in \mathbb{R}^*\}$. Prove that: $(G,\circ) \cong (H,\circ)$.

Proposed by Daniel Sitaru, Claudia Nănuți -Romania

S.2349 Prove without any software:

$$9\cdot 4^{\sqrt{5}}\cdot 5^{\sqrt{6}}\cdot 6^{\sqrt{7}} < 25\cdot 2^{\sqrt{3}}\cdot 3^{\sqrt{2}}\cdot 6^{\sqrt{5}}\cdot 7^{\sqrt{6}}$$

Proposed by Daniel Sitaru-Romania

S.2350 If $m \ge 0$ and $x, y, z > 0$ then in ΔABC holds:

$$3m + \left(\frac{xa^2}{y+z}\right)^{m+1} + \left(\frac{yb^2}{z+x}\right)^{m+1} + \left(\frac{zc^2}{x+y}\right)^{m+1} \ge 2(m+1)\sqrt{3}\cdot F$$

Proposed by D.M. Bătinețu-Giurgiu, Alina Tigae-Romania

S.2351 If $M \in Int(\Delta ABC)$ such that $x = MA, y = MB, z = MC$ then:

$$\frac{xa}{\sqrt{yz}} + \frac{yb}{\sqrt{zx}} + \frac{zc}{\sqrt{xy}} \ge 2\cdot\sqrt[4]{27}\cdot\sqrt{F}$$

Proposed by D.M. Bătinețu-Giurgiu, Cristina Ene-Romania

S.2352 If $m \geq 0$ and $x, y, z > 0$ then in ΔABC holds:

$$\left(\frac{x}{\sqrt{yz}}\right)^{m} \cdot \frac{a^{m+2}}{h_a^m} + \left(\frac{y}{\sqrt{zx}}\right)^{m+1} \cdot \frac{b^{m+2}}{h_b^m} + \left(\frac{z}{\sqrt{xy}}\right)^{m+1} \cdot \frac{c^{m+2}}{h_c^m} \geq 2^{m+2}\left(\sqrt{3}\right)^{1-m} \cdot F$$

Proposed by D.M. Bătinețu-Giurgiu, Doina Cristina Călina-Romania

S.2353 If $m \geq 0$ and $x, y, z > 0$ then on ΔABC holds:

$$\frac{x^{m+1} \cdot a^{2m+1}}{(y+z)^{m+1} \cdot h_a} + \frac{y^{m+1} \cdot b^{2m+1}}{(z+x)^{m+1} \cdot h_b} + \frac{z^{m+1} \cdot c^{2m+1}}{(x+y)^{m+1} \cdot h_c} \geq 2^m \cdot \left(\sqrt{3}\right)^{1-m} \cdot F^m$$

Proposed by D.M. Bătinețu-Giurgiu, Daniela Dîrnu-Romania

S.2354 If $x, y, z \geq 0, u, v \in \mathbb{R}$ then in $\Delta A_k B_k C_k$ with areas $F_k, k = \overline{1,3}$ holds:

$$\frac{m_{a_1}^u \cdot a_1^{x+u} \cdot b_2^{y-v} \cdot c_3^z}{h_{b_2}^v} + \frac{m_{b_1}^u \cdot b_1^{x+u} \cdot c_2^{y-v} \cdot a_3^z}{h_{c_2}^v} + \frac{m_{c_1}^u \cdot c_1^{x+u} \cdot a_2^{y-v} \cdot b_3^z}{h_{a_2}^v} \geq$$

$$\geq 2^{x+y+z+u-v} \cdot \left(\sqrt[4]{3}\right)^{4-(x+y+z)} \cdot \left(\sqrt{F_1}\right)^{x+2u} \cdot \left(\sqrt{F_2}\right)^{y-2v} \cdot \left(\sqrt{F_3}\right)^{z}$$

Proposed by D.M. Bătinețu-Giurgiu, Gilena Dobrică-Romania

S.2355 If $x, y, z > 0$ then in ΔABC holds: $x\left(\frac{1}{y} + \frac{1}{z}\right) \cdot \frac{a}{h_b} + y\left(\frac{1}{z} + \frac{1}{x}\right) \cdot \frac{b}{h_b} + z\left(\frac{1}{x} + \frac{1}{y}\right) \cdot \frac{c}{h_a} \geq 4\sqrt{3}$

Proposed by D.M. Bătinețu-Giurgiu, Oana Simona Dascălu-Romania

S.2356 If $m, n \geq 0, m \neq n$ and in $\Delta ABC, A_1, A_2 \in (BC), B_1, B_2 \in (CA), C_1, C_2 \in (AB)$ such that

$BA_1 = mB_1C, BA_2 = nA_2C, CB_1 = mB_1A, CB_2 = nB_2A, AC_1 = mC_1B, AC_2 = nC_2B$ then if $a_1 = B_1C_1, b_1 = C_1A_1, c_1 = A_1B_1$ and $a_2 = B_2C_2, b_2 = C_2A_2, c_2 = A_2B_2$ holds:

$$a_1a_2 + b_1b_2 + c_1c_2 \geq \frac{4\sqrt{3} \cdot \sqrt{mn}}{(m+1)(n+1)} \cdot F$$

Proposed by D.M. Bătinețu-Giurgiu, Dorina Goiceanu-Romania

S.2357 If $m \geq 0$ and $x, y, z > 0$ then in ΔABC holds:

$$3m + \left(\frac{xa^2}{y+z}\right)^{m+1} + \left(\frac{yb^2}{z+x}\right)^{m+1} + \left(\frac{zc^2}{x+y}\right)^{m+1} \geq 2(m+1) \cdot r(r_a + r_b + r_c)$$

Proposed by D.M. Bătinețu-Giurgiu, Ramona Nălbaru-Romania

S.2358 If in $\Delta ABC, a, b, c \geq 0$ or $a, b, c \leq 1$ then holds:

$$\sqrt{(a^4 + a^2 + 1)(b^4 + b^2 + 1)} + \sqrt{(b^4 + b^2 + 1)(c^4 + c^2 + 1)} +$$

$$+\sqrt{(c^4 + c^2 + 1)(a^4 + a^2 + 1)} \geq 12\sqrt{3} \cdot F$$

Proposed by D.M. Bătinețu-Giurgiu, Ileana Duma-Romania

S.2359 If $x,y,z>0$ then in ΔABC holds:

$$(x+y)ab+(y+z)bc+(z+x)ca\geq 8\sqrt{xy+yz+zx}\cdot F$$

Proposed by D.M. Bătinețu-Giurgiu, Maria Lavinia Popa-Romania

S.2360 If $m\geq 0, M\in Int(\Delta ABC)$ such that $x=MA, y=MB, z=MC$, then:

$$\frac{x\cdot a^{m+1}}{\sqrt{yz}}+\frac{y\cdot b^{m+1}}{\sqrt{zx}}+\frac{z\cdot c^{m+1}}{\sqrt{xy}}\geq 2^{m+1}\cdot\left(\sqrt[4]{3}\right)^{3-m}\cdot\left(\sqrt{F}\right)^{m+1}$$

Proposed by D.M. Bătinețu-Giurgiu, Sorin Pîrlea-Romania

S.2361 If $M\in Int(\Delta ABC), d_a=d(M,BC), d_c=d(M,CA), d_c=d(M,AB)$ then:

$$\frac{a^5}{d_a^3}+\frac{b^5}{d_b^3}+\frac{c^5}{d_c^3}\geq 108F$$

Proposed by D.M. Bătinețu-Giurgiu, Nicolae Radu-Romania

S.2362 If $M\in Int(\Delta ABC), d_a=d(M,BC), d_b=d(M,CA), d_c=d(M,AB)$ then:

$$\frac{a^2b}{d_b}+\frac{b^2c}{d_c}+\frac{c^2a}{d_a}\geq 24F$$

Proposed by D.M. Bătinețu-Giurgiu, Mioara Mihaela Mirea-Romania

S.2363 If $m\geq 0$ and $M\in Int(\Delta ABC), d_a=d(M,BC), d_b=d(M,CA), d_c=d(M,AB)$ then:

$$\frac{(ab)^{m+2}}{(d_ad_b)^m}+\frac{(bc)^{m+2}}{(d_bd_c)^m}+\frac{(ca)^{m+2}}{(d_cd_a)^m}\geq 3\cdot 4^{m+2}\cdot F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Cătălin Pană-Romania

S.2364 If $x,y,z>0$ then in ΔABC holds: $\frac{x^2+1}{y+z}\cdot a^4+\frac{y^2+1}{z+x}\cdot b^2+\frac{z^2+1}{x+y}\cdot c^4\geq 16F^2$

Proposed by D.M. Bătinețu-Giurgiu, Alecu Orlando-Romania

S.2365 If $f:\mathbb{R}_+^*\times\mathbb{R}_+^*\to\mathbb{R}_+^*$ and $x,y>0$ then:

$$(x^2+2y)\left(y^2+2f(x,y)\right)\left(f(x,y)^2+2x\right)\geq 3\left(x\sqrt{xf(x,y)}+y\sqrt{xy}+f(x,y)\sqrt{f(x,y)}\right)^2$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți -Romania

S.2366 If $t,x,y,z>0$ then:

$$\left(\left(\frac{x}{y+z}+\frac{y}{z+x}\right)^2+t^2\right)\left(\left(\frac{y}{z+x}+\frac{z}{x+y}\right)^2+t^2\right)\left(\left(\frac{z}{x+y}+\frac{x}{y+z}\right)^2+t^2\right)\geq\frac{27}{4}t^4$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți -Romania

S.2367 Solve for real numbers: $x^3(x+1)^3+8x^3+8=12x^2(x+1)$

Proposed by Daniel Sitaru, Luiza Dumitrescu-Romania

S.2368 If $S\in Int[ABCD]$ −rectangle, S −fixed, $M\in(AB),N\in(BC),P\in(CD),Q\in(DA),AM=PC,BN=DQ$ then find the probability that a random point from $Int[ABCD]$ lies in: $[AMSQ]\cup[NSPC]$.

Proposed by Daniel Sitaru, Ileana Stanciu-Romania

S.2369 Solve for real numbers: $x^3+y^3+2=2(x+y)+\log(x^x y^y)$

Proposed by Daniel Sitaru, Lavinia Trincu-Romania

S.2370 Solve for real numbers: $2^{\arctan x}+2^{\text{arccot}\, x}=2^2$

Proposed by Daniel Sitaru, Mihaela Nascu-Romania

S.2371 Find:

$$\Omega=\lim_{n\to\infty}\frac{1}{n^3}\sum_{1\le i<j<k\le n}\sqrt[5]{\frac{ijk}{n^3}}$$

Proposed by Daniel Sitaru, Luiza Cremeneanu-Romania

S.2372 Find without any software:

$$\Omega=512\left(\int_1^2\frac{x\cdot\log x}{4+x^4}dx\right)\left(\int_2^4\frac{x\cdot\log x}{64+x^4}dx\right)\left(\int_3^6\frac{x\cdot\log x}{324+x^4}dx\right)$$

Proposed by Daniel Sitaru, Roxana Vasile-Romania

S.2373 If $0<a\le b,n\in\mathbb{N}^*$ then:

$$\int_a^{\sqrt{ab}}t^2(t^3-a^3)^{n-1}\,dt\le\frac{1}{24n}(b^3+3b^2a+3a^2b-7a^3)^n$$

Proposed by Daniel Sitaru, Elena Grigore-Romania

S.2374 AA',BB',CC' −internal bisectors in ΔABC with I −incenter. Prove that:

$$216\cdot[IBA']\cdot[ICB']\cdot[ICA']\le F^3$$

Proposed by Daniel Sitaru, Meda Iacob-Romania

S.2375 Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(x^2+y^2)=f(xy)+f(x)+f(y),(\forall)x,y\in\mathbb{R}$

Proposed by Nguyen Van Canh-Vietnam

S.2376 Let $n\in\mathbb{N},n\ge1$. Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x^n)=f((x+1)^{n+1})+nx^n,(\forall)x\in\mathbb{R}$$

Proposed by Nguyen Van Canh-Vietnam

S.2377 In ΔABC the following relationship holds:

$$\sqrt{\frac{a^{4n}+b^{4n}+c^{4n}}{a^{2n}+b^{2n}+c^{2n}}}\cdot\sqrt{\frac{A^{4n}+B^{4n}+C^{4n}}{A^{2n}+B^{2n}+C^{2n}}}\cdot\sqrt{\frac{r_a^{4n}+r_b^{4n}+r_c^{4n}}{r_a^{2n}+r_b^{2n}+r_c^{2n}}}\geq[2\pi s(4R+r)]^n 3^{3n}$$

Proposed by Radu Diaconu-Romania

S.2378 O –circumcentre, G –centroid in ΔABC. Prove that:

$$\frac{a^2+b^2+c^2}{(a-b)^2+(b-c)^2+(c-a)^2+4\sqrt{3}F}>\frac{OG}{R}$$

Proposed by Khaled Abd Imouti-Syria

S.2379 Solve for real numbers: $\left(x^2+\sqrt{3+x}\right)\left(x+\sqrt{3+x^2}\right)=4$

Proposed by Sakthi Vel-India

S.2380 If $a,b,c>0$ then:

$$2^{a-b}+2^{b-c}+2^{c-a}\geq\frac{2^a+2^b+2^c}{\sqrt[3]{2^{a+b+c}}}$$

Proposed by Daniel Sitaru, Simona Radu-Romania

S.2381 If $a\in\left(0,\frac{\pi}{2}\right)$ then:

$$\frac{sin2a}{sina+cosa}+\frac{1}{\sqrt{2}}\geq\sqrt{\frac{sin2a}{2}+\frac{sina+cosa}{2}}$$

Proposed by Daniel Sitaru, Mihaela Dăianu-Romania

S.2382 If in $\Delta ABC, a\neq b\neq c\neq a$ then:

$$\sum_{cyc}\frac{b+c}{b+c-a}>\frac{9}{w_a+w_b+w_c}$$

Proposed by Daniel Sitaru, Elena Alexie-Romania

S.2383 If $x,y,z,a,b>0$ then:

$$\left(x^{\sqrt{ab}}+y^{\sqrt{ab}}+z^{\sqrt{ab}}\right)^{\frac{1}{\sqrt{ab}}}\geq\left(\sqrt{x^{a+b}}+\sqrt{y^{a+b}}+\sqrt{z^{a+b}}\right)^{\frac{2}{a+b}}$$

