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SOME GENERALIZATIONS FOR LANGLEY'S PROBLEM

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Abstract: This paper presents other generalizations for Langley's problem. **Keywords: isosceles triangle, adventitious angles.MSC: 51M04**

Langley's Adventitious Angles is a puzzle in which one must infer an angle in a geometric diagram from other given angles. It was posed by [Edward Mann Langley] in *The Mathematical Gazette* in 1922.

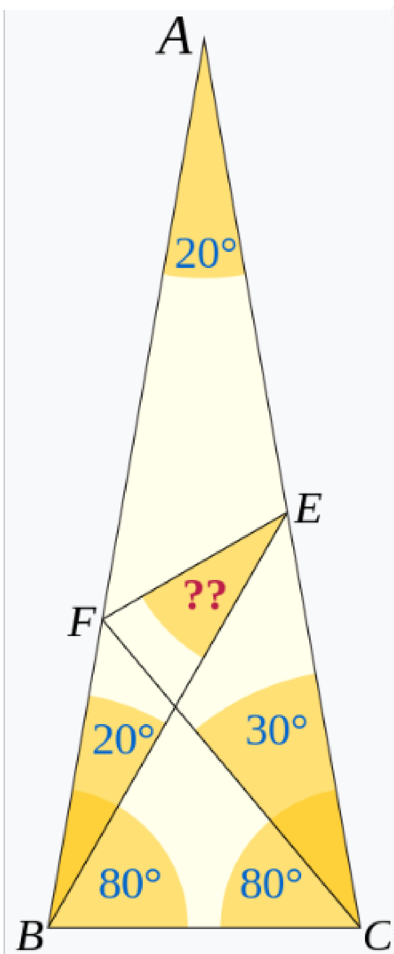
In its original form the problem was as follows:

ABC is an isosceles triangle with $\angle CBA = \angle ACB = 80^\circ$.

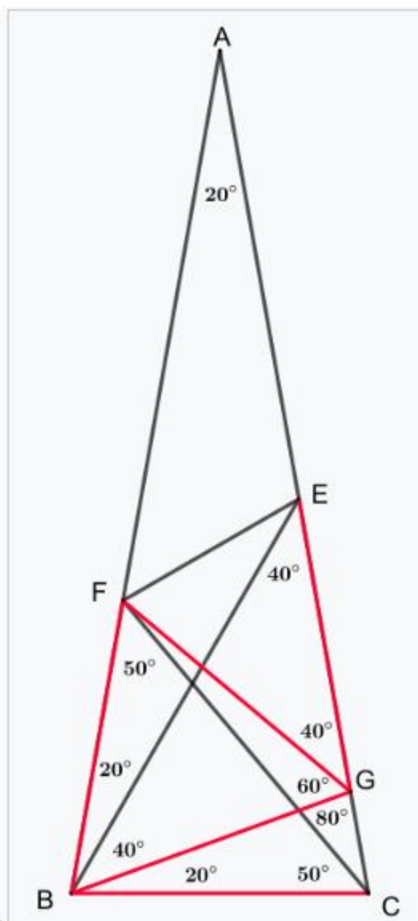
CF at 30° to AC cuts AB in F.

BE at 20° to AB cuts AC in E.

Prove $\angle BEF = 30^\circ$. [1][2][3]



Langley's Adventitious Angles

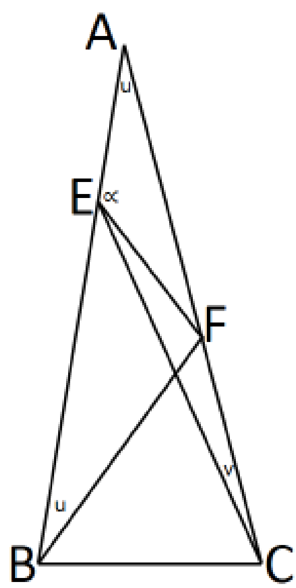


Solution to Langley's 80-80-20 triangle problem

Generalizations of Langley's problem. If ABC **is a isosceles triangle with** $AB = AC$**,** $E \in (AB)$**,** $F \in (AC)$ **such that** $\angle ABF = \angle FAB = u$ **and** $\angle ACE = v$**, then:**
$\alpha = \angle AEF = \arcsin\ f(u,v)$ **or** $\alpha = \angle AEF = 180^0 - \arcsin\ f(u,v)$**, where**

$$f(u,v) = \frac{\sin u \sin(u+v)}{\sqrt{4\cos^2 u \sin^2 v - 4\cos^2 u \sin v \sin(u+v) + \sin^2(u+v)}}.$$

Proof. $AB = AC \overset{not.}{=} y$**,** $BF = AF \overset{not.}{=} x$**,** $\angle AEF \overset{not.}{=} \alpha$**.**



By Sine Law in ΔAEC **:** $\frac{y}{\sin(u+v)} = \frac{AE}{\sin v} \Leftrightarrow AE = \frac{y \sin v}{\sin(u+v)}$.

