SP.547 Prove that if $x, y > 1$ then:

$$\ln x \cdot \ln y\left(\sqrt[3]{\log_x y} + \sqrt[3]{\log_y x}\right)^3 \le 2\ln^2(xy)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\left(\sqrt[3]{\log_x y} + \sqrt[3]{\log_y x}\right)^3 \le \frac{2(\ln x + \ln y)^2}{\ln x \cdot \ln y}$$

$$\left(\sqrt[3]{\frac{\ln y}{\ln x}} + \sqrt[3]{\frac{\ln x}{\ln y}}\right)^3 \le \frac{2(\ln x + \ln y)^2}{\ln x \cdot \ln y}$$

Denote: $\ln x = u^3; \ln y = v^3$

$$\left(\sqrt[3]{\frac{v^3}{u^3}} + \sqrt[3]{\frac{u^3}{v^3}}\right)^3 \le \frac{2(u^3 + v^3)^2}{u^3 \cdot v^3}$$

$$\left(\frac{v}{u} + \frac{u}{v}\right)^3 \le \frac{2(u^3 + v^3)^2}{(uv)^3}$$

$$(u^2 + v^2)^3 \le 2(u^3 + v^3)^2 \quad \text{(to prove)}$$

$$(u^2 + v^2)^3 = (u^2 + v^2)(u^2 + v^2)^2 = (u^2 + v^2)\left(\sqrt{u} \cdot \sqrt{u^3} + \sqrt{v} \cdot \sqrt{v^3}\right)^2 \le$$

$$\overset{CBS}{\le} (u^2 + v^2)(u + v)(u^3 + v^3) \le$$

$$\overset{CEBYSHEV}{\le} 2(u^2 \cdot u + v^2 \cdot v)(u^3 + v^3) = 2(u^3 + v^3)^2$$

Equality holds for $x = y$.

Solution 2 by Marin Chirciu-Romania

With the substitution $(a, b) = (\ln x, \ln y)$ the inequality can be written:

$$ab\left(\sqrt[3]{\frac{a}{b}} + \sqrt[3]{\frac{b}{a}}\right)^3 \le 2(a + b)^2$$

Denoting $(a, b) = (u^3, v^3)$ the above inequality becomes:

$$u^3v^3\left(\frac{u}{v}+\frac{v}{u}\right)^3 \le 2(u^3+v^3)^2 \Leftrightarrow u^3v^3\left(\frac{u^3}{v^3}+3\frac{u}{v}+3\frac{v}{u}+\frac{v^3}{u^3}\right) \le 2(u^6+2u^3v^3+v^6)$$

$$\Leftrightarrow$$

$$\Leftrightarrow u^6+3u^4v^2+3u^2v^4+v^6 \le 2u^6+4u^3v^3+2v^6 \Leftrightarrow$$

$$\Leftrightarrow u^6-3u^4v^2+4u^3v^3-3u^2v^4+v^6 \ge 0 \Leftrightarrow$$

$\Leftrightarrow (u-v)^2(u^4+2u^3v+2uv^3+v^4) \ge 0$, with equality for $u=v \Leftrightarrow a=b \Leftrightarrow$

$$\Leftrightarrow \ln x = \ln y \Leftrightarrow x = y$$

Equality holds if and only if $x=y$.

Remark.

The problem can be developed.

If $x, y > 1$ then:

$$\ln x \ln y\left(\sqrt[4]{\log_x y}+\sqrt[4]{\log_y x}\right)^4 \le 4\ln^2(xy).$$

Marin Chirciu

Solution

With the substitution $(a,b)=(\ln x, \ln y)$ the inequality can be written:

$$ab\left(\sqrt[4]{\frac{a}{b}}+\sqrt[4]{\frac{b}{a}}\right)^4 \le 4(a+b)^2.$$

Denoting $(a,b)=(u^4,v^4)$ the above inequality becomes:

$$u^4v^4\left(\frac{u}{v}+\frac{v}{u}\right)^4 \le 4(u^4+v^4)^2 \Leftrightarrow u^4v^4\left(\frac{u^4}{v^4}+4\frac{u^2}{v^2}+6+4\frac{v^2}{u^2}+\frac{v^4}{u^4}\right) \le$$

$$\le 4(u^8+2u^4v^4+v^8) \Leftrightarrow$$

$$\Leftrightarrow u^8+4u^6v^2+6u^4v^4+4u^2v^6+v^8 \le 4u^8+8u^4v^4+4v^8 \Leftrightarrow$$

$$\Leftrightarrow 3u^8-4u^6v^2+2u^4v^4-4u^2v^6+3v^8 \ge 0 \Leftrightarrow (u^2-v^2)^2(3u^4+2u^2v^2+3v^4) \ge 0$$

with equality for $u=v \Leftrightarrow a=b \Leftrightarrow \ln x = \ln y \Leftrightarrow x=y$.

Equality holds if and only if $x=y$.