Proposed by Daniel Sitaru, Iulia Sanda-Romania

S.2384 $ABCD$ –convex quadrilateral, $AB=a, BC=b, CD=c, DA=d, AC=e$,

$BD=f, AC\perp BD$. Prove that:

$$\frac{a^3b}{c^2}+\frac{b^3c}{d^2}+\frac{c^3d}{a^2}+\frac{d^3a}{b^2}\geq\frac{8e^2f^2}{(e+f)^2}$$

Proposed by Daniel Sitaru, Claudia Nănuți-Romania

S.2385 If I −incenter in ΔABC then holds:

$$\sum_{cyc}\sqrt{\frac{a}{b+c-a}}=\sqrt{\frac{1}{r}\sum_{cyc}AI+\frac{2R}{r}-1}$$

Proposed by Bogdan Fuștei-Romania

S.2386 If $a_1,a_2,\dots,a_n>0, n\in\mathbb{N}^*$ then:

$$\pi log(a_1a_2\cdot\ldots\cdot a_n)\leq nlog\left(\frac{a_1^\pi+a_2^\pi+\cdots+a_n^\pi}{n}\right)$$

Proposed by Khaled Abd Imouti-Syria

S.2387 Prove that:

$$\left|\sum_{k=0}^{n}(-1)^{[\sqrt{k}]}\right|\leq\sqrt{n+1}, n\in\mathbb{N}, [*]-GIF$$

Proposed by Khaled Abd Imouti-Syria

S.2388 In ΔABC the following relationship holds:

$$\frac{\sqrt{r_ar_b}+\sqrt{r_br_c}+\sqrt{r_cr_a}}{h_a+h_b+h_c}\cdot\sqrt{\frac{r_a+r_b+r_c}{w_a+w_b+w_c}}\leq\sqrt{\frac{R}{2r}}$$

Proposed by Bogdan Fuștei-Romania

S.2389 In ΔABC the following relationship holds:

$$2\sum_{cyc}\sqrt{a}\leq 2\sum_{cyc}\frac{a}{\sqrt{b}+\sqrt{c}-\sqrt{a}}\leq\frac{abc}{8r^2(4R+r)^2}\cdot\left(\sum_{cyc}\sqrt{a}\right)^3$$

Proposed by Daniel Sitaru, Cătălin Nicola-Romania

S.2390 $a,b>0, a\neq b, f_m(x)=a+(x-a)^m, g_m(x)=b+(x-b)^m$

$$f_m:(0,\infty)\to\mathbb{R}, g_m:(0,\infty)\to\mathbb{R}, G=\{f_m|m\in\mathbb{R}^*\}, H=\{g_m|m\in\mathbb{R}^*\}$$

Prove that: $(G,\circ)\cong(H,\circ)$.

Proposed by Daniel Sitaru, Claudiu Ciulcu-Romania

S.2391 In ΔABC the following relationship holds:

$$(2a+b)(2c+b)+(2b+c)(2a+c)+(2c+a)(2c+b)\le 81R^2$$

Proposed by Daniel Sitaru, Elena Nedelcu-Romania

S.2392

$$\Omega_1=1-\sum_{cyc}\frac{x}{2x+z},\qquad \Omega_2=1-\sum_{cyc}\frac{x}{x+2y},\qquad x,y>0$$

Make a choice:

$$A.\Omega_1\,\Omega_2\ge 0\qquad B.\Omega_1\,\Omega_2\le 0$$

Proposed by Daniel Sitaru, Camelia Dană-Romania

S.2393 Find:

$$\Omega=\lim_{n\to\infty}\frac{1}{n^5}\left(4!+\frac{5!}{1!}+\frac{6!}{2!}+\cdots+\frac{(n+4)!}{n!}\right)$$

Proposed by Daniel Sitaru, Mihaela Duță-Romania

S.2394 In ΔABC the following relationship holds:

$$\frac{a^3}{n_b+n_c}+\frac{b^3}{n_c+n_a}+\frac{c^3}{n_a+n_b}\ge\frac{48\sqrt{3}r^3}{3R-2r}$$

Proposed by Mehmet Șahin-Turkiye

S.2395 If $a,b,c>0$ then:

$$\left(\frac{a+b+c}{3\sqrt[3]{abc}}+\frac{3\sqrt[9]{abc}}{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}\right)\left(\frac{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{3\sqrt[9]{abc}}+\frac{3\sqrt[27]{abc}}{\sqrt[9]{a}+\sqrt[9]{b}+\sqrt[9]{c}}\right)\ge 4$$

Proposed by Daniel Sitaru, Dan Mitricoiu-Romania

S.2396 If $a,b,c,d,e,f>0,n>1$ then:

$$12\cdot\sqrt[3]{abcdef}\le\left(\sum_{cyc(a,b,c)}(a+b)^n\right)^{\frac{1}{n}}\cdot\left(\sum_{cyc(d,e,f)}(d+e)^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}$$

Proposed by Khaled Abd Imouti-Syria

S.2397 In ΔABC the following relationship holds:

$$\frac{a}{sin\frac{A}{3}}+\frac{b}{sin\frac{B}{3}}+\frac{c}{sin\frac{C}{3}}<18R$$

Proposed by Daniel Sitaru, Laura Zaharia-Romania

S.2398 $x,y,z>0, n\in\mathbb{N}, n\geq 3, [*]-GIF$

$$x^n=n+[x],\qquad y^{n+1}=n+1+[y],\qquad z^{n+2}=n+2+[z]$$

In these conditions: $A. x\leq y\leq z,\quad B. x\geq y\geq z$

Proposed by Daniel Sitaru, Claudia Nănuți-Romania

S.2399 In ΔABC the following relationship holds:

$$R^4\geq\frac{(a^2+b^2+c^2)^2}{81}$$

Proposed by Daniel Sitaru, Claudia Nănuți-Romania

S.2400 If $I-$incenter in ΔABC then:

$$IA^4+IB^4+IC^4\leq\frac{(a^2+b^2+c^2)^2}{27}$$

Proposed by Daniel Sitaru, Dan Nănuți-Romania

S.2401 If $\lambda,\mu>0$ $then\ in$ ΔABC the following relationship holds:

$$4\sqrt{3}(\lambda+\mu)\cdot\frac{r}{R}\leq\sum_{cyc}\frac{\lambda a+\mu b}{r_c}\leq\frac{3R(\lambda+\mu)}{F}\cdot\sqrt{9R^2-s^2}$$

Proposed by Mehmet Șahin-Turkiye

S.2402 $K-$Lemoine's point, $I-$incenter in ΔABC. Prove that:

$$KA^4+KB^4+KC^4\geq IA^4+IB^4+IC^4$$

Proposed by Daniel Sitaru, Claudia Nănuți -Romania

S.2403 Prove that: $logx-sinx<\frac{x^2}{2}-\frac{x}{2}+\sqrt{x},\ \ x\in[1,\infty)$

Proposed by Khaled Abd Imouti-Syria

S.2404 If $m\geq 0$ then in ΔABC holds:

$$\frac{a^m b}{h_a\cdot h_a^m}+\frac{b^m c}{h_b\cdot h_c^m}+\frac{c^m a}{h_c\cdot h_a^m}\geq 2^{m+1}\cdot\left(\sqrt{3}\right)^{1-m}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

S.2405 If $m\geq 0$ and $x,y,z>0$ then in ΔABC holds:

$$\frac{x^{m+1}\cdot a^{m+2}}{(y+z)^{m+1}\cdot h_a^m}+\frac{y^{m+1}\cdot b^{m+2}}{(z+x)^{m+1}\cdot h_b^m}+\frac{z^{m+1}\cdot c^{m+2}}{(x+y)^{m+1}\cdot h_c^m}\geq 2\cdot\left(\sqrt{3}\right)^{1-m}\cdot F$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

S.2406 In ΔABC the following relationship holds:

$$\frac{a^2\cdot m_b}{\sqrt{m_a m_a}}+\frac{b^2\cdot m_c}{\sqrt{m_a m_b}}+\frac{c^2\cdot m_a}{\sqrt{m_b m_c}}\ge 4\sqrt{3}\cdot F$$

Proposed by D.M. Bătinețu-Giurgiu, Monica Velea-Romania

S.2407 In $\Delta ABC, g_a, g_b, g_c$ —Gergonne's cevians, the following relationship holds:

$$\frac{a^3\cdot g_a^2}{\sqrt{g_b g_c}}+\frac{b^3\cdot g_b^2}{\sqrt{g_c g_a}}+\frac{c^3\cdot g_c^2}{\sqrt{g_a g_b}}\ge 8\sqrt{3}\cdot F$$

Proposed by D.M. Bătinețu-Giurgiu, Marin Laura Nicoleta-Romania

S.2408 If $x,y,z>0$ then in ΔABC holds:

$$\frac{x}{h_a h_b\sqrt{yz}}+\frac{y}{h_b h_c\sqrt{zx}}+\frac{z}{h_c h_a\sqrt{xy}}\ge\frac{\sqrt{3}}{F}$$

Proposed by D.M. Bătinețu-Giurgiu, Dan Grigorie-Romania

S.2409 If $x\ge 0$ and in $\Delta ABC, A_1\in(BC), B_1\in(CA), C_1\in(AB)$ such that $AA_1=xA_1C$,

$CB_1=xB_1A, AC_1=xC_1B$ and $a_1=B_1C_1, b_1=C_1A_1, c_1=A_1B_1$, then:

$$a_1b_1+b_1c_1+c_1a_1\ge\frac{4x\cdot\sqrt{3}}{(x+1)^2}\cdot F$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

S.2410 In ΔABC the following relationship holds:

$$\frac{r_a}{\sqrt{r_b r_c}}\cdot bc+\frac{r_b}{\sqrt{r_c r_a}}\cdot ca+\frac{r_c}{\sqrt{r_a r_b}}\cdot ab\ge 4\sqrt{3}\cdot F$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți -Romania

S.2411 If $m\ge 0$ and $x,y>0$ then in ΔABC holds:

$$\frac{a^{m+1}\cdot b}{(ax+by)^m\cdot h_b^m}+\frac{b^{m+1}\cdot c}{(bx+cy)^m\cdot h_c^m}+\frac{c^{m+1}\cdot a}{(cx+ay)^m\cdot h_a^m}\ge\frac{2\left(\sqrt{3}\right)^{m+1}\cdot r}{(x+y)^m\cdot s^{m+1}}$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți -Romania

S.2412 If $m\ge 0$ and $M\in Int(\Delta ABC), d_a=d(M,BC), d_b=d(M,CA), d_c=d(M,AB)$ then:

$$\frac{ab^m}{d_a^m\cdot h_b}+\frac{bc^m}{d_b^m\cdot h_c}+\frac{ca^m}{d_c^m\cdot h_a}\ge\left(2\sqrt{3}\right)^{m+1}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

S.2413 If $m \geq 0, u, v > 0$ and $X \in Int(\Delta ABC), x_a = d(M, BC), x_b = d(M, CA), x_c = d(M, AB)$ then:

$$\frac{a}{(u \cdot h_a + v \cdot x_a)^m} + \frac{b}{(u \cdot h_b + v \cdot_b)^m} + \frac{c}{(u \cdot h_c + v \cdot x_c)^m} \geq \frac{6\sqrt{3}}{(3u + v)r^{m-1}}$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți -Romania

S.2414 If $t > 0$ then in ΔABC holds:

$$\left(\left(\frac{a^3}{bR + cr}\right)^2 + t^2\right)\left(\left(\frac{b^3}{cR + ar}\right)^2 + t^2\right)\left(\left(\frac{c^3}{aR + br}\right)^2 + t^2\right) \geq \frac{36t^4}{(R + r)^2} \cdot F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți -Romania

S.2415 If $x, y > 0$ and in ΔABC, $x \cdot s \geq y \cdot \max\{a, b, c\}$, then:

$$\frac{a^3}{xs - ya} + \frac{b^3}{xs - yb} + \frac{c^3}{xs - yc} \geq \frac{4\sqrt{3}}{3x - y} \cdot F$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

S.2416 If $u, v > 0$ and $X \in Int(\Delta ABC), x_a = d(M, BC), x_b = d(M, CA), x_c = d(M, AB)$ then:

$$\frac{a}{u \cdot h_a + v \cdot x_a} + \frac{b}{u \cdot h_b + v \cdot x_b} + \frac{c}{u \cdot h_c + v \cdot x_c} \geq \frac{6\sqrt{3}}{3u + v}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

S.2417 If $M \in Int(\Delta ABC), d_a = d(M, BC), d_b = d(M, CA), d_c = d(M, AB)$ then:

$$\frac{a^2 b}{d_b + h_b} + \frac{b^2 c}{d_c + h_c} + \frac{c^2 a}{d_a + h_a} \geq 6F$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

S.2418 If $x > 0$ then in ΔABC holds:

$$\left(\frac{m_a^{2x+4}}{(m_b R + m_c r)^{2x}} + 1\right)\left(\frac{m_b^{2x+4}}{(m_c R + m_a r)^{2x}} + 1\right)\left(\frac{m_c^{2x+4}}{(m_a R + m_b r)^{2x}} + 1\right) \geq \frac{81}{4(R + r)^{2x}} \cdot F^2$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

S.2419 If $x \geq 0$ then in ΔABC holds:

$$\left(\frac{m_a^{2x+4}}{(m_b R + m_c r)^{2x}} + 2\right)\left(\frac{m_b^{2x+4}}{(m_c R + m_a r)^{2x}} + 2\right)\left(\frac{m_c^{2x+4}}{(m_a R + m_b r)^{2x}} + 2\right) \geq \frac{81}{(R + r)^{2x}} \cdot F^2$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

S.2420 If $M \in Int(\Delta ABC), x = [MBC], y = [MCA], z = [MAB]$ then:

$$(x^{1+x}+y^{1+x}+z^{1+x})+(x^{1+y}+y^{1+y}+z^{1+y})+(x^{1+z}+y^{1+z}+z^{1+z}) \geq 3^{1-\frac{F}{3}} \cdot F^{1+\frac{F}{3}}$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți -Romania