By Cosine Law in $\Delta AEF : EF^2 = AE^2 + AF^2 - 2 \cdot AE \cdot AF \cdot \cos u \Leftrightarrow$

$$\Leftrightarrow EF^2 = \frac{y^2 \sin^2 v}{\sin^2(u+v)} + x^2 - \frac{2xy \sin v \cos u}{\sin(u+v)} \Leftrightarrow \frac{EF^2}{x^2} = \left(\frac{y}{x}\right)^2 \frac{\sin^2 v}{\sin^2(u+v)} + 1 - \frac{2y}{x} \cdot \frac{\sin v \cos u}{\sin(u+v)}.$$

By Sine Law in ΔABF **:** $\frac{y}{\sin 2u} = \frac{x}{\sin u} \Leftrightarrow \frac{y}{x} = 2\cos u$.

$$\frac{EF^2}{x^2} = \frac{4\cos^2 u \sin^2 v}{\sin^2(u+v)} + 1 - \frac{4\sin v \cos^2 u}{\sin(u+v)} = \frac{4\cos^2 u \sin^2 v + \sin^2(u+v) - 4\sin v \cos^2 u \sin(u+v)}{\sin^2(u+v)}.$$

By Sine Law in ΔAEF **:** $\frac{EF^2}{x^2} = \frac{\sin^2 u}{\sin^2 \alpha}$.

$$\frac{\sin^2 u}{\sin^2 \alpha} = \frac{4\cos^2 u \sin^2 v + \sin^2(u+v) - 4\sin v \cos^2 u \sin(u+v)}{\sin^2(u+v)}.$$

Hence: $\alpha = \angle AEF = \arcsin f(u,v)$ **or** $\alpha = \angle AEF = 180^0 - \arcsin f(u,v)$**, where**

$$f(u,v) = \frac{\sin u \sin(u+v)}{\sqrt{4\cos^2 u \sin^2 v - 4\cos^2 u \sin v \sin(u+v) + \sin^2(u+v)}}.$$

Next: If u **and** v **are by the form** $u = \overline{m0}^0$**,** $v = \overline{n0}^0$**;** $m,n \in \{1,2,...,8\}$ **we will determine the angles** α **which satisfy the Langley's generalizations from above, where** α **is of the form** $\overline{k0}^0$**,** $k \in \{1,2,...,8\}$**. Since** $u < \frac{180^0 - u}{2}$**, yields that** $u < 60^0$**, so** $m \in \{1,2,3,4,5\}$**.**

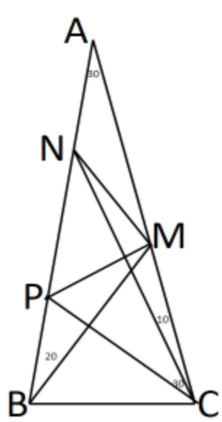
We will obtain the following cases:

1. $(u,v,\alpha) = (10^0,10^0,85^0)$ **- trivial;**
2. $(u,v,\alpha) = (20^0,10^0,180^0 - \arcsin f(20^0,10^0)) = (20^0,10^0,130^0)$**;**
3. $(u,v,\alpha) = (20^0,30^0,\arcsin f(20^0,30^0)) = (20^0,30^0,50^0)$ **- the problem of Langley;**
4. $(u,v,\alpha) = (20^0,50^0,\arcsin f(20^0,30^0)) = (20^0,50^0,30^0)$ **;**
5. $(u,v,\alpha) = (40^0,40^0,\arcsin f(40^0,40^0)) = (40^0,40^0,70^0)$ **- trivial;**
6. $(u,v,\alpha) = (50^0,50^0,\arcsin f(50^0,50^0)) = (50^0,50^0,65^0)$ **- trivial.**

For other cases $k \notin N^*$**, and** $(u,v,\alpha) = (20^0,80^0,\alpha)$ **and** $(u,v,\alpha) = (40^0,70^0,\alpha)$ **doesn't work.**

Problem. If ABC **is an isosceles triangle with** $\angle A = 20^0$**,** $N,P \in (AB)$**,** $M \in (AC)$**,** $\angle ABM = 20^0$**,** $\angle PCA = 30^0$ **şi** $\angle NCA = 10^0$**, then** $MN = MP$**.**

Solution.



By the case 2, yields $\angle ANM = 130^0$**, and by the case 3, yields** $\angle APM = 50^0$**.**

Hence, $\angle MNP = \angle APM = 50^0$**, so** $MN = MP$**.**

References:

WIKIPEDIA:

1. ^ a b Langley, E. M. (1922), "Problem 644", *The Mathematical Gazette*, **11**: 173.
2. ^ a b c Darling, David (2004), *The Universal Book of Mathematics: From Abracadabra to Zeno's Paradoxes*, John Wiley & Sons, p. 180, ISBN 9780471270478.
3. ^ Tripp, Colin (1975), "Adventitious angles", *The Mathematical Gazette*, **59** (408): 98–106, doi:10.2307/3616644, JSTOR 3616644.