SP.541 Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that:

$$f\left(\frac{x}{3}\right)-3f(x)=15x;\ (\forall)x\in\mathbb{R}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

Replace $x=0$ in statement:

$$f(0)-3f(0)=0\Rightarrow -2f(0)=0\Rightarrow f(0)=0$$

$$\frac{1}{3}f\left(\frac{x}{3}\right)-f(x)=5x\quad (1)$$

Replace $x\to\frac{x}{3}$ in (1):

$$\frac{1}{3}f\left(\frac{x}{3^2}\right)-f\left(\frac{x}{3}\right)=5\cdot\frac{x}{3}$$

$$\frac{1}{3^2}f\left(\frac{x}{3^2}\right)-\frac{1}{3}f\left(\frac{x}{3}\right)=5\cdot\frac{x}{3^2}\quad (2)$$

Replace $x\to\frac{x}{3}$ in (2):

$$\frac{1}{3^2}f\left(\frac{x}{3^3}\right)-\frac{1}{3}f\left(\frac{x}{3^2}\right)=5\cdot\frac{x}{3^3}$$

$$\frac{1}{3^3}f\left(\frac{x}{3^3}\right)-\frac{1}{3^2}f\left(\frac{x}{3^2}\right)=5\cdot\frac{x}{3^4}\quad (3)$$

Replace $x\to\frac{x}{3}$ in (3):

$$\frac{1}{3^3}f\left(\frac{x}{3^4}\right)-\frac{1}{3^2}f\left(\frac{x}{3^3}\right)=5\cdot\frac{x}{3^5}$$

$$\frac{1}{3^4}f\left(\frac{x}{3^4}\right)-\frac{1}{3^3}f\left(\frac{x}{3^3}\right)=5\cdot\frac{x}{3^6}\quad (4)$$

Analogous:

$$\frac{1}{3^{n+1}}f\left(\frac{x}{3^{n+1}}\right)-\frac{1}{3^n}f\left(\frac{x}{3^n}\right)=5\cdot\frac{x}{3^{2n}}\quad (n)$$

By adding: (1); (2); (3); (4);…;(n):

$$\frac{1}{3^{n+1}}f\left(\frac{x}{3^{n+1}}\right)-f(x)=5x\left(1+\frac{1}{3^2}+\frac{1}{3^4}+\cdots+\frac{1}{3^{2n}}\right)$$

$$\frac{1}{3^{n+1}}f\left(\frac{x}{3^{n+1}}\right)-f(x)=5x\cdot\frac{\frac{1}{3^{2n+2}}-1}{\frac{1}{3^2}-1}$$

$$\lim_{n\to\infty}\left(\frac{1}{3^{n+1}}f\left(\frac{x}{3^{n+1}}\right)-f(x)\right)=15x\lim_{n\to\infty}\frac{\frac{1}{3^{2n+2}}-1}{\frac{1}{3^2}-1}$$

$$\frac{1}{\infty}\cdot f(0)-f(x)=5x\cdot\frac{\frac{1}{\infty}-1}{\frac{1}{9}-1}$$

$$0\cdot 0-f(x)=5x\cdot\frac{0-1}{-\frac{8}{9}}$$

$$-f(x)=\frac{45x}{8}$$

$$f(x)=-\frac{45x}{8}$$

Solution 2 by Marin Chirciu-Romania

$$f\left(\frac{x}{3}\right)-3f(x)=15x\Leftrightarrow f(x)=\frac{1}{3}f\left(\frac{x}{3}\right)-5x$$

Given to x the values: $\frac{x}{3},\frac{x}{3^2},\dots,\frac{x}{3^n}$ we obtain successively:

$$f(x)=\frac{1}{3}f\left(\frac{x}{3}\right)-5x=\frac{1}{3}\left[\frac{1}{3}f\left(\frac{x}{3^2}\right)-\frac{5x}{3}\right]-5x=\frac{1}{3^2}f\left(\frac{x}{3^2}\right)-5x\left(1+\frac{1}{3^2}\right)=$$

$$=\frac{1}{3^2}\left[\frac{1}{3}f\left(\frac{x}{3^3}\right)-5\frac{x}{3^2}\right]-5x\left(1+\frac{1}{3^2}\right)=\frac{1}{3^3}f\left(\frac{x}{3^3}\right)-5\frac{x}{3^4}-5x\left(1+\frac{1}{3^2}\right)=$$

$$=\frac{1}{3^3}f\left(\frac{x}{3^3}\right)-5x-\left(1+\frac{1}{3^2}+\frac{1}{3^4}\right)=\dots=\frac{1}{3^n}f\left(\frac{x}{3^n}\right)-5x\left(1+\frac{1}{3^2}+\frac{1}{3^4}+\dots+\frac{1}{3^{2n}}\right)$$

Passing the limit for $n\to\infty$ and taking into account that the function is continuous and $f(0)=0$ we obtain:

$$f(x)=-5x\lim_{n\to\infty}\left(1+\frac{1}{3^2}+\frac{1}{3^4}+\dots+\frac{1}{3^{2n}}\right)=-5x\cdot\frac{9}{8}=\frac{-45}{8}x.$$

We deduce that the searched function is $f:\mathbb{R}\to\mathbb{R}, f(x)=\frac{-45}{8}x$

Remark: The problem can be developed.

Let $a>1, b\in\mathbb{R}$ fixed. Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that:

$$f\left(\frac{x}{a}\right)-af(x)=abx,(\forall)x\in\mathbb{R}$$

Marin Chirciu

Solution

$$f\left(\frac{x}{a}\right)-af(x)=abx\Leftrightarrow f(x)=\frac{1}{a}f\left(\frac{x}{a}\right)-bx$$

Given to x the values: $\frac{x}{a},\frac{x}{a^2},\dots,\frac{x}{a^n}$ we obtain successively:

$$f(x)=\frac{1}{a}f\left(\frac{x}{a}\right)-bx=\frac{1}{a}\left[\frac{1}{a}f\left(\frac{x}{a^2}\right)-b\frac{x}{a}\right]-bx=\frac{1}{a^2}f\left(\frac{x}{a^2}\right)-ax\left(1+\frac{1}{a^2}\right)=$$

$$=\frac{1}{a^2}\left[\frac{1}{a}f\left(\frac{x}{a^3}\right)-b\frac{x}{a^2}\right]-bx\left(1+\frac{1}{a^2}\right)=\frac{1}{a^3}f\left(\frac{x}{a^3}\right)-b\frac{x}{a^4}-bx\left(1+\frac{1}{a^2}\right)=$$

$$=\frac{1}{a^3}f\left(\frac{x}{a^3}\right)-bx\left(1+\frac{1}{a^2}+\frac{1}{a^4}\right)=\dots=\frac{1}{a^n}f\left(\frac{x}{a^n}\right)-bx\left(1+\frac{1}{a^2}+\frac{1}{a^4}+\dots+\frac{1}{2^n}\right)$$

Passing to the limit for $n\to\infty$ and taking into account that the function is continuous and $f(0)=0$ we obtain:

$$f(x)=-bx\lim_{n\to\infty}\left(1+\frac{1}{a^2}+\frac{1}{a^4}+\dots+\frac{1}{a^{2n}}\right)=-bx\cdot\frac{a^2}{a^2-1}=\frac{a^2b}{1-a^2}x$$

We deduce that the searched function is $f:\mathbb{R}\to\mathbb{R}, f(x)=\frac{a^2b}{1-a^2}x$

Note:

For $a=3, b=5$ we obtain Problem SP.541. from RMM Nr. 37 – Summer 2025