SP.546 If $a,b,c\in(0,1)$ then:

$$\sqrt{3a-a^2}+\sqrt{5b-b^2}+\sqrt{7c-c^2}\le\sqrt{15(a+b+c)-(a+b+c)^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

Let be $x,y,z>0$ such that:

$$\frac{a}{3}=x;\frac{b}{5}=y;\frac{c}{7}=z$$

$$\sqrt{3a-a^2}+\sqrt{5b-b^2}+\sqrt{7c-c^2}=$$

$$=\sqrt{9\left(\frac{a}{3}-\frac{a^2}{9}\right)}+\sqrt{25\left(\frac{b}{5}-\frac{b^2}{25}\right)}+\sqrt{49\left(\frac{c}{7}-\frac{c^2}{49}\right)}$$

$$=3\sqrt{\frac{a}{3}-\left(\frac{a}{3}\right)^2}+5\sqrt{\frac{b}{5}-\left(\frac{b}{5}\right)^2}+7\sqrt{\frac{c}{7}-\left(\frac{c}{7}\right)^2}$$

Let be $f:(0,1)\to\mathbb{R};f(x)=\sqrt{x-x^2}$

$$f'(x)=\frac{-1}{4\sqrt{x-x^2}(x-x^2)}<0\Rightarrow f\text{ concave}$$

By Jensen's inequality:

$$f\left(\frac{3}{15}x+\frac{5}{15}y+\frac{7}{15}z\right)\ge\frac{3}{15}f(x)+\frac{5}{15}f(y)+\frac{7}{15}f(z)$$

$$15f\left(\frac{3x+5y+7z}{15}\right)\ge 3f(x)+5f(y)+7f(z)$$

$$15\sqrt{\frac{3x+5y+7z}{15}-\left(\frac{3x+5y+7z}{15}\right)^2}\ge 3\sqrt{x-x^2}+5\sqrt{y-y^2}+7\sqrt{z-z^2}$$

$$3\sqrt{x-x^2}+5\sqrt{y-y^2}+7\sqrt{z-z^2}\le 15\sqrt{\frac{3x+5y+7z}{15}-\left(\frac{3x+5y+7z}{15}\right)^2}$$

$$\sqrt{9x-9x^2}+\sqrt{25y-25y^2}+\sqrt{49z-49z^2}\le$$

$$\le\sqrt{225\left(\frac{3x+5y+7z}{15}-\left(\frac{3x+5y+7z}{15}\right)^2\right)}$$

$$\sqrt{3\cdot 3x-(3x)^2}+\sqrt{5\cdot 5y-(5y)^2}+\sqrt{7\cdot 7z-(7z)^2}\le$$
$$\le\sqrt{15(3x+5y+7z)-(3x+5y+7z)^2}$$
$$\sqrt{3a-a^2}+\sqrt{5b-b^2}+\sqrt{7c-c^2}\le$$
$$\le\sqrt{15(a+b+c)-(a+b+c)^2}$$

Solution 2 by Marin Chirciu-Romania

$$\sqrt{3a-a^2}+\sqrt{5b-b^2}+\sqrt{7c-c^2}\le\sqrt{15(a+b+c)-(a+b+c)^2}$$

Using CBS inequality, we obtain:

$$LHS=\sqrt{3a-a^2}+\sqrt{5b-b^2}+\sqrt{7c-c^2}=\sqrt{a}\sqrt{3-a}+\sqrt{b}\sqrt{5-b}+\sqrt{c}\sqrt{7-c}\overset{CBS}{\le}$$
$$\overset{CBS}{\le}\sqrt{(a+b+c)(3-a+5-b+7-c)}=\sqrt{(a+b+c)(15-a-b-c)}=$$
$$=\sqrt{15(a+b+c)-(a+b+c)^2}=RHS$$

Observation: $a,b,c\in(0,1)\Rightarrow\sqrt{a},\sqrt{b},\sqrt{c}>0$ **and** $\sqrt{3-a},\sqrt{5-b},\sqrt{7-c}>0$

Equality holds if and only if $\frac{a}{3-a}=\frac{b}{5-b}=\frac{c}{7-c}\Leftrightarrow\frac{a}{3}=\frac{b}{5}=\frac{c}{7}$

Remark: The problem can be developed.

If $a,b,c\in(0,1)$ **and** $x,y,z\in(1,\infty)$ **then:**

$$\sqrt{xa-a^2}+\sqrt{yb-b^2}+\sqrt{zc-c^2}\le\sqrt{(x+y+z)(a+b+c)-(a+b+c)^2}$$

Marin Chirciu

Solution: Using CBS inequality, we obtain:

$$LHS=\sqrt{xa-a^2}+\sqrt{yb-b^2}+\sqrt{zc-c^2}=\sqrt{a}\sqrt{x-a}+\sqrt{b}\sqrt{y-b}+\sqrt{c}\sqrt{z-c}\overset{CBS}{\le}$$
$$\overset{CBS}{\le}\sqrt{(a+b+c)(x-a+y-b+z-c)}=\sqrt{(a+b+c)(x+y+z-a-b-c)}=$$
$$=\sqrt{(x+y+z)(a+b+c)-(a+b+c)^2}=RHS$$

Observation: $a,b,c\in(0,1)\Rightarrow\sqrt{a},\sqrt{b},\sqrt{c}>0$ **and** $\sqrt{x-a},\sqrt{y-b},\sqrt{z-c}>0$

Equality holds if and only if $\frac{a}{x-a}=\frac{b}{y-b}=\frac{c}{z-c}\Leftrightarrow\frac{a}{x}=\frac{b}{y}=\frac{c}{z}$.

Note: For $x=3,y=5,z=7$ **we obtain Problem SP.546 from RMM Nr. 37 – Summer 2025**