S.2421 Let $a, b, c > 0$. Prove that:

$$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b} \leq \frac{3}{2}+\frac{2022^2(a^2+b^2+c^2)}{a^2b+b^2c+c^2a}(|a-b|+|b-c|+|c-a|)$$

Proposed by Nguyen Van Canh-Vietnam

S.2422 Let $n \geq 2$. In ΔABC, prove that:

$$\sum \sqrt[3]{\frac{a}{2b+3c}}+\left(\frac{R}{2r}\right)^n \geq 1+\sum \sqrt[3]{\frac{a}{2c+3b}}$$

Proposed by Nguyen Van Canh-Vietnam

S.2423 In ΔABC:

$$\sqrt[3]{\frac{h_a^4}{h_b^2+h_c(h_a+h_b)}}+\sqrt[3]{\frac{h_b^4}{h_c^2+h_a(h_b+h_c)}}+\sqrt[3]{\frac{h_c^4}{h_a^2+h_b(h_c+h_a)}} \geq 3\sqrt[3]{3r^2}$$

Proposed by Marin Chirciu – Romania

S.2424 In ΔABC the following relationship holds:

$$\frac{\left(m_a^5+2m_a^2m_b^2(m_a+m_b)+m_b^5\right)^3}{m_a^4+2m_am_b(m_a^2+m_b^2)+m_b^4}+\frac{\left(m_b^5+2m_b^2m_c^2(m_b+m_c)+m_c^5\right)^3}{m_b^4+2m_bm_c(m_b^2+m_c^2)+m_c^4}+$$

$$+\frac{(m_c^5+2m_c^2m_a^2(m_c+m_a)+m_a^5)^3}{m_c^4+2m_cm_a(m_c^2+m_a^2)+m_a^4} \geq 4 \cdot 3^{14} \cdot r^{11}$$

Proposed by Zaza Mzhavanadze – Georgia

S.2425 If $a, b, c, n, k > 0$ then:

$$\frac{a^3}{nab^2+kc^3}+\frac{b^3}{nbc^2+ka^3}+\frac{c^3}{nca^2+cb^3} \geq \frac{3}{n+k}$$

Proposed by Marin Chirciu – Romania

S.2426 If $n \in \mathbb{N}$. Then prove that:

$$\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{4n}\right) < \left(1+\frac{1}{n+1}\right)^{n+1}\left(1+\frac{1}{4(n+1)}\right)$$

Proposed by Hikmat Mammadov-Aze rbaijan

S.2427 1. Let $a, b > 0$. Prove that $|a-b|+\sqrt{a^2-ab+b^2} \geq \sqrt{2(a^2+b^2)-3ab}$

2. Find all value of $\alpha, \beta \in \mathbb{R}$ such that $|a-b|+\sqrt{\alpha(a^2-ab+b^2)} \geq \beta\sqrt{2(a^2+b^2)-3ab}$, $\forall a, b \geq 0$

Proposed by Nguyen Van Canh-Vie tnam

S.2428 Let $a, b, c > 0$. Prove that

$$(ab+bc+ca)\left(\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right) \leq$$

$$\leq \frac{9}{4}+\frac{2023^2(a^2+b^2+c^2)}{abc}(|a-b|+|b-c|+|c-a|)$$

Proposed by Nguyen Van Canh-Vietnam

S.2429 If $0 < a_1 \leq a_2 \leq \cdots \leq a_n$ then prove that:

$$\sum_{i=1}^{n}\frac{1}{a_i} \leq \frac{1}{a_1 a_n}\left(n(a_1+a_n)-\sum_{i=1}^{n}a_i\right)$$

Proposed by Hikmat Mammadov-Azerbaijan

S.2430 If $a, b, c > 0, a+b+c=1$ then: $\sum\frac{a(b+c)^2}{a+1} \leq \frac{1}{3}$

Proposed by Marin Chirciu – Romania

S.2431 In acute ΔABC

$$\sum\sqrt{\cot A}\sec\frac{A}{2} \leq \frac{3R}{2}\sqrt{\frac{1}{F}\left(2+\frac{R}{r}\right)}$$

Proposed by Marin Chirciu – Romania

S.2432 If $f(f(x+y)+x+y)=xf(y)+yf(x)$ then: $f(x) \geq x\ \forall x, y \in \mathbb{N}$ and $f(2)=2, f(1)=1$

Proposed by Sidi Abdallah Lemrabott

S.2433 1. Find: $\frac{10}{\sqrt{x}+\sqrt{\pi+3}}+\frac{10}{\sqrt{\pi+3}+\sqrt{\pi+6}}+\cdots+\frac{10}{\sqrt{\pi+2022}+\sqrt{\pi+2025}}$

2. Find x: $\sqrt{x+\sqrt{x+\sqrt{x}}}=2022$

Proposed by Nguyen Van Canh-Vietnam

S.2434 In ΔABC:

$$\sum h_b h_c+\sum(p-b)(p-c) \leq \lambda\sum m_a^2, \lambda \geq \frac{2}{3}$$

Proposed by Marin Chirciu – Romania

S.2435 Prove that for $n \in \mathbb{N}$:

$$\left\lfloor \sqrt{n^2+4n+5}+\sqrt{n^2+4n+3} \right\rfloor + \left\lfloor \sqrt{n^2+4n} \right\rfloor =$$
$$= \left\lfloor \sqrt{n^2+4n}+\sqrt{n^2+4n+6} \right\rfloor + \left\lfloor \sqrt{n^2+4n-1} \right\rfloor$$

Proposed by Kerimov Elsen-Azerbaijan

S.2436 If $a,b,c>0$ and $a+b+c=3$ then find:

$$\max\left\{\frac{a}{a+2}+\frac{b}{b+2}+\frac{c}{c+2}\right\}$$

Proposed by Sidi Abdallah Lemrabott-Mauritania

S.2437 Let $a,b,c>0: ab+bc+ca=a+b+c$. Prove that:

$$\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}+33 \ge 4\left(\sqrt{8a+1}+\sqrt{8b+1}+\sqrt{8c+1}\right)$$

Proposed by Phan Ngoc Chau-Vietnam

S.2438 In ΔABC the following relationship holds:

$$\frac{r_a^4+r_b^4}{r_c r_a}+\frac{r_b^4+r_c^4}{r_a r_b}+\frac{r_c^4+r_a^4}{r_b r_c} \le \frac{27(81R^5-2560r^5)}{16r^3}$$

Proposed by Zaza Mzhavanadze – Georgia

S.2439 Find all values of $\alpha,\beta \in \mathbb{R}$ such that:

$$\alpha|a-b|+\beta\sqrt{2a^2-ab+2b^2} \ge \sqrt{3(a^2-ab+b^2)}, \forall a,b \ge 0$$

Proposed by Nguyen Van Canh-Vietnam

S.2440 In $\Delta ABC, \Delta A'B'C', \Delta A''B''C''$. Prove that:

$$\min\left\{\sum\frac{a^3}{a^3+b^3}, \sum\frac{a'^3}{a'^3+b'^3}, \sum\frac{a''^2}{a''^2+b''^2}\right\}+\frac{RR'R''}{rr'r''} \ge 8+\sum\frac{a}{b+c}$$

Proposed by Nguyen Van Canh-Vietnam

S.2441 In ΔABC the following relationship holds:

$$r_a^3\cdot\sqrt[3]{\frac{r_a}{r_b+r_c}}+r_b^3\cdot\sqrt[3]{\frac{r_b}{r_c+r_a}}+r_c^3\cdot\sqrt[3]{\frac{r_c}{r_a+r_b}} \ge \frac{81}{\sqrt[3]{2}}\cdot r^3$$

Proposed by Zaza Mzhavanadze – Georgia

S.2442 Solve in $\mathbb{R}$:

$$\sqrt[3]{x+\sqrt{x+\sqrt{x}}} \geq 2022+\sqrt{x}$$

Proposed by Nguyen Van Canh-Vietnam

S.2443 If $x_i > 0$ $\left(i=\overline{1,2023}\right)$ such that $x_1+x_2+\cdots+x_{2023}=x_1x_2\ldots x_{2023}$, find

$$\min\left(\max\{x_i|i=\overline{1,2023}\}\right)$$

Proposed by Neculai Stanciu – Romania

S.2444 O-the circumcenter of ΔABC lies on the incircle of ΔABC. Prove that:

$$8\sqrt{2}+cos\frac{A-B}{2}cos\frac{B-C}{2}cos\frac{C-A}{2}>12$$

Proposed by Daniel Sitaru , Claudia Nănuți – Romania

S.2445 Let be:

$$A=\begin{pmatrix} 5 & -3 & -1 \\ -3 & 16 & 5 \\ -1 & 5 & -3 \end{pmatrix}$$

Find $B,C\in M_3(\mathbb{R})$ such that: $A^{2023}=B^{2023}+C^{2023}$

Proposed by Daniel Sitaru , Claudia Nănuți – Romania

S.2446 If $p>1, n\in\mathbb{N}^*$ then:

$$n^{\frac{1}{p!}-1}\cdot\sum_{k=1}^{n}(k+k^2+k^3)\leq\left(\sum_{k=1}^{n}k^{p!}\right)^{\frac{1}{2^p-1}}\cdot\left(\sum_{k=1}^{n}k^{\frac{p!}{p!-1}}\right)^{\frac{p!-1}{2^p-1}}\cdot\left(\sum_{k=1}^{n}k^{\frac{2p!}{p!-1}}\right)^{\frac{p!-1}{2^p-1}}$$

Proposed by Khaled Abd Imouti-Syria

S.2447 $u_0=2, u_{n+1}=\frac{1}{2}u_n+n^2+n, n\in\mathbb{N}$. Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{n^3}\sum_{k=1}^{n}u_k\right)$$

Proposed by Khaled Abd Imouti-Syria

S.2448 Let $a,b,c>0$ such that $a+b+c=3abc$. Prove that

$$\sqrt{a^2+1}+\sqrt{b^2+1}+\sqrt{c^2+1}\leq\sqrt{a^2+b^2+c^2+4(ab+bc+ca)+3}.$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

S.2449 In non– obtuse $\Delta ABC, \omega$– Brocard's angle. Prove that

$\csc^2\omega + 4(\sin^2 A + \sin^2 B + \sin^2 C) \geq 13.$ When does equality holds?

Proposed by Mohamed Amine Ben Ajiba-Morocco

S.2450 If $x, y, z > 0$, then:

$$\frac{x}{(x+1)(z+1)} + \frac{y}{(y+1)(x+1)} + \frac{z}{(z+1)(y+1)} + \frac{xyz+1}{(x+1)(y+1)(z+1)} = 1$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

S.2451 In ΔABC the following relationship holds:

$$(\sin A + 2\sin B)^4 + (\sin B + 2\sin C)^4 + (\sin C + 2\sin A)^4 \leq \frac{3^7}{2^3}\left(1 - \frac{r}{R}\right)$$

Proposed by Marian Ursărescu – Romania

S.2452 In $\Delta ABC, BA', CB', AC'$ - external bisectors,

$CB' = CB, BA' = BA, CB' = CB, BC' = a', CA' = b', AB' = c', R_1, R_2, R_3, R$ – circumradies of $\Delta A'BC, \Delta B'CA, \Delta C'AB, \Delta ABC$. Prove that: $Ra'b'c' = R_1R_2R_3(a+b+c)$

$$\frac{a'}{R_3} + \frac{b'}{R_1} + \frac{c'}{R_2} \geq \frac{s}{2R}$$

Proposed by Mehmet Șahin – Turkey

S.2453 Solve in $\mathbb{R}$:

$$\sqrt{1 + \sqrt{x + \sqrt{x}}} \geq \pi + \sqrt{\pi - x}$$

Proposed by Nguyen Van Canh – Vietnam

S.2454 If $a, b, c > 0$, $a + b + c = m$ and $ab + bc + ca = n$. Prove that:

$$\max(a,b,c) - \min(a,b,c) \leq \frac{4}{3}\sqrt{m^2 - 3n}$$

Proposed by Hikmat Mammadov-Azerbaijan

S.2455 In ΔABC the following relationship holds:

$$\frac{a}{b+c}\sqrt{\tan A} + \frac{b}{c+a}\sqrt{\tan B} + \frac{c}{a+b}\sqrt{\tan C} \geq \frac{3\sqrt[4]{3}}{2}$$

Proposed by Vasile Mircea Popa-Romania

S.2456 If $m, n \in \mathbb{R}_+^*$, then in any triangle ABC holds

$$\frac{\cot\frac{A}{2}}{m+n\cdot\tan\frac{B}{2}\cdot\tan\frac{C}{2}}+\frac{\cot\frac{B}{2}}{m+n\cdot\tan\frac{C}{2}\cdot\tan\frac{A}{2}}+\frac{\cot\frac{C}{2}}{m+n\cdot\tan\frac{A}{2}\cdot\tan\frac{B}{2}}\ge\frac{9p}{4mR+(m+3n)r}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu

S.2457 In acute ΔABC

$$\sum\frac{\tan A}{bc}\ge 3\sum\frac{\cot A}{bc}$$

Proposed by Marin Chirciu – Romania

S.2458 In ΔABC the following relationship holds:

$$\frac{\sum_{cyc}\frac{m_a^4}{b^2+c^2}(b+c)^2\cdot\sum_{cyc}\frac{w_a^3}{w_bw_c}\cdot\sum_{cyc}\frac{w_a^2}{w_bw_c}\cdot\sum_{cyc}w_bw_c}{\left(\sum_{cyc}\left((b+c)^3-a^2(b+c)\right)\right)^2\cdot\left(\sum_{cyc}w_a\right)^3}\cdot\frac{R^4}{\sum_{cyc}a^2}\gtrless\frac{1}{2592}$$

Proposed by Elsen Kerimov & Murad Shikmatov-Azerbaijan

S.2459 In ΔABC the following relationship holds:

$$\frac{a}{2r}\ge\frac{m_a}{h_a}+\frac{2r_a}{s+n_a}+\frac{\sqrt{g_a}\left(\sqrt{n_a}-\sqrt{g_a}\right)}{h_a}$$

Proposed by Bogdan Fuștei – Romania

S.2460 In any triangle ABC, inequalities occur:

$$\frac{p^3r+pr^3-8Rr^2p}{2R^3}\le\sum_{cyc}h_a^2\cdot\frac{p-a}{bc}\cdot\cos^2\frac{\hat{B}-\hat{C}}{2}\le\frac{p^3+r^2p-8Rrp}{4R^2}$$

