ROMANIAN MATHEMATICAL MAGAZINE

SP.548 If $a,b,c>0$ then:

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2+2\left(\frac{ab}{a+b}+\frac{bc}{c+a}+\frac{ca}{a+b}\right)\geq 4\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

For $a,b>0; \left(a+b-2\sqrt{ab}\right)^2\geq 0$

$$(a+b)^2-2(a+b)\cdot 2\sqrt{ab}+4ab\geq 0$$

$$(a+b)^2+4ab\geq 4(a+b)\sqrt{ab}$$

$$\frac{(a+b)^2}{2(a+b)}+\frac{4ab}{2(a+b)}\geq 2\sqrt{ab}$$

$$\frac{a+b}{2}+\frac{2ab}{a+b}\geq 2\sqrt{ab}$$

$$\sum_{cyc}\frac{a+b}{2}+2\sum_{cyc}\frac{ab}{a+b}\geq 2\sum_{cyc}\sqrt{ab}$$

$$a+b+c+2\sum_{cyc}\frac{ab}{a+b}\geq 2\sum_{cyc}\sqrt{ab}$$

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2-2\sum_{cyc}\sqrt{ab}+2\sum_{cyc}\frac{ab}{a+b}\geq 2\sum_{cyc}\sqrt{ab}$$

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2+2\sum_{cyc}\frac{ab}{a+b}\geq 4\sum_{cyc}\sqrt{ab}$$

Equality holds for $a=b=c$.

Solution 2 by Marin Chirciu-Romania

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2+2\sum\frac{ab}{a+b}\geq 4\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)\Leftrightarrow$$

$$\Leftrightarrow a+b+c+2\sum\sqrt{ab}+2\sum\frac{ab}{a+b}\geq 4\sum\sqrt{ab}\Leftrightarrow a+b+c+2\sum\frac{ab}{a+b}\geq$$

$$\geq 2\sum\sqrt{ab}\Leftrightarrow\sum\frac{a+b}{2}+2\sum\frac{ab}{a+b}\geq 2\sum\sqrt{ab}$$

which follows from $\frac{a+b}{2}+\frac{2ab}{a+b}\geq 2\sqrt{ab}$

Indeed:

$$\frac{a+b}{2}+\frac{2ab}{a+b}\geq 2\sqrt{ab}\Leftrightarrow (a+b)^2+4ab\geq 4\sqrt{ab}(a+b)\Leftrightarrow$$

$$\Leftrightarrow a^2+6ab+b^2\geq 4\sqrt{ab}(a+b)\Leftrightarrow$$

$$\Leftrightarrow (a^2+6ab+b^2)^2\geq 16ab(a+b)^2\Leftrightarrow a^4-4a^3b+6a^2b^2-4ab^3+b^4\geq 0\Leftrightarrow$$

$$\Leftrightarrow (a-b)^4\geq 0 \text{ with equality for } a=b.$$

Summing the inequalities $\frac{a+b}{2}+\frac{2ab}{a+b}\geq 2\sqrt{ab}$ **we obtain** $\sum\frac{a+b}{2}+2\sum\frac{ab}{a+b}\geq 2\sum\sqrt{ab}$

Equality holds if and only if $a=b=c$.

Remark.

The problem can be developed.

If $a,b,c>0$ **then:**

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2+\lambda\sum\frac{ab}{a+b}\geq\left(3+\frac{\lambda}{2}\right)\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)$$

Marin Chirciu

Solution

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2+\lambda\sum\frac{ab}{a+b}\geq\left(3+\frac{\lambda}{2}\right)\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)\Leftrightarrow$$

$$\Leftrightarrow a+b+c+2\sum\sqrt{ab}+\lambda\sum\frac{ab}{a+b}\geq\left(3+\frac{\lambda}{2}\right)\sum\sqrt{ab}\Leftrightarrow$$

$$a+b+c+\lambda\sum\frac{ab}{a+b}\geq\left(1+\frac{\lambda}{2}\right)\sum\sqrt{ab}\Leftrightarrow$$

