SP.543 Let be $f:[0,1]\to[0,20];\ f(x)=2\cdot 3^x+4\cdot 5^x-6$. Find:

$$\Omega=\int_0^{20} f^{-1}(x)dx$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$f'(x)=2\cdot 3^x\log 3+4\cdot 5^x\log 5>0$$

f strictly increasing

$$f(0)=2+4-6=0;\ f(1)=6+20-6=20$$

f continuous $\Rightarrow f([0,1])=[0,20]$

By Young's theorem:

$$\int_0^1 f(x)\,dx+\int_0^{20} f^{-1}(x)dx=1\cdot 20$$

$$\int_0^1(2\cdot 3^x+4\cdot 5^x-6)\,dx+\int_0^{20} f^{-1}(x)\,dx=20$$

$$\Omega=\int_0^{20} f^{-1}(x)\,dx=20-2\int_0^1 3^x\,dx-4\int_0^1 5^x\,dx+6(1-0)=$$

$$=20-2\left(\frac{3^1}{\log 3}-\frac{3^0}{\log 3}\right)-4\left(\frac{5^1}{\log 5}-\frac{5^0}{\log 5}\right)+6=26-\frac{4}{\log 3}-\frac{16}{\log 5}$$

Solution 2 by Marin Chirciu-Romania

Using Young equality:

If $f:[0,a]\to[0,b]$ is a derivable function strictly increasing with $f(0)=0$ and $f(a)=b$, then the following equality holds $\int_0^a f(x)\,dx+\int_0^b f^{-1}(x)\,dx=ab$.

In our case $f:[0,1]\to[0,20],\ f(x)=2\cdot 3^x+4\cdot 5^x-6$ is a derivable function, strictly increasing (exponentials with a superunit base) with $f(0)=0$ and $f(1)=20$.

Applying Young equality we obtain:

$$\int_0^1 f(x)\,dx+\int_0^{20} f^{-1}(x)\,dx=20$$

We have $\int_0^1 f(x)\,dx=\int_0^1(2\cdot 3^x+4\cdot 5^x-6)\,dx=\left(2\cdot\frac{3^x}{\ln 3}+4\cdot\frac{5^x}{\ln 5}-6x\right)\Big|_0^1=$

$$=\left(2\cdot\frac{3}{\ln 3}+4\cdot\frac{5}{\ln 5}-6\right)-\left(2\cdot\frac{1}{\ln 3}+4\cdot\frac{1}{\ln 5}\right)=\frac{4}{\ln 3}+\frac{16}{\ln 5}-6$$

We obtain:

$$\Omega=\int_0^{20} f^{-1}(x)\,dx=20-\left(\frac{4}{\ln 3}+\frac{16}{\ln 5}-6\right)=26-\frac{4}{\ln 3}-\frac{16}{\ln 5}$$

Remark: In the same way.

Let be $f:[0,1]\to[0,25], f(x)=3\cdot 4^x+4\cdot 5^x-7$. Find:

$$\Omega=\int_0^{25} f^{-1}(x)\,dx$$

Marin Chirciu

Solution: **Using Young equality:**

If $f:[0,a]\to[0,b]$ is a derivable function, strictly increasing with $f(0)=0$ and $f(a)=b$, then the following equality $\int_0^a f(x)\,dx+\int_0^b f^{-1}(x)\,dx=ab$.

In our case $f:[0,1]\to[0,25], f(x)=3\cdot 4^x+4\cdot 5^x-7$ is a derivable function, strictly increasing (exponentials with a superunit base), with $f(0)=0$ and $f(1)=25$.

Applying Young's equality we obtain:

$$\int_0^1 f(x)\,dx+\int_0^{25} f^{-1}(x)\,dx=25.$$

We have $\int_0^1 f(x)\,dx=\int_0^1(3\cdot 4^x+4\cdot 5^x-7)\,dx=\left(3\cdot\frac{4^x}{\ln 4}+4\cdot\frac{5^x}{\ln 5}-6x\right)\Big|_0^1=$

$$=\left(3\cdot\frac{4}{\ln 4}+4\cdot\frac{5}{\ln 5}-7\right)-\left(3\cdot\frac{1}{\ln 4}+4\cdot\frac{1}{\ln 5}\right)=\frac{9}{\ln 3}+\frac{16}{\ln 5}-7.$$

We obtain:

$$\Omega=\int_0^{20} f^{-1}(x)\,dx=25-\left(\frac{9}{\ln 4}+\frac{16}{\ln 5}-7\right)=32-\frac{9}{\ln 4}-\frac{16}{\ln 5}.$$