Proposed by Radu Diaconu – Romania

S.2461 If $x, y, z > 0$, then in ΔABC holds:

$$\frac{x}{h_a^2}+\frac{y}{h_b^2}+\frac{z}{h_c^2}\ge\frac{1}{2F}\sqrt{\frac{xy}{\sin^2\frac{C}{2}}+\frac{yz}{\sin^2\frac{A}{2}}+\frac{zx}{\sin^2\frac{B}{2}}}$$

Proposed by D.M.Bătinețu-Giurgiu, Dorin Mărghidanu-Romania

S.2462 If $x, y, z \ge 0$, then in ΔABC holds:

$$\frac{ayz}{h_a}+\frac{bxz}{h_b}+\frac{cxy}{h_c}\le\frac{R}{2F}(x+y+z)^2$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți-Romania

S.2463 If $m \ge 0$ and $x, y, z > 0$, then in ΔABC holds:

$$\frac{x^{m+1}a^2}{(y+z)^{m+1}}+\frac{y^{m+1}b^2}{(z+x)^{m+1}}+\frac{z^{m+1}c^2}{(x+y)^{m+1}}\geq 2^{1-m}\sqrt{3}\cdot F$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți-Romania

S.2464 If $m\geq 0$ and $M\in Int(\Delta ABC), d_a=d(M,BC), d_b=d(M,CA), d_c)=d(M,AB)$ then:

$$\frac{a^{2m+1}}{d_a}+\frac{b^{2m+1}}{d_b}+\frac{c^{2m+1}}{d_c}\geq 2^{2m+1}\left(\sqrt{3}\right)^{3-m}\cdot F^m$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți-Romania

S.2465 If $m\geq 0$, then in ΔABC holds:

$$\frac{a^{m+2}}{h_a^m}+\frac{b^{m+2}}{h_b^m}+\frac{c^{m+2}}{h_c^m}>2^{m+2}\left(\sqrt{3}\right)^{1-m}\cdot F$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți-Romania

S.2466 In ΔABC the following relationship holds:

$$h_a^2\cdot\cot\frac{A}{2}+h_b^2\cdot\cot\frac{B}{2}+h_c^2\cdot\cot\frac{C}{2}\geq 9F$$

Proposed by D.M.Bătinețu-Giurgiu, Dan Nănuți-Romania

S.2467 If $x,y,z>0$, then in ΔABC holds:

$$\frac{a^4e^{x^2}}{y+z}+\frac{b^4e^{y^2}}{z+x}+\frac{c^4e^{z^2}}{x+y}>16F^2$$

Proposed by D.M.Bătinețu-Giurgiu, Dan Nănuți-Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

UNDERGRADUATE PROBLEMS

U.2341 Let $t>0$ and $(a_n)_{n\geq 1},(b_n)_{n\geq 1}$ sequences of real numbers strictly positive such that

$\lim\limits_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n}=a, \lim\limits_{n\to\infty}\frac{b_{n+1}}{b_n\cdot n^t}=b>0$. Find:

$$\Omega = \lim_{n\to\infty} \frac{\sqrt[n+1]{a_{n+1}^{t+1}} - \sqrt[n]{a_n^{t+1}}}{\sqrt[n]{b_n}}$$

Proposed by D.M.Bătinețu-Giurgiu, Gheorghe Stoica-Romania

U.2342 Let $(a_n)_{n\geq 1}, a_n \in (0,\infty), n \in \mathbb{N}^*$ such that $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n\cdot\sqrt[n]{(2n-1)!!}} = a > 0$. Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}} - \frac{n^2}{\sqrt[n]{a_n}}\right)$$

Proposed by D.M.Bătinețu-Giurgiu, Ionel Tudor-Romania

U.2343 Let $(a_n)_{n\geq 1}, a_1 = 1, a_{n+1} = a_n \cdot \sqrt[n+1]{(2n+1)!!}$ and $(b_n)_{n\geq 1}, b_n \in (0,\infty)$ such that

$\lim\limits_{n\to\infty} \frac{b_{n+1}}{b_n\cdot\sqrt[n]{a_n}} = b > 0$. Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{b_{n+1}}} - \frac{n^2}{\sqrt[n]{b_n}}\right)$$

Proposed by D.M.Bătinețu-Giurgiu, Nicolae Mușuroia-Romania

U.2345 Find:

$$\Omega = \lim_{n\to\infty}\left(\sum_{k=2}^{n} \frac{1}{\sqrt[k]{k!}} \cdot H_n^{-1}\right)^{H_n - \log n}$$

Proposed by Khaled Abd Imouti-Syria

U.2346 Find a closed form:

$$\Omega = \sum_{n=0}^{\infty}((-\varphi)^n - (\varphi-1)^n)\cdot\left(\frac{-1}{4}\right)^n$$

Proposed by Khaled Abd Imouti-Syria

U.2347 If $\lambda \in (0,1), \lambda$ −fixed then solve for complex numbers:

$$e^{i\pi\lambda}(z+1)^{2n} - e^{-i\pi\lambda}(z-1)^{2n} = 0, n \in \mathbb{N}, n \geq 2$$

Proposed by Khaled Abd Imouti-Syria

U.2348 If $n \in \mathbb{N}, n \geq 2$ then:

$$\sum_{k=1}^{n} \pi^k > \frac{ne^2}{2}$$

Proposed by Khaled Abd Imouti-Syria

U.2349

$$\omega = \lim_{n\to\infty}\left(H_n + log\left(\frac{e}{n}\right) - \gamma\right)^n$$

Find:

$$\Omega = \lim_{x\to\infty}\left(\frac{\omega x}{1+\omega x}\right)^{\frac{\omega x^2}{2+\omega x}}$$

Proposed by Daniel Sitaru-Romania

U.2350 Find all functions $y(x) \in C^3(\mathbb{R})$ such that: $y''' + y'' = (6y^2 + 2y + 2)y'$

Proposed by Khaled Abd Imouti-Syria

U.2351 For $x, y, z > 0$ prove that: $(x^3 + y^3 + z^3) - (x + y + z) \geq 2\log(xyz)$

$\boxed{\textbf{Proposed by Jalil Hajimir} - \textbf{Canada}}$

U.2352 Find:

$$\Omega = \int \frac{\sin(\log x)}{x^n}dx$$

$\boxed{\textbf{Proposed by Jalil Hajimir} - \textbf{Canada}}$

U.2354 Evaluate:

$$\Omega = \lim_{x\to\infty}\frac{2^{\frac{1}{x^2}} - \cos\frac{1}{x}}{\frac{1}{x}\left(\log\left(1+\frac{1}{x}\right) - \sin\frac{1}{x}\right)}$$

$\boxed{\textbf{Proposed by Jalil Hajimir} - \textbf{Canada}}$

U.2355 $u_0 > 0, u_{n+1} = u_n + \frac{1}{u_n^2}$

Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{logn}\cdot\sum_{k=1}^{n-1}(u_{k+1}^3 - u_k^3 - 3)\right)$$

Proposed by Khaled Abd Imouti-Syria

U.2356 Find:

$$\Omega = \sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}\left(\frac{1}{4n+1} + \frac{1}{4n+2} + \frac{1}{8n+6}\right)$$

Proposed by Khaled Abd Imouti-Syria

U.2357 Find the general solution for:

$$\frac{dy}{dx} = 1 + \frac{x^2 + y^2}{xy} \cdot 2^{-\frac{y}{x}}$$

Proposed by Jalil Hajimir – Canada

U.2358 Evaluate:

$$\Omega = \int_0^1 \frac{\sin^3(\log x)}{\log^3 x} dx$$

Proposed by Jalil Hajimir – Canada

U.2359 Prove:

$$\Omega = \int_0^{\frac{1}{2}} (1+x)^{\frac{1}{2}+x}(1-x)^{\frac{1}{2}-x}\, dx > \frac{1}{2}$$

Proposed by Jalil Hajimir – Canada

U.2360 Solve for x:

$$\sum_{k=1}^{10} \left[\frac{1}{x}\right] = 45, [x] - GIF$$

Proposed by Jalil Hajimir – Canada

U.2361 Let x, y be real numbers. Prove:

$$\left|\log\left(\frac{x+\sqrt{1+x^2}}{y+\sqrt{1+y^2}}\right)\right| \le |x-y|$$

Proposed by Jalil Hajimir – Canada

U.2362 Prove that:

$$\int_0^a \sqrt[m]{a - x^n}\, dx = \int_0^a \sqrt[n]{a - x^m}\, dx, m, n \in \mathbb{N}$$

Proposed by Jalil Hajimir – Canada

U.2363 Calculate the surface area of: $S = \{(x,y,z) | x^2 + y^2 + z^2 = 1, -1 \le z \le 1\}$

Proposed by Jalil Hajimir – Canada

U.2364 Prove without softs: $\int_0^{\frac{\pi}{2}} (\sin x)^{\sqrt{2}(1+\sqrt{2}\cos x)\cos x} dx < \frac{\pi}{4}$

Proposed by Jalil Hajimir – Canada

U.2365 Let $x_1, x_2, \dots, x_m$ be non-negative real numbers. Prove:

$$\sum_{n=1}^{m} x_n^2 \geq \left(\sum_{n=1}^{m} \frac{n}{2^n} x_n\right)^2$$

Proposed by Jalil Hajimir – Canada

U.2366 Prove:

$$\int_0^1 \frac{xdx}{\sqrt{(1-x)(16-x^2)}} \leq \frac{1}{4}$$

Proposed by Jalil Hajimir – Canada

U.2367 Prove:

$$\log_{1011} 2 \cdot \log_{1011} 4 \cdot \dots \cdot \log_{1011} 2020 < 1$$

Proposed by Jalil Hajimir – Canada

U.2368 If $P(z) = z^3 + 7z^2 - 2z - 8, S = \{Z \in \mathbb{C} | |z+1| \leq 1\}$. Prove that $|P(z)| \leq 18$.

Proposed by Jalil Hajimir – Canada

U.2369 Find:

$$\int_0^{2020} (x - [2x])\sqrt{x - [x]}\, dx, [*] - GIF$$

Proposed by Jalil Hajimir – Canada

U.2370 For what values of A, B, C and D is the value of:

$\int_0^{\pi} (A + B\cos x + C\cos 2x + D\cos 3x - \cos^6 x)^2 dx$ minimum?

Proposed by Jalil Hajimir – Canada

U.2371 Find without softs:

$$\Omega = \int_0^{\infty} \frac{x \sin 5x}{x^2 + 25} dx$$

Proposed by Jalil Hajimir – Canada

U.2372 Find a closed form:

$$\Omega = \int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+1)} dx$$

Proposed by Jalil Hajimir – Canada

U.2373 Find:

$$\Omega = \lim_{x\to 0} \frac{4\cdot 81^x - 27^x - 9^x - 3^x - 1}{1 - \sqrt[3]{1+x+x^2}}$$

Proposed by Mohamad Hamed Nasery-Afghanistan

U.2374 Prove that:

$$\int_0^1 \frac{1}{x}\log^2\left(x+\sqrt{1+x^2}\right)\cos^{-1}x\,dx = \frac{3\pi}{32}\zeta(3)$$

Proposed by Ose Favour-Nigeria

U.2375 Prove that:

$$\int_0^1 \frac{\cos^{-1}x}{\sqrt{1+ax^2}}dx = \frac{4Li_2\left(\sqrt{-a}\right) - Li_2(-a)}{4\sqrt{-a}}, a \ge -1$$

Proposed by Le Thu-Vietnam

U.2376 Prove that:

$$\int_0^1 \log\left(\frac{1+x}{1-x}\right)\log\left(\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\right)dx = \frac{1}{2}(16G - \pi^2)$$

Proposed by Ose Favour-Nigeria

U.2377 Find:

$$\Omega = \lim_{x\to\infty} \frac{log\left(\frac{logx+\sqrt{log^2x+1}}{logx+\sqrt{log^2x-1}}\right)}{log^2\left(\frac{logx+1}{logx-1}\right)}$$

Proposed by Daniel Sitaru, Claudia Nănuți -Romania

U.2378 If $a \in \left(0, \frac{\pi}{2}\right), b \in (0, \pi)$,

$sin^2a + 2sina\cdot cosa\cdot sinb + 2sina\cdot cosa\cdot cosb + 2cos^2a = 0$ then find:

$$\text{a, b and } \Omega = \int_{-a}^{a}\int_{-b}^{b} \frac{u^3v^5}{(1+u^{10})(1+v^{16})}dudv$$

Proposed by Daniel Sitaru, Claudia Nănuți -Romania

U.2379 $V = \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix}$ $V_1 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^4 & b^4 & c^4 & d^4 \end{vmatrix}$

$$V_2=\begin{vmatrix}1&1&1&1\\a&b&c&d\\a^3&b^3&c^3&d^3\\a^4&b^4&c^4&d^4\end{vmatrix}\quad V_3=\begin{vmatrix}1&1&1&1\\a^2&b^2&c^2&d^2\\a^3&b^3&c^3&d^3\\a^4&b^4&c^4&d^4\end{vmatrix}, a\neq b\neq c\neq d\neq a.$$

Solve for real numbers: $V_1V_2V_3x^3-VV_2V_3x^2+VV_1V_3x-VV_1V_2=0$

Proposed by Daniel Sitaru-Romania

U.2380 Find $\alpha\in\mathbb{R}$ such that: $x_1^{16}+x_2^{16}+x_3^{16}=90$ where x_1,x_2,x_3 –are the roots of the equation: $x^3+\alpha x+1=0$.

