ROMANIAN MATHEMATICAL MAGAZINE

SP.544 Find the maximum value of $n \in \mathbb{N}^*$ such that:

$$\sum_{k=1}^{n} \frac{1}{(k+1)\sqrt{k+1}+k\sqrt{k}} < \frac{31}{32}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$1 > 0 \Rightarrow n+1 > n \Rightarrow \sqrt{n+1} > \sqrt{n}$$

$$\Rightarrow \sqrt{n+1}-\sqrt{n} > 0 \Rightarrow \left(\sqrt{n+1}-\sqrt{n}\right)(n+1-n) > 0$$

$$(n+1)\left(\sqrt{n+1}-\sqrt{n}\right)-\left(\sqrt{n+1}-\sqrt{n}\right)\cdot n > 0$$

$$(n+1)\sqrt{n+1}-(n+1)\sqrt{n}-n\sqrt{n+1}+n\sqrt{n} > 0$$

$$(n+1)\sqrt{n+1}+n\sqrt{n} > (n+1)\sqrt{n}+n\sqrt{n+1}$$

$$\frac{1}{(n+1)\sqrt{n+1}+n\sqrt{n}} < \frac{1}{(n+1)\sqrt{n}+n\sqrt{n+1}} =$$

$$= \frac{(n+1)\sqrt{n}-n\sqrt{n+1}}{(n+1)^2\cdot n-n^2(n+1)} = \frac{(n+1)\sqrt{n}-n\sqrt{n+1}}{n(n+1)(n+1-n)} =$$

$$= \frac{(n+1)\sqrt{n}}{n(n+1)}-\frac{n\sqrt{n+1}}{n(n+1)} = \frac{\sqrt{n}}{n}-\frac{\sqrt{n+1}}{n+1}$$

$$\sum_{k=1}^{n} \frac{1}{(k+1)\sqrt{k+1}+k\sqrt{k}} < \sum_{k=1}^{n}\left(\frac{\sqrt{k}}{k}-\frac{\sqrt{k+1}}{k+1}\right) =$$

$$= 1-\frac{\sqrt{n+1}}{n+1} = 1-\frac{1}{\sqrt{n+1}} < \frac{31}{32} \Leftrightarrow \frac{1}{\sqrt{n+1}} > \frac{1}{32}$$

$$\sqrt{n+1} < 32 \Rightarrow n+1 < 1024 \Rightarrow n < 1023$$

Solution: $n = 1022$

Solution 2 by Marin Chirciu-Romania

$$\sum_{k=1}^{n} \frac{1}{(k+1)\sqrt{k+1}+k\sqrt{k}} < \sum_{k=1}^{n} \frac{1}{\sqrt{k+1}+\sqrt{k}} = \sum_{k=1}^{n}\left(\frac{1}{\sqrt{k}}-\frac{1}{\sqrt{k+1}}\right) =$$

$$= \frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}} = 1-\frac{1}{\sqrt{n+1}} \le 1-\frac{1}{32}$$

From $1-\frac{1}{\sqrt{n+1}}<1-\frac{1}{32}\Leftrightarrow\frac{1}{\sqrt{n+1}}>\frac{1}{32}\Leftrightarrow\sqrt{n+1}<32\Leftrightarrow n+1<32^2\Leftrightarrow$

$$\Leftrightarrow n<1023\Rightarrow n_{\max}=1022$$

Remark: The problem can be developed.

Let $m\geq 2$ fixed. Find the maximum value of $n\in\mathbb{N}^*$ such that:

$$\sum_{k=1}^{n}\frac{1}{(k+1)\sqrt{k+1}+k\sqrt{k}}<\frac{m-1}{m}$$

Marin Chirciu

Solution

$$\sum_{k=1}^{n}\frac{1}{(k+1)\sqrt{k+1}+k\sqrt{k}}<\sum_{k=1}^{n}\frac{1}{\sqrt{k+1}+\sqrt{k}}=\sum_{k=1}^{n}\left(\frac{1}{\sqrt{k}}-\frac{1}{\sqrt{k+1}}\right)=$$

$$=\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}=1-\frac{1}{\sqrt{n+1}}<1-\frac{1}{m}$$

From $1-\frac{1}{\sqrt{n+1}}<1-\frac{1}{m}\Leftrightarrow\frac{1}{\sqrt{n+1}}>\frac{1}{m}\Leftrightarrow\sqrt{n+1}<m\Leftrightarrow n+1<m^2\Leftrightarrow$

$$\Leftrightarrow n<m^2-1\Rightarrow n_{\max}=m^2-2.$$

Note: For $m=32$ we obtain Problem SP.544 from RMM Nr. 37 – Summer 2025