$\sum\frac{a+b}{2}+\lambda\sum\frac{ab}{a+b}\geq\left(1+\frac{\lambda}{2}\right)\sum\sqrt{ab}$**, which follows from** $\frac{a+b}{2}+\frac{\lambda}{a+b}\geq\left(1+\frac{\lambda}{2}\right)\sqrt{ab}$

Indeed:

$$\frac{a+b}{2}+\frac{\lambda ab}{a+b}\geq\left(1+\frac{\lambda}{2}\right)\sqrt{ab}\Leftrightarrow (a+b)^2+2\lambda ab\geq(\lambda+2)(a+b)\sqrt{ab}\Leftrightarrow$$

$$a^2+2(\lambda+1)ab+b^2\geq(\lambda+2)(a+b)\sqrt{ab}\Leftrightarrow$$

$$\Leftrightarrow (a^2+2(\lambda+1)ab+b^2)^2\geq(\lambda+2)^2(a+b)^2ab\Leftrightarrow$$

$$a^4-\lambda^2a^3b+2(\lambda^2-1)a^2b^2-\lambda^2ab^3+b^4\geq 0\Leftrightarrow$$

$$(a-b)^2(a^2+(2-\lambda^2)ab+b\hat{\ }2)\geq 0$$

with equality for $a=b$.

Summing the inequalities $\frac{a+b}{2}+\frac{2ab}{a+b}\ge 2\sqrt{ab}$ we obtain $\sum\frac{a+b}{2}+2\sum\frac{ab}{a+b}\ge 2\sum\sqrt{ab}$

Equality holds if and only if $a=b=c$.

Solution 3 by Tapas Das – India

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2+2\left(\frac{ab}{a+b}+\frac{bc}{c+a}+\frac{ca}{a+b}\right)\overset{Bergsgtrom}{\ge}$$

$$\ge(a+b+c)+2\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)+2\frac{\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)^2}{2(a+b+c)}$$

$$=2\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)+\frac{\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)^2}{(a+b+c)}+(a+b+c)$$

$$\overset{AM-GM}{\ge}2\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)+2\sqrt{\frac{(a+b+c)\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)^2}{a+b+c}}\ge$$

$$\ge 4\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)$$

Equality holds if and only if $a=b=c$.

Solutions 4, 5 by Kunihiko Chikaya-Tokyo-Japan

Solution 4:

Using the AM-GM inequality, $\frac{a+b}{2}+\frac{2ab}{a+b}\ge 2\sqrt{\frac{a+b}{2}\cdot\frac{2ab}{a+b}}=2\sqrt{ab}$.

Equality $\frac{a+b}{2}=\frac{2ab}{a+b}$ and $a>0,b>0\Leftrightarrow(a-b)^2=0\ (a,b>0)\Leftrightarrow a=b$.

Analogously, we have $\frac{b+c}{2}+\frac{2bc}{b+c}\ge 2\sqrt{bc}$ and $\frac{c+a}{2}+\frac{2ca}{b+c}\ge 2\sqrt{ca}$.

Adding these three inequalities gives

$$a+b+c+2\left(\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a}\right)\ge 2\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)\Leftrightarrow$$

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2+2\left(\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a}\right)\ge 4\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right).$$

Equality attains if only if $a=b=c$.

Solution 5:

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2+2\left(\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a}\right)-4\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)$$

$$=a+b+c+2\left(\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a}\right)-2\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)$$

$$= \left(\sqrt{\frac{a+b}{2}} - \sqrt{\frac{2ab}{a+b}}\right)^2 + \left(\sqrt{\frac{b+c}{2}} - \sqrt{\frac{bc}{b+c}}\right)^2 + \left(\sqrt{\frac{c+a}{2}} - \sqrt{\frac{2ca}{c+a}}\right)^2 \geq 0$$

Equality attains if only if

$\frac{a+b}{2} = \frac{2ab}{a+b}, \frac{b+c}{2} = \frac{2bc}{b+c}, \frac{c+a}{2} = \frac{2ca}{c+a}$ **and**

$$a > 0, b > 0, c > 0 \Leftrightarrow a = b = c.$$