Proposed by Daniel Sitaru-Romania

U.2381 If $p>1, 0<\alpha,\beta<1$ then:

$$\sqrt[p]{\sum_{k=1}^{n}\left(\frac{2-\alpha-\beta+(\beta-1)\alpha^{k+1}+(\alpha-1)\beta^{k+1}}{1-\alpha-\beta+\alpha\beta}\right)^p}\leq\sqrt[p]{n}\left(\frac{2-\alpha-\beta}{1-\alpha-\beta+\alpha\beta}\right)$$

Proposed by Khaled Abd Imouti-Syria

U.2382 Find:

$$\Omega=\int_0^1\int_0^1\frac{\arcsin(xy)\arccos(xy)}{y}dxdy$$

Proposed by Togrul Ehmedov-Azerbaijan

U.2383 $A=\begin{pmatrix}1&0\\1&1\end{pmatrix}, B=\begin{pmatrix}1&1\\0&1\end{pmatrix}$. Solve for real numbers:

$$e^A\cdot\begin{pmatrix}x\\0\end{pmatrix}+e^B\cdot\begin{pmatrix}0\\y\end{pmatrix}=\begin{pmatrix}2e-1\\2e-1\end{pmatrix},\ e^A\text{ –exponential matrix}$$

Proposed by Daniel Sitaru, Claudia Nănuți -Romania

U.2384 $f(0)=0, f(1)=e^3, f\in C^2(\mathbb{R}), f''(x)-5f'(x)+6f(x)=0, \forall x\in\mathbb{R}$

Find:

$$\Omega=\lim_{x\to\infty}\left(1-\frac{1}{f(x)}\right)^x$$

Proposed by Daniel Sitaru, Claudia Nănuți -Romania

U.2385 $\Omega_n=\begin{vmatrix}1&1&1&\dots&1\\1&2^2&2^3&\cdots&2^n\\1&3^2&3^3&\cdots&3^n\\\dots&\dots&\dots&\cdots&\dots\\1&n^2&n^3&\dots&n^n\end{vmatrix}, n\in\mathbb{N}^*$. Find:

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\frac{\Omega_{n+1}}{\Omega_n}}$$

Proposed by Daniel Sitaru, Dan Grigorie-Romania

U.2386 $u_0 = \sqrt{5}, u_{n+1} = u_n^2 - 2, n \in \mathbb{N}$. Find:

$$\Omega = \sum_{n=0}^{\infty} \prod_{k=0}^{n} \frac{1}{u_k}$$

Proposed by Khaled Abd Imouti-Syria

U.2387 Find a closed form:

$$\Omega = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (n+1)}{(n+2) \cdot n! \cdot 2^{n-1}}$$

Proposed by Khaled Abd Imouti-Syria

U.2388

$$\Omega_1 = \sum_{n=0}^{\infty} \frac{1}{(2n+1)(4n+1)(4n+3)}, \Omega_2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)(n+2)(2n+3)},$$

$$A. \Omega_1 > \Omega_2, \quad B. \Omega_1 = \Omega_2, \quad C. \Omega_1 < \Omega_2$$

Proposed by Daniel Sitaru-Romania

U.2389

$$\Omega_1 = \sum_{n=1}^{\infty} \frac{(-1)^n (2n-1)!!}{(2n)!!}, \Omega_2 = \sum_{n=1}^{\infty} \frac{(-1)^n (2n-1)!!}{(2n+2)!!}$$

Solve for real numbers: $4^x + 23\Omega_1 = 27^x + 23\Omega_2$

Proposed by Daniel Sitaru-Romania

U.2390 Find all values of $m \in \mathbb{R}$ such that the equation

$\int_0^x \frac{arctany}{y} dy = mx$ has two real roots: $x_1 \in (-\infty, 0), x_2 \in (0, \infty)$.

Proposed by Daniel Sitaru-Romania

U.2391 $A_n = \left(a_{ij}\right)_{1\le i,j\le n}, a_{ij} = \frac{1}{i+j-1}, n \in \mathbb{N}, n \ge 2$. Find:

$$\Omega = \lim_{n\to\infty} \left(\frac{det(A_{n+1})}{(n!)^3 \cdot det(A_n)} \right)$$

Proposed by Daniel Sitaru-Romania

U.2392 Solve for natural numbers:

$$\begin{cases}\binom{x}{2x}+2\binom{x}{2x-1}+4\binom{x}{2x-2}+\dots+2^x\binom{x}{x}=4^y\\ x\cdot 2^y=8\end{cases}$$

Proposed by Daniel Sitaru-Romania

U.2393 If $a,b,c,d>0, A\in M_{7,5}(\mathbb{C}), B\in M_{5,9}(\mathbb{C}), C\in M_{7,3}(\mathbb{C}), D\in M_{3,9}(\mathbb{C})$ then:

$$32(a^2+b^2+c^2+d^2)\geq(a+b+c+d)^2\cdot rank(AB+CD)$$

Proposed by Daniel Sitaru-Romania

U.2394 If $A\in M_n(\mathbb{C}), n\in\mathbb{N}, n\geq 2$ then: $rank(A^5)+rank(A)\geq rank(A^4)+rank(A^2)$

Proposed by Daniel Sitaru-Romania

U.2395 $n\in\mathbb{N},\lambda\in\mathbb{R},\lambda$ –fixed. Find $\alpha\in\mathbb{R}$ such that:

$$x^{n-\lambda+1}\cdot\left(x^\lambda\cdot lnx\right)^{(n+1)}-(\lambda-n)\cdot x^{n-\lambda}=\alpha,\quad \forall x>0$$

Proposed by Khaled Abd Imouti-Syria

U.2396 If $x>1, n\in\mathbb{N}^*$ then:

$$\sum_{k=1}^{n}\left(\frac{logx}{x}\right)^{2k}\leq\frac{6(xlogx)^2}{n(n+1)(n+2)(x-logx)^4}$$

Proposed by Khaled Abd Imouti-Syria

U.2397 Find a closed form:

$$\Omega(x,p)=\sum_{n=0}^{\infty}\left(\frac{(n+p)(n+p-1)\cdot\ldots\cdot(p-1)}{n!}\left(1-\frac{2}{1+e^x}\right)^n\right)$$

Proposed by Khaled Abd Imouti-Syria

U.2398 $A^1(x)=\begin{pmatrix}x_1 & y_1\\ -y_1 & x_1\end{pmatrix}, A^n(x)=\begin{pmatrix}x_n & y_n\\ -y_n & x_n\end{pmatrix}, n\in\mathbb{N}-\{0\},$

$x_1=\sqrt{\frac{x}{3(x^2+1)}}, y_1=\sqrt{\frac{x}{4(x^2+1)}}, x\geq 0$. Find: $\Omega=\lim_{n\to\infty}(x_n+y_n)$.

Proposed by Daniel Sitaru-Romania

U.2399 $A,B,C\in M_2(\mathbb{R}), detA=1, A^2=BC, B^2=CA, C^2=AB$. Solve for complex numbers: $x^4+6(xdetB)^3-14(xdetC)^2+6x+1=0$.

Proposed by Daniel Sitaru-Romania

U.2400 If $x_i \in [e^2\sqrt{e}, \infty), i \in \overline{1,n}, n \in \mathbb{N}^*$ then:

$$\sqrt[n]{\prod_{k=1}^{n} x_k}\left(\sum_{k=1}^{n}\frac{1}{x_k}-\frac{n}{e}\right)+log\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{x_k}\right)\le 1$$

Proposed by Khaled Abd Imouti-Syria

U.2401 Find a closed form:

$$\Omega=\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{\sqrt{4n-2\sqrt{4n^2-1}}}{1+\sqrt{4n^2-1}}\right)$$

Proposed by Neculai Stanciu-Romania

U.2402 Let $(F_n)_{\ge 0}, F_0 = F_1 = 1, F_{n+2} = F_{n+1} + F_n$ be the Fibonacci's sequence, and T_k is the k^{th} triangular number defined by $T_k = \binom{k+1}{2}$ for all $k \ge 1$. Prove that:

$$\sum_{k=1}^{n}\frac{F_k^{2m+2}}{T_k^m}\ge\frac{3^m(F_nF_{n+1})^{m+1}}{n^mT_{n+1}^m}, n\ge 1, m>0$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

U.2403 Let $a, b \in \mathbb{R}_+^*, f: \mathbb{R} \to \mathbb{R}$ is an even continuous function such that $\int_0^a f(x)\,dx = c$. Find:

$$\Omega=\int_{-a}^{a}\frac{f(x)}{b^2+\arctan x+\sqrt{b^4+\arctan^2 x}}dx$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

U.2404 If $(a_n)_{n\ge 0}, (b_n)_{n\ge 0}$ are given by $a_0 = b_0 = 1, a_{n+1} = a_n + b_n, b_{n+1} = (n^2+n+1)a_n + b_n$, $n \ge 1$ then compute: $\Omega = \lim_{n\to\infty}\frac{n}{\sqrt[n]{\frac{b_1b_2\dots b_n}{a_1a_2\dots a_n}}}$

Proposed by Neculai Stanciu-Romania

U.2405 If $a, b \in \mathbb{R}, a < b, c \in \mathbb{R}_+^*$ and $f: \mathbb{R} \to \mathbb{R}_+^*$ is a continuous function, then prove that:

$$\int_a^b\frac{e^{f(x-a)}\big(f(x-a)\big)^{\frac{1}{c}}}{e^{f(x-a)}\big(f(x-a)\big)^{\frac{1}{c}}+e^{f(b-x)}\big(f(b-x)\big)^{\frac{1}{c}}}dx=\frac{b-a}{2}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

U.2406 If $A_1A_2 \dots A_n, n \ge 3$ is a polygon, and $M \in Int(A_1A_2 \dots A_n)$, with $pr_{A_kA_{k+1}}M = T_k \in [A_kA_{k+1}]$, for any $k \in \{1,2 \dots, n\}, A_{n+1} \equiv A_1$, then: $\sum_{k=1}^{n}\frac{A_kA_{k+1}}{MT_k} \ge 2n\tan\frac{\pi}{n}$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

U.2407 Prove that:

$$\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\frac{H_{a-1}H_{b-1}}{ab}f(a+b)$$

$$=\sum_{n=1}^{\infty}\left(\frac{H_n^3}{n}+2\frac{H_n^{(3)}}{n}-3\frac{H_n^{(2)}H_n}{n}-3\frac{H_n^2}{n^2}+3\frac{H_n^{(2)}}{n^2}+6\frac{H_n}{n^3}-6\frac{1}{n^4}\right)f(n)$$

Where: H_n → Harmonic number and $H_n^{(m)}$ → Generalized harmonic number of order "m"

Proposed by Izumi Ainsworth-Japan

U.2408 Prove that:

$$\Omega=\int_0^1\int_0^1\left(x\ln(\arccos(1-y))+\arctan(1-y)^2\right)dxdy=$$

$$=\frac{1}{16}\left(\pi\ln(16)+\pi^2+8Ci\left(\frac{\pi}{2}\right)-8\gamma-16G\right)$$

Proposed by Tahirov Shirvan-Azerbaijan

U.2409 If we have the recurrence relation $5a(n+2)+3a(n+1)+a(n)=2n+1$

$a(0)=1, a(1)=3$ then show that

$$\sum_{n=0}^{\infty}a(n)\,x^{2n+1}=\frac{x(19x^6-30x^4+8x^2+5)}{x^8+x^6-7x^2+5}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.2410 Show that

$$\int_0^1\frac{O_{\left\lfloor\frac{1}{x}\right\rfloor}}{x}dx=\ln(4)-\frac{\pi^2}{12}$$

where O_n is the sum of the reciprocal of the first n odd numbers.

Proposed by Vincent Nguyen-USA

U.2411 Find:

$$\int_0^1\frac{\ln(1+x)}{1+\pi x}$$

Proposed by Basir Ahmad Alizada-Afghanistan

U.2412 Prove the below closed form.

$$\int_0^1\frac{\ln^2(1+x)}{(1+x)(2+x)(3+x)}dx=2Li_3(-2)+Li_3\left(-\frac{1}{2}\right)-2\ln(2)\,Li_2(-2)+\frac{9}{4}\zeta(3)-$$

$$-\frac{\pi^2}{12}\ln(2)+\frac{2}{3}\ln^3(2)-\ln^2(2)\ln(3)$$

Where, $Li_s(z)$ is a polylogarithm of Jonquiere's function and $\zeta(3)$ is the Apery's constant

Proposed by Ankush Kumar Parcha-India

U.2413 Find a closed – form:

$$\Omega=\int_0^1\frac{\ln(x+1)}{x+1}Li_2\left(\frac{x^2}{1+x^2}\right)dx$$

Where: $Li_2(z)$ denotes the dilogarithm function and $Li_2(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^2}(|z|<1)$

Proposed by Pham Duc Nam-Vietnam

U.2414 Prove the summation:

$$\sum_{n=1}^{\infty}\frac{(F_{n+1}-\phi^{n-1})(F_{n-1}+\phi^{n+1})}{\phi^{4n}}=\frac{1}{4}(2\phi-3)$$

where F_n is Fibonacci number & ϕ is Golden ratio.

Proposed by Srinivasa Raghava-AIRMC-India

U.2415 Prove that:

$$\left(\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot 5^{n-1}}{26^n}\right)\left(\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot 7^{n-1}}{50^n}\right)\left(\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot 11^{n-1}}{122^n}\right)<\frac{64}{27}$$

Proposed by Daniel Sitaru-Romania

U.2416 Find a closed form:

$$\Omega=\sum_{n=0}^{\infty}\frac{cos\left(\frac{n\pi}{2}\right)\cdot e^{i\cdot\frac{n\pi}{2}}}{F_nF_{n+2}}$$

Proposed by Khaled Abd Imouti-Syria

U.2417 Find:

$$\Omega=\lim_{n\to\infty}\left(n\sum_{k=1}^{n}k^2sin^{k-1}\left(\frac{1}{n}\right)\cdot\left(\sum_{k=1}^{n}k^2\right)^{-1}\right)$$

Proposed by Khaled Abd Imouti-Syria

U.2418 If $n>1$ then:

$$\int_{\frac{1}{2}-\frac{1}{2n}}^{\frac{1}{2}+\frac{1}{2n}} logx \cdot log(1-x)dx \le \frac{1}{n} log^2 2$$

Proposed by Khaled Abd Imouti-Syria

U.2419 Find: $\Omega(x) = \lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} arctan\left(\frac{1+cosx+cos(2x)+\cdots+cos(ix)}{sinx+sin(2x)+\cdots+sin(ix)}\right)$

Proposed by Khaled Abd Imouti-Syria

U.2420 If $n \ge 1$ then:

$$2 + log\left(\frac{6\left((n+1)!-1\right)^2}{n(2n^2+3n+1)}\right) \le 2\sqrt{\sum_{k=1}^{n}(k!)^2}$$

Proposed by Khaled Abd Imouti-Syria

U.2421 Solve for natural numbers:

$$31 + \sum_{k=1}^{n}\left(\binom{n}{k} \cdot \sum_{m=1}^{n} m^{n-k}\right) = 31^{30}$$

Proposed by Daniel Sitaru-Romania

U.2422 Prove that:

$$\sum_{k=0}^{\infty} \frac{\left((2k)!\right)^2}{2^{2k} \cdot (k!)^4} > 16\left(log\frac{e}{2}\right)^2 \cdot \left(\sum_{k=0}^{\infty}\frac{1}{k^4}\right)^{-1}$$

Proposed by Khaled Abd Imouti-Syria

U.2423 If $x \in [0, \frac{\pi}{2})$ then: $2e^{tanx} \cdot log\left(\frac{1+sinx}{cosx}\right) + 1 \le 2tanx \cdot e^{tanx} + e^{tanx}$

Proposed by Khaled Abd Imouti-Syria

U.2424 If $x, y, z > 0, x^2 + y^2 + z^2 \le 1$ then:

$$\sqrt{\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} + \frac{1}{\sqrt{1+z^2}}} > log\left(\frac{3e}{2}\right)$$

Proposed by Khaled Abd Imouti-Syria

U.2425 Let be: $A = \begin{pmatrix} a+2b & 2b \\ -b & a-2b \end{pmatrix}$. Find $a, b \in \mathbb{Z}$ such that: $A^3 + 9A = 6A^2 + 27I_2$

Proposed by Daniel Sitaru-Romania

U.2426 Find:

$$\Omega = \lim_{n\to\infty}\left(1+\frac{280}{n^9}\sum_{k=1}^{n}k^4(n-k)^3\right)^n$$

Proposed by Daniel Sitaru-Romania

U.2427 $A \in M_5(\mathbb{Z}), A^4 = O_5$. Solve for complex numbers:

$$\left(\frac{z-i}{z+i}\right)^{5det\left(A^2+I_5\right)} = -1$$

Proposed by Daniel Sitaru-Romania

U.2428 Find:

$$\Omega = \lim_{x\to\infty}\left(\sqrt[4]{x^3(x+1)}-\sqrt[4]{x^3(x-1)}\right)^{x\cdot\left[\frac{1}{x}\right]}, [*]-GIF$$

Proposed by Daniel Sitaru-Romania

U.2429 Find:

$$\Omega = \lim_{n\to\infty}\sum_{k=0}^{n}\left(cos\frac{\sqrt{k}}{n}-cos\frac{1}{\sqrt{n+k}}\right)$$

Proposed by Daniel Sitaru-Romania

U.2430 If $a, b, c \geq 1$ then:

$$e^{\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}} \geq \frac{e\sqrt{e}}{\sqrt[4]{abc}}$$

Proposed by Daniel Sitaru-Romania

U.2431 If $x, y \in [0,1], p, q \in (1,\infty), \frac{1}{p}+\frac{1}{q} = 1$ then:

$$\sum_{k=1}^{n}(x+y)^k \leq 2^n\left(\frac{x}{\sqrt[p]{1-x^p}}+\frac{y}{\sqrt[q]{1-x^q}}\right)$$

Proposed by Daniel Sitaru-Romania

U.2432 If $0 < a \leq \frac{\pi}{2}, n \in \mathbb{N}^*$ then:

$$\int_a^x sint(cosa-cost)^{n-1}dt \leq \frac{1}{n}(cosa-cosx)^n$$

Proposed by Daniel Sitaru-Romania

U.2433 Find:

$$\Omega=\lim_{n\to\infty}\frac{1}{log(n+1)}\sqrt[n]{\prod_{k=1}^{n}\frac{1}{2k+1}}$$

Proposed by Daniel Sitaru-Romania

U.2434 In ΔABC the following relationship holds:

$$e^a+e^b+e^c+2\sqrt{e^{a+b}}+2\sqrt{e^{b+c}}+2\sqrt{e^{c+a}}\geq 9\sqrt[3]{e^{2s}}$$

Proposed by Daniel Sitaru-Romania

U.2435 Find:

$$\Omega=\lim_{n\to\infty}\frac{1}{3^n}\prod_{k=2}^{n}\left(\sqrt[k]{\frac{e+1}{e}}+\sqrt[k]{\frac{e-1}{e}}\right)$$

Proposed by Daniel Sitaru-Romania

U.2436 If $a<b$ then:

$$\frac{arctanb-arctana}{b-a}<\frac{4}{(a+b)^2+4}+\frac{(b-a)^2}{48}$$

Proposed by Daniel Sitaru-Romania

U.2437 If $x\in\left(0,\frac{\pi}{2}\right)$ then:

$$\left(\frac{(sinx)^{sinx}+x^{sinx}+(tanx)^{sinx}}{3}\right)^{\frac{1}{sinx}}<\left(\frac{(sinx)^{x}+x^{x}+(tanx)^{x}}{3}\right)^{\frac{1}{x}}<$$
$$<\left(\frac{(sinx)^{tanx}+x^{tanx}+(tanx)^{tanx}}{3}\right)^{\frac{1}{tanx}}$$

Proposed by Daniel Sitaru-Romania

U.2438 If $a,b,c\geq 1$ then: $(abc)^2\cdot e^{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}\leq e^{a+b+c}$

Proposed by Daniel Sitaru-Romania

U.2439 Prove that:

$$\prod_{n=1}^{\infty}\left(\tanh\left(\frac{\pi}{2\sqrt{3}}(n+1)\right)\coth\left(\frac{\pi}{2\sqrt{3}}n\right)\right)^n=\sqrt[8]{\frac{48\pi^2(7+4\sqrt{3})\Gamma^4\left(\frac{2}{3}\right)}{\Gamma^4\left(\frac{1}{6}\right)}}$$

Proposed by Toubal Fethi -Algeria

U.2440 Show that:

$$\int_{\mathbb{R}} \frac{\arctan(x^2+1)}{x^2+1}\,dx = \pi \arctan\sqrt{2+2\sqrt{2}}$$

Proposed by Vincent Nguyen-USA

U.2441 Prove the below closed form

$$\int_0^1\int_0^1\int_0^1 \ln\left(\frac{1}{1+xy}+\frac{1}{1-xyz}\right)dx\,dy\,dz = -\frac{7}{4}\zeta(3)-\frac{\pi^2}{4}-\ln(2)+6$$

Proposed by Ankush Kumar Parcha-India

U.2442 Prove that

$$\sum_{n=1}^{\infty}\frac{n}{(2n-1)^2 64^n}\binom{2n}{n}\binom{4n}{2n} = \frac{7\pi}{48}+\frac{1}{\sqrt{2}\pi}-\frac{1}{\pi}Li_2\left(\frac{1}{\sqrt{2}}\right)-\frac{\sinh^{-1}(1)}{2\pi}-\frac{\log^2(2)}{8\pi},$$

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}\binom{4n}{2n}}{(2n-1)^2 64^n} = \frac{7\pi}{24}-1+\frac{2\sqrt{2}}{\pi}-\frac{2}{\pi}Li_2\left(\frac{1}{\sqrt{2}}\right)-\frac{\log^2(2)}{4\pi},$$

Proposed by Narendra Bhandari –Nepal

U.2443 Prove the summation

$$2\sum_{n=1}^{\infty}\frac{(H_n)^2}{(2n)^3-2n} = \zeta(2)-\frac{\zeta(3)}{2}+\big(4-2\zeta(2)\big)\log(2)-\log^2(4)+\frac{\log^3(4)}{3}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.2444 Find a closed form of:

$$\Omega = \int_{-\infty}^{\infty}\left(\frac{1}{2}-\frac{\sin^2(x)\sin^{-1}\left(\frac{\sinh(x+\tan x-\csc x-\sec x)}{\sqrt{\cosh\left(2x+\tan\left(\frac{x}{2}\right)-\cot\left(\frac{x}{2}\right)\right)}}\right)}{(1-\cos(2x))x-\sin(2x)}\right)\frac{\sin^4(x)}{\big((1-\cos(2x))x-\sin(2x)\big)^2}\,dx.$$

Proposed by Toubal Fethi -Algeria

U.2445 Prove that:

$$\Omega = \int_0^1\left(\arccos(x^2)+\ln\left(\sqrt{\frac{1}{1+x}+\frac{1}{1-x}}\right)\right)dx = 1-\frac{\ln(2)}{2}+\frac{2\sqrt{\pi}\,\Gamma\left(\frac{7}{4}\right)}{3\Gamma\left(\frac{5}{4}\right)}$$

Proposed by Tahirov Shirvan-Azerbaijan

U.2446 Show that

$$\int_0^1 \left(\frac{\pi^2 x}{12\ln(x)} - \frac{Li_2(-x)}{1-x}\right)dx = \frac{\pi^2}{12}(\Gamma + \ln(2)) - \frac{5}{8}\zeta(3)$$

Proposed by Vincent Nguyen-USA

U.2447 Find a closed form of:

$$\Omega = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{\Gamma(n+k+1)(2k-1)!!}{\Gamma\left(n+k+\frac{3}{2}\right)\Gamma(k+1)\sqrt{2^{n+3k}}}\right)^2$$

Proposed by Fethi Toubal-Algeria

U.2448 Find:

$$\Omega = \int_0^{\infty} \frac{\ln(1+x)}{x(x^4-x^2+1)}dx$$

Proposed by Vasile Mircea Popa – Romania

U.2449

$$\Omega(m;n;r) = \int_0^1 \frac{mr\tan^{-1}\frac{1}{n\sqrt{1+m^2x^2}}}{(1+(1+r^2)m^2x^2)\sqrt{1+m^2x^2}}dx$$

$$\Omega(m;n;r) + \Omega\left(\frac{1}{n};\frac{1}{m};\frac{1}{r}\right) = \tan^{-1}\frac{\sqrt{1+r^2}}{nr}\tan^{-1} m\sqrt{1+r^2}$$

Proposed by Hikmat Mammadov, Nuran Mammadi -Azerbaijan

U.2450 Inspired by professor Daniel Sitaru

If $X = \begin{pmatrix} a+2b & 2b \\ -b & a-2b \end{pmatrix}$ such that $X^3 + 9X = 6X^2 + 27I_2$ then find $a, b \in \mathbb{R}$.

Proposed by Hikmat Mammadov-Azerbaijan

U.2451 Prove the integral

$$\int_{-\infty}^{\infty} \frac{\tanh(2x)}{\left(\sinh^2\left(\frac{x}{2}\right)+1\right)^2}\frac{dx}{x} = 8\left(8\log\left(4-2\sqrt{2}\right) - \frac{7\zeta(3)}{\pi^2}\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

U.2452 Show that

$$\int_0^{\infty} \frac{\left(\sqrt{x^4+1}-x^2\right)\log(x)}{\sqrt[3]{x}}dx = \frac{\sqrt{3}\left(\frac{2\pi}{\sqrt{3}}+3+4\log(2)\right)\Gamma\left(-\frac{2}{3}\right)\Gamma\left(\frac{1}{3}\right)^2}{64\pi\sqrt[3]{2}}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.2453 Show that

$$\int_0^1 \int_0^1 \left\{\frac{2x}{y}\right\}\left\{\frac{2y}{x}\right\}\left\lfloor\frac{2x}{y}\right\rfloor\left\lfloor\frac{2y}{x}\right\rfloor dx\,dy = \frac{127}{36} + 2\ln\left(\frac{3}{16}\right)$$

where $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part.

Proposed by Vincent Nguyen-USA

U.2454 Prove the below closed form

$$\int_0^\infty \int_0^\infty \left(1 - \frac{1}{\sqrt{x}}\right)\left(1 - \frac{1}{\sqrt{y}}\right)\left(1 - \frac{x^2}{1+x^2}\right)\left(1 - \frac{y^2}{1+y^2}\right)\frac{\ln(x)}{\ln(y)}dx\,dy = \frac{\pi^2}{2\sqrt{2}}\ln\left(1+\sqrt{2}\right)$$

Proposed by Ankush Kumar Parcha-India

U.2455 Find the value of a:

$$\sqrt{a} = 6 + \cfrac{11}{12 + \cfrac{11}{12 + \cfrac{11}{12 + \cfrac{11}{12 + \cdots}}}}$$

Proposed by Jay Jay Oweifa-Nigeria

U.2456 Show that

$$\sum_{n=1}^{\infty}(-1)^n\, n\left(H_n - \log\left(n + \frac{1}{3}\right) - \gamma - \frac{1}{6n}\right) =$$

$$= \frac{\gamma}{4} - \log(A) - \frac{1}{12} - \frac{5\pi}{18\sqrt{3}} + \frac{19\log(2)}{36} + \frac{\log(\pi)}{6} - \frac{7\log(3)}{24} + \frac{5\psi^{(1)}\left(\frac{1}{3}\right)}{12\sqrt{3}\pi} - \frac{\log(\varpi_6)}{3}$$

where γ is the Euler – Mascheroni constant, A is the Glaisher-Kinkelin constant, H_n denotes the n – th harmonic number, $\psi^{(1)}$ denotes the trigamma function, and ϖ_n: $\frac{2\sqrt{\pi}\Gamma\left(1+\frac{1}{n}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}$ denotes the n – th clover constant

Proposed by Vincent Nguyen-USA

U.2457 Show that:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\, n\left(H_n - \log\sqrt{n(n+1)} - \gamma\right) = \frac{1}{4}\left(1 - \gamma + \log\left(\frac{\pi}{4}\right)\right)$$

where γ is the Euler – Mascheroni constant and H_n is the n – th harmonic number.

Proposed by Vincent Nguyen-USA

U.2458 Find:

$$\Omega=\int_0^{\infty}\frac{\arctan(x)}{x(x^4+x^2+1)}dx$$

Proposed by Vasile Mircea Popa – Romania

U.2459 If we have the recurrence relation $\varphi(n+1)+\varphi(n+2)+\varphi(n)=(-1)^{\frac{n(n+1)}{2}}$

$\varphi(0)=1,\varphi(1)=-1$ then show that

$$\sum_{n=1}^{\infty}\frac{(-1)^{\frac{n(n+1)}{2}}\varphi(n)}{\frac{1}{2}n(n+1)}=\pi+2-\frac{2\pi}{\sqrt{3}}+\log\left(\frac{1}{16}\left(2+\sqrt{3}\right)^{\frac{2}{\sqrt{3}}}\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

U.2460 If $\int_0^1\frac{L^2(x)}{x}dx=\frac{\zeta(3)}{\eta(a)}-\frac{\zeta(5)}{\zeta(b)}$. Prove that:

$$\sum_{n=1}^{\frac{a+b}{2}}\int_0^{\frac{\pi}{2}}\log(\cos(x))\sin^n(2x)\,dx=\frac{\pi}{16}-\frac{\pi\ln(2)}{4}-\frac{7}{9}$$

Proposed by Amin Hajiyev-Azerbaijan

U.2461 Prove that:

$$\int_0^{\infty}\left(\frac{1-e^{2x}}{1+e^{2x}}\right)^2\frac{e^{-x}-e^{-3x}}{x}dx=\log_e\left(\frac{12288\pi^8e^{\frac{8G}{\pi}}}{\Gamma^{16}\left(\frac{1}{4}\right)}\right)$$

Proposed by Amin Hajiyev-Azerbaijan

U.2462 Prove:

$$\int_0^1\frac{\ln\ln\frac{1}{x}}{(x+1)^5}dx=$$

$$=\frac{15}{32\pi^4}\zeta(4)+\frac{7}{16\pi^2}\zeta(3)+\frac{11}{8}\ln A-\frac{271}{1152}\gamma+\frac{23}{192}\ln\pi-\frac{799}{2880}\ln 2-\frac{121}{1152}$$

Proposed by Fao Ler-Iraq

U.2463 Determine the value of α

$$\int_0^1\int_0^1\frac{dx\,dy}{\alpha^2-\alpha(x+y-xy)-(x+y)}=Li_2\left(-\frac{1}{3}\right)+\frac{\pi^2}{12}+\frac{\ln^2(3)}{2}$$

Proposed by Ankush Kumar Parcha-India

U.2464 Show that

$$\int_0^{\infty}\frac{e^{3z}-e^z}{z(e^{2z}+1)^2}dz=\frac{2G}{\pi}$$

Proposed by Vincent Nguyen-USA

U.2465 If

$$\Omega:=\int_1^2\int_2^3\int_3^1\frac{\frac{xy}{(1+x)(2+y)}}{\frac{(2+x)(1+y)}{yz}}\cdot\frac{\frac{yz}{(1+y)(2+z)}}{\frac{(2+y)(1+z)}{zx}}\cdot\frac{\frac{zx}{(1+z)(3+x)}}{\frac{(2+z)(1+x)}{xy}}dx\,dy\,dz$$

Then, show that:

$$\Omega=\frac{789}{5}\ln^2(3)-432\ln^3(2)-216\ln(2)\ln^2(3)+648\ln^2(2)\ln(3)+\frac{1017}{5}\ln^2(2)-$$

$$-\frac{2256}{5}\ln(2)\ln(3)-\frac{9731}{600}\ln(3)+\frac{66131}{600}\ln(2)-\frac{29719}{1440}$$

Proposed by Ankush Kumar Parcha-India

U.2466 Find:

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\sum_{i=1}^{k}i\left(i+\frac{1}{3}\right)\right]^{-1}$$

Proposed by Vasile Mircea Popa – Romania

U.2467 Show that

$$\lim_{n\to\infty}16^n n^{2n+1}\,a_n=\frac{e^{\frac{2G}{\pi}-1}}{4}$$

where G is Catalan's constant.

Proposed by Vincent Nguyen-USA

U.2468 Let $u_1=\frac{2023}{2024}, nu_{n+1}=(n+1)u_n+n+2, (n\in\mathbb{N}, n\geq 1)$.Find: $\lim\limits_{n\to+\infty}\frac{u_n}{2023}$

Proposed by Nguyen Van Canh-Vietnam

U.2469 Prove the below closed form

$$\int_0^1\int_0^1\int_0^1\frac{(1+xyz)(3+xyz)(5+xyz)}{(2+xyz)(4+xyz)(6+xyz)}dx\,dy\,dz=$$

$$=1+\frac{3}{8}Li_3\left(-\frac{1}{2}\right)+\frac{3}{4}Li_3\left(-\frac{1}{4}\right)+\frac{15}{8}Li_3\left(-\frac{1}{6}\right)$$

Where, $Li_3(z)$ is a trilogarithm function

Proposed by Ankush Kumar Parcha-India

U.2470 If $a,b,c > 0, a^2 + b^2 + c^3 = 3abc$ then find max of

$$Q = \sum \frac{a}{2a^3 + 1}$$

Proposed by Marin Chirciu – Romania

U.2471 In ΔABC, g_a —Gergonne's cevian, $M \in Int(\Delta ABC)$ holds:

$$xyh_a g_b + yzh_b g_c + zxh_c g_a \geq 4F^2$$

Proposed by D.M.Bătinețu-Giurgiu, Gheorghe Boroica-Romania

U.2472 If $t,u \geq 0$ and $x,y,z > 0$, then in ΔABC holds:

$$\frac{x+y}{z} \cdot a^t b^u + \frac{y+z}{x} \cdot b^t c^u + \frac{z+x}{y} \cdot c^t a^u \geq 2^{1+t+u} \left(\sqrt[4]{3}\right)^{4-t-u} \left(\sqrt{F}\right)^{t+u}$$

Proposed by D.M.Bătinețu-Giurgiu, Flaviu Cristian Verde-Romania

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ROMANIAN MATHEMATICAL MAGAZINE-R.M.M.-AUTUMN 2024

PROBLEMS FOR JUNIORS

JP.496 In ΔABC the following relationship holds:

$$2(4R+r)^2 \geq \frac{r_a^7 + r_b^7}{(r_a + r_b)^5} + \frac{r_b^7 + r_c^7}{(r_b + r_c)^5} + \frac{r_c^7 + r_a^7}{(r_c + r_a)^5} \geq 6s^2$$

Proposed by Alex Szoros-Romania

JP.497 If $a,b,c>0$, then:

$$\frac{a^3}{b^2}+\frac{b^3}{c^2}+\frac{c^3}{a^2}\ge\frac{1}{2}\left(\sqrt[3]{7a^3+b^3}+\sqrt[3]{7b^3+c^3}+\sqrt[3]{7c^3+a^3}\right)$$

Proposed by Marin Chirciu-Romania

JP.498 If $a,b>0$ then:

$$\sum_{cyc}\frac{(a+1)(b+1)}{a+b+2}\ge\frac{3}{2}+\sum_{cyc}\frac{ab}{a+b}$$

Proposed by Daniel Sitaru-Romania

JP.499 Find $\lambda>0$ so that the double inequality

$$R^2\ge\frac{(a+b+c)^3-a^3-b^3-c^3}{\lambda(a+b+c)}\ge 2Rr$$

holds in any triangle ABC.

Proposed by Alex Szoros-Romania

JP.500 If $a,b,c>0, abc=1$ then:

$$\frac{a^2+1}{a+1}+\frac{b^2+1}{b+1}+\frac{c^2+1}{c+1}\ge 3$$

Proposed by Daniel Sitaru-Romania

JP.501 If $a,b,c>0$, then:

$$\frac{a^{10}}{(b+c)^5(2a+b+c)^5}+\frac{b^{10}}{(c+a)^5(a+2b+c)^5}+\frac{c^{10}}{(a+b)^5(a+b+2c)^5}\ge\frac{3}{8^5}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

JP.502 If $m,n\ge 0, m+n=4$ and $x,y,z>0$ then in ΔABC holds:

$$\frac{x\cdot a^m}{(y+z)h_a^n}+\frac{y\cdot b^n}{(z+x)h_b^n}+\frac{z\cdot c^n}{(x+y)h_c^n}\ge 2^{m-1}\cdot F^{2-n}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

JP.503 In acute ΔABC the following relationship holds:

$$\sum_{cyc}\frac{a+b}{ab}\cdot h_c\ge\frac{a+b+c}{R}$$

Proposed by Marian Ursărescu-Romania

JP.504 In ΔABC the following relationship holds:

$$\frac{A}{\pi A+BC+12}+\frac{B}{\pi B+CA+12}+\frac{C}{\pi C+AB+12}\le\frac{3+\pi}{32}$$

Proposed by Radu Diaconu-Romania

JP.505 For $n\in\mathbb{N}^*, x\in(0,\pi)$ prove that:

$$\sum_{k=1}^{n}\left(\frac{4k+(k-1)^2\sin^2 2x}{(k-1)^2+4k\csc^2 2x}\right)\le(n+1)^2$$

Proposed by Florică Anastase, Flavius Pacionea-Romania

JP.506 For $a,b,c>0, a+b+c=1$ and $k\in\mathbb{N}$ prove:

$$\frac{a}{(a^2+abc)^{k+1}}+\frac{b}{(b^2+abc)^{k+1}}+\frac{c}{(c^2+abc)^{k+1}}\ge\left(\frac{27}{4}\right)^{k+1}$$

Proposed by Florică Anastase, Andreea Lixandru-Romania

JP.507 Let $ABCD$ be a cyclic quadrilateral with circumradius R

and area F. Prove: $\frac{\sum\sec^2\frac{A}{2}}{\sum\sec^{-2}\frac{A}{2}}\le\frac{16R^4}{F^2}$

where the sums are taken over all angles of the quadrilateral.

Proposed by George Apostolopoulos-Greece

JP.508 Let $ABCD$ be a cyclic quadrilateral with circumradius R and area F. Prove

$$\csc A+\csc B+\csc C+\csc D\le\frac{8R^2}{F}$$

Proposed by George Apostolopoulos-Greece

JP.509 Let ABC be a triangle with inradius r and circumradius R. Prove that:

$$\left(1+\cot\frac{A}{2}\right)\left(1+\cot\frac{B}{2}\right)\left(1+\cot\frac{C}{2}\right)\ge\left(1+\sqrt{3}\cdot\frac{2r}{R}\right)^3$$

Proposed by George Apostolopoulos-Greece

JP.510 In ΔABC the following relationship holds:

$$9\sqrt{2}r\le\sum_{cyc}\sqrt{r_a^2+r_b^2}\le(4R+r)\sqrt{6\left(\frac{R-r}{R+r}\right)}$$

Proposed by Marin Chirciu-Romania

PROBLEMS FOR SENIORS

SP.496 If $f:\mathbb{R}\to\mathbb{R}$, f −derivable, $f(0)=0, f'(0)=1$ then find:

$$\Omega=\lim_{n\to\infty}\left(\lim_{x\to 0}\frac{1}{x}\left(f(x)+f\left(\frac{x}{2}\right)+f\left(\frac{x}{3}\right)+\cdots+f\left(\frac{x}{n}\right)-lnn\right)\right)$$

Proposed by Marin Chirciu-Romania

SP.497 Let $D\ ABC$ be a triangle with circumradius R. Let r_a, r_b, r_c be the exradii. Prove that:

$$\frac{r_a^4}{\sin(2A)}+\frac{r_b^4}{\sin(2B)}+\frac{r_c^4}{\sin(2C)}\geq\frac{81\sqrt{3}}{8}R^4$$

Proposed by George Apostolopoulos-Greece

SP.498 Let $A, B\in M_3(\mathbb{R})$ such that $AB=BA$. Prove that

$\det(A^2+B^2)=0$ if and only if $\det(A+B)=2(\det A+\det B)$ and

$$\det(A-B)=2(\det A-\det B).$$

Proposed by Florentin Vișescu-Romania

SP.499 Let $A, B\in M_3(\mathbb{R})$ such that $AB=BA$. Prove that

$\det(A^2+AB+B^2)=0$ if and only if $\det(A+B)=\det A+\det B$ and $\det(A-B)=3(\det A-\det B)$.

Proposed by Florentin Vișescu-Romania

SP.500 Find:

$$\Omega=\int_0^{\frac{\pi}{2}}\frac{3\cos x-4\sin x}{12\sin 2x+7\cos^2 x+4\cos x+3\sin x+9}dx$$

Proposed by Daniel Sitaru-Romania

SP.501 If $M\in Int(\Delta ABC)$ such that $x=MA, y=MB, z=MC$, then:

$$\frac{x}{h_a h_b\sqrt{yz}}+\frac{y}{h_b h_c\sqrt{zx}}+\frac{z}{h_c h_a\sqrt{xy}}\geq\frac{\sqrt{3}}{F}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

SP.502 If $x, y, z>0$, then:

$$(2x^2+1)(2y^2+1)(2z^2+1)\geq\frac{9}{2}(xy+yz+zx)$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți-Romania

SP.503 Solve for real positive numbers:

$$e\cdot(x^x+(1+\log x)^2)^e=1$$

Proposed by Daniel Sitaru-Romania

SP.504 If $x,y,z>0,\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=3$, then:

$$4\sum_{cyc}x^2+9\ge\sum_{cyc}\left(5x+\frac{4}{x+1}\right)$$

Proposed by Daniel Sitaru-Romania

SP.505 If $x,y,z>0,\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{y}}+\frac{1}{\sqrt{z}}=3$, then:

$$x^2\sqrt{x}+y^2\sqrt{y}+z^2\sqrt{z}\ge x^2+y^2+z^2$$

Proposed by Daniel Sitaru-Romania

SP.506 Determine the sequence $(a_n)_{n\ge1},a_n\in\mathbb{R}^*$ such that:

$$\binom{n-1}{0}\cdot\binom{n-1}{1}a_1+\binom{n-1}{1}\cdot\binom{n}{2}a_2+\cdots+\binom{n-1}{n-1}\cdot\binom{n}{n}a_n=\left[\binom{2n}{n}-1\right]a_n$$

$$(\forall)n\ge1$$

Proposed by Marian Ursărescu-Romania

SP.507 In acute ΔABC the following relationship holds:

$$\sum_{cyc}\frac{a+b}{ab}\cdot r_c\ge\frac{a+b+c}{R}$$

Proposed by Marian Ursărescu-Romania

SP.508 Let $A_1A_2\dots A_n$ a convex polygon, $n\in\mathbb{N},n\ge3$. Prove that:

$$\sum_{i=1}^{n}\left(\frac{A_i^{p+1}+1}{A_i^p+1}\right)^k\ge\frac{[(n-2)\pi+n]^k}{2^k\cdot n^{k-1}},p,k\in\mathbb{N}$$

Proposed by Radu Diaconu-Romania

SP.509 Let ABC be an acute triangle. Prove that:

$$\left(9\sqrt{3}\cot^3A+2\right)\left(9\sqrt{3}\cot^3B+2\right)\left(9\sqrt{3}\cot^3C+2\right)\ge125$$

Proposed by George Apostolopoulos-Greece

SP.510 Let ABC be an arbitrary triangle having the sides a, b, c. Denote by m_a, s_a the lengths of the median and the symmedian corresponding to the side a, and the analogs. Let ω Brocard's angle and M be the set

$$M = \left\{\frac{m_a}{s_a}, \frac{m_b}{s_b}, \frac{m_c}{s_c}\right\}. \text{ Prove that:}$$

$$\frac{2R}{r}\max M \geq \frac{1}{\sin^2 \omega} \geq \frac{2R}{r}\min M$$

Proposed by Vasile Jiglău-Romania

UNDERGRADUATE PROBLEMS

UP.496 Prove that:

$$\int_0^1 \int_0^1 \frac{\log(1+x^2)\log(1+y)}{xy^{\frac{3}{2}}} dxdy = \frac{\zeta(2)}{4}(\pi - 2\log(2))$$

Proposed by Said Attaoui– Algerie

UP.497 Prove that:

$$F(x) = \int \frac{x-1}{x^2}\log\left(1+\frac{1}{x^2}\right)dx = \frac{1}{2}Li_2\left(-\frac{1}{x^2}\right) + \frac{\log\left(1+\frac{1}{x^2}\right)}{x} - \frac{2}{x} + 2\arctan\left(\frac{1}{x}\right)$$

Deduce

$$\int_1^\infty \frac{x-1}{x^2}\log\left(1+\frac{1}{x^2}\right)dx$$

Proposed by Said Attaoui –Algerie

UP.498 Prove that:

$$\int_0^1 \int_0^1 \frac{(1+2x-x^2)(1-2y+y^2)}{\sqrt{x^2y^2-2yx^2-2xy^2+4xy}} dxdy = \frac{9}{8}\zeta(2)$$

Proposed by Said Attaoui-Algerie

UP.499 If $0 < a \leq b$ then:

$$\int_a^b \int_a^b \frac{dxdy}{1+xy} \leq \frac{2(b-a)^2}{(1+a)(1+b)}$$

Proposed by Daniel Sitaru-Romania

UP.500 Let $0 < a < b$ and $f: [a,b] \to \mathbb{R}$. If f −differentiable on $[a,b]$ and $f(a) = f(b)$, then $(\exists)\ c_1, c_2 \in (a,b)$ such that $af(c_1) + bf(c_2) = 0$.

Proposed by Marian Ursărescu-Romania

UP.501 Determine all functions $f:\mathbb{R}\to(0,\infty)$ such that

$$f(x)\cdot f(3x)\cdot f(9x)\cdot f(27x)=3^x,(\forall)\ x\in\mathbb{R}$$

Proposed by Marian Ursărescu-Romania

UP.502 Let $A_1A_2\dots A_n$ a convex polygon, $n\geq 3, n\in\mathbb{N}$. Prove that:

$$\begin{vmatrix} -1 & a & a & \dots & a \\ a & -1 & a & \dots & a \\ a & a & -1 & \cdots & a \\ \dots & \dots & \dots & \cdots & \dots \\ a & a & a & \cdots & -1 \end{vmatrix}\cdot\begin{vmatrix} -1 & b & b & \dots & b \\ b & -1 & b & \dots & b \\ b & b & -1 & \cdots & b \\ \dots & \dots & \dots & \cdots & \dots \\ b & b & b & \cdots & -2 \end{vmatrix}=$$

$$=(2p+1)^{n-1}[(n-1)2p-1][(n-2)\pi+1]^{n-1}[(n-1)(n-2)\pi-1].$$

where $a=a_1+a_2+\dots+a_n=2p,(n-1)2p-1>0, b=\widehat{A_1}+\widehat{A_2}+\dots+\widehat{A_n}$ and the order of the determinants is n.

Proposed by Radu Diaconu-Romania

UP.503 Let $f:[n-1,n]\to[n,n+1]$ continuous function such that

$$\int_{n-1}^{n}\left(1+xf'(x)\right)dx\leq nf(n)-(n-1)f(n-1),\text{then prove}:$$

$$\int_{n-1}^{n}\frac{dx}{f(x)}\leq\frac{2}{n+1},n\in\mathbb{N}^*$$

Proposed by Florică Anastase-Romania

UP.504 Let $a,b,c,d>1$ and $f:[a,b]\to[c,d]$ continuous function for which

$$(\exists)\lambda\in(a,b)\text{ such that }a\int_a^{\lambda}f(x)\,dx+b\int_{\lambda}^{b}f(x)\,dx\geq a+c,\text{then prove}:$$

$$\int_a^b\frac{x}{f(x)}dx\leq\left(\frac{1}{a}+\frac{1}{c}\right)\cdot\frac{b^2-a^2-2}{2}$$

Proposed by Florică Anastase-Romania

UP.505 Find:

$$\Omega=\lim_{x\to\infty}\left([2x](2x)^{\frac{1}{[2x]}}-[x]x^{\frac{1}{[x]}}-\left[x+\frac{1}{2}\right]\right)$$

where $[a]$ is the greatest integer less than a.

Proposed by Cristian Miu-Romania

UP.506 Let R and r be the circumradius and inradius, respectively, of triangle ABC. Let D, E and F be chosen on sides BC, CA and AB so that AD, BE, and CF bisect the angle of ABC. Prove that:

$$\left(\frac{EF}{BC}\right)^4+\left(\frac{FD}{CA}\right)^4+\left(\frac{DE}{AB}\right)^4+\frac{3}{16}\le\frac{3}{8}\left(\frac{R}{2r}\right)^2$$

Proposed by George Apostolopoulos-Messolonghi-Greece

UP.507 Prove that:

$$\int_0^1\int_0^1\frac{\log x}{(x^2-1)(1+x^2y)^2}dxdy=\frac{1}{2}G+\frac{3}{8}\zeta(2)$$

where $G=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$ is Catalan′s constant and $\zeta(k)=\sum_{n=1}^{\infty}\frac{1}{n^k}$ is a zeta function.

Find the value of the series: $\Omega=\sum_{n=0}^{\infty}\frac{1}{(4n+1)^2}$.

Proposed by Said Attaoui-Algerie

UP.508 Find: $\Omega=\lim_{n\to\infty}e^{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}\cdot\left(\sqrt[n]{\pi^3}-\sqrt[n]{e^3}\right)$

Proposed by D.M.Bătinețu-Giurgiu, Daniel Sitaru-Romania

UP.509 Find:

$$\Omega=\lim_{n\to\infty}\left(e^{3H_{n+1}}-e^{3H_n}\right)\left(\sqrt[n]{\pi}-1\right)^2$$

Proposed by D.M.Bătinețu-Giurgiu, Daniel Sitaru-Romania

UP.510 Find:

$$\Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{n^4}\frac{1}{n^2+k}\right)\cdot\frac{1}{\log n}$$

Proposed by D.M.Bătinețu-Giurgiu, Daniel Sitaru-Romania

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INDEX OF AUTHORS RMM-42

Nr.crt.	Numele și prenumele	Nr.crt.	Numele și prenumele
1	DANIEL SITARU-ROMANIA	24	MARIA CARINA VIESPESCU-ROMANIA
2	D.M.BĂTINEȚU-GIURGIU-ROMANIA	25	MIHAI IONESCU-ROMANIA
3	ALINA TIGAE-ROMANIA	26	OLIVIA BERCEA-ROMANIA
4	CRISTINA ENE-ROMANIA	27	JIPESCU ANA-ROMANIA
5	DOINA CRISTINA CĂLINA-ROMANIA	28	CARMEN TERHECI-ROMANIA
6	DANIELA DÎRNU-ROMANIA	29	SORINA TUDOR-ROMANIA
7	GILENA DOBRICĂ-ROMANIA	30	CORINA IONESCU-ROMANIA
8	OANA SIMONA DASCĂLU-ROMANIA	31	CARMEN VLAD-ROMANIA
9	DORINA GOICEANU-ROMANIA	32	DANIELA BARBU-ROMANIA
10	RAMONA NĂLBARU-ROMANIA	33	GIGI ZAHARIA-ROMANIA
11	ILEANA DUMA-ROMANIA	34	MIHAELA STĂNCELE-ROMANIA
12	LUIZA DUMITRESCU-ROMANIA	35	CĂTĂLINA STAN-ROMANIA
13	ILEANA STANCIU-ROMANIA	36	CRISTIAN CATANĂ-ROMANIA
14	LAVINIA TRINCU-ROMANIA	37	SEBASTIAN ILINCA-ROMANIA
15	MIHAELA NASCU-ROMANIA	38	SIMONA CHIRIȚĂ-ROMANIA
16	LUIZA CREMENEANU-ROMANIA	39	IONUȚ IVĂNESCU-ROMANIA
17	ROXANA VASILE-ROMANIA	40	RAREȘ TUDORAȘCU-ROMANIA
18	ELENA GRIGORE-ROMANIA	41	MARIA LAVINIA POPA-ROMANIA
19	MEDA IACOB-ROMANIA	42	SORIN PÎRLEA-ROMANIA
20	SIMONA RADU-ROMANIA	43	NICOLAE RADU-ROMANIA
21	MIHAELA DĂIANU-ROMANIA	44	MIOARA MIHAELA MIREA-ROMANIA
22	ELENA ALEXIE-ROMANIA	45	CĂTĂLIN PANĂ-ROMANIA
23	IULIA SANDA-ROMANIA	46	ALECU ORLANDO-ROMANIA

47	ELENA NEDELCU-ROMANIA	90	DAN MITRICOIU-ROMANIA
48	CAMELIA DANĂ-ROMANIA	91	CĂTĂLIN NICOLA-ROMANIA
49	MIHAELA DUȚĂ-ROMANIA	92	CLAUDIU CIULCU-ROMANIA
50	CLAUDIA NĂNUȚI-ROMANIA	93	LAURA ZAHARIA-ROMANIA
51	NECULAI STANCIU-ROMANIA	94	MONICA VELEA-ROMANIA
52	MARIAN URSĂRESCU-ROMANIA	95	MARIN LAURA NICOLETA-ROMANIA
53	BOGDAN FUȘTEI-ROMANIA	96	DAN GRIGORIE-ROMANIA
54	DAN NĂNUȚI-ROMANIA	97	ANDREEA LIXANDRU-ROMANIA
55	MARIN CHIRCIU-ROMANIA	98	FLORENTIN VIȘESCU-ROMANIA
56	RADU DIACONU-ROMANIA	99	VASILE JIGLĂU-ROMANIA
57	ALEX SZOROS-ROMANIA	100	FLAVIU CRISTIAN VERDE-ROMANIA
58	DANIEL HARDISKY-USA	101	PANAGIOTIS DANOUSIS-GREECE
59	CRISTIAN MIU-ROMANIA	102	BENNY LE VAN-VIETNAM
60	ILIR DEMIRI-AZERBAIJAN	103	KHALED ABD IMOUTI-SYRIA
61	SRINIVASA RAGHAVA-INDIA	104	NAREN BHANDARI-NEPAL
62	ELSEN KERIMOV-AZERBAIJAN	105	HOANG LE NHAT TUNG-VIETNAM
63	ALBERT STADLER-SWITZERLAND	106	SAKTHI VEL-INDIA
64	TOGRUL EHMEDOV-AZERBAIJAN	107	OSE FAVOUR-NIGERIA
65	MURAD SHIKMATOV-AZERBAIJAN	108	TRAN QUOC THINH-VIETNAM
66	MOHAMED HAMED NASERY-AFGHANISTAN	109	HIKMAT MAMMADOV-AZERBAIJAN
67	LE THU-VIETNAM	110	GHEORGHE BOROICA-ROMANIA
68	ANKUSH KUMAR PARCHA-INDIA	111	NICOLAE MUȘUROIA-ROMANIA
69	LAZAROS ZACHARIADIS-GREECE	112	BRIAN D. BEASLEY-USA
70	IZUMI AINSWORTH-JAPAN	113	VINCENT NGUYEN-USA
71	SHIRVAN TAHIROV-AZERBAIJAN	114	NURAN MAMMADI-AZERBAIJAN

72	BASIR AHMED ALIZADA-AFGHANISTAN	115	JAY JAY OWEIFA-NIGERIA
73	NGUYEN VAN CANH-VIETNAM	116	FAO LER-IRAQ
74	SAID ATTAOUI-ALGERIA	117	GEORGE APOSTOLOPOULOS-GREECE
75	TOUBAL FETHI-ALGERIA	118	IONEL TUDOR-ROMANIA
76	PHAN NGOC CHAU-VIETNAM	119	ZAZA MZHAVANADZE-GEORGIA
77	MOHAMED AMINE BEN AJIBA-MOROCCO	120	SIDI ABDALLAH LEMRABOTT-MAURITANIA
78	FLORICĂ ANASTASE-ROMANIA	121	MEHMET ȘAHIN-TURKIYE
79	JALIL HAJIMIR-CANADA	122	AMIN HAJIYEV-AZERBAIJAN
80	HYUNBIN YOO-SOUTH KOREA	123	TRAN QUOC ANH-VIETNAM
81	ANDREW SIEFKER-USA	124	MOHAMMED BOURAS-MOROCCO
82	THE EAGLE PROBLEM SOLVERS-USA	125	PHAM DUC NAM-VIETNAM
83	DIONNE BAILEY –USA	126	VASILE MIRCEA POPA-ROMANIA
84	ELSIE CAMPBELL-USA	127	GEORGE FLORIN ȘERBAN-ROMANIA
85	CHARLES DIMINNIE-USA	128	MARIUS DRĂGAN-ROMANIA
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87	DAVID A. HUCKABY-USA	130	KUNIHIKO CHIKAYA-JAPAN
88	MICHEL BATAILLE-FRANCE	131	GHEORGHE STOICA-ROMANIA
89	FLAVIUS PACIONEA-ROMANIA	132	DORIN MĂRGHIDANU-ROMANIA

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