SP.549 Let be ΔABC with sides a, b, c. Let be $x, y, z \in \mathbb{R}$ such that:

$$\cos x = \frac{a}{b+c}; \cos y = \frac{b}{c+a}; \cos z = \frac{c}{a+b}$$

Prove that:

$$\tan\frac{x}{2}\tan\frac{y}{2}\tan\frac{z}{2} \le \frac{\sqrt{3}}{9}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\tan\frac{x}{2} = \sqrt{\frac{1-\cos x}{1+\cos x}} = \sqrt{\frac{1-\frac{a}{b+c}}{1+\frac{a}{b+c}}} = \sqrt{\frac{b+c-a}{a+b+c}} =$$

$$= \sqrt{\frac{a+b+c-2a}{a+b+c}} = \sqrt{\frac{2s-2a}{2s}} = \sqrt{\frac{s-a}{s}}$$

$s = \frac{a+b+c}{2}$ – semiperimeter; F – area

$$\tan\frac{x}{2}\tan\frac{y}{2}\tan\frac{z}{2} = \sqrt{\frac{s-a}{s}} \cdot \sqrt{\frac{s-b}{s}} \cdot \sqrt{\frac{s-c}{s}} =$$

$$= \sqrt{\frac{(s-a)(s-b)(s-c)}{s^3}} = \sqrt{\frac{s(s-a)(s-b)(s-c)}{s^4}} =$$

$$\overset{HERON}{=} \frac{F}{s^2} = \frac{rs}{s^2} = \frac{r}{s} \overset{MITRINOVIC}{\le} \frac{r}{3\sqrt{3}r} = \frac{1}{3\sqrt{3}} = \frac{\sqrt{3}}{9}$$

Solution 2 by Marin Chirciu-Romania

Using $\tan^2\frac{x}{2} = \frac{1-\cos x}{1+\cos x} = \frac{1-\frac{a}{b+c}}{1+\frac{a}{b+c}} = \frac{b+c-a}{b+c+a} = \frac{2s-2a}{2s} = \frac{s-a}{s}$ we obtain:

$$LHS = \prod\sqrt{\frac{s-a}{s}} = \sqrt{\frac{r^2 s}{s^3}} \overset{Mitrinovic}{\le} \frac{r}{3\sqrt{3}r} = \frac{1}{3\sqrt{3}} = \frac{\sqrt{3}}{9} = RHS$$

Equality holds if and only if the triangle is equilateral.

Remark.

In the same way.

Let be ΔABC with a, b, c sides and $x, y, z \in \mathbb{R}$ such that

$$x = \frac{a}{b+c}, \cos y = \frac{b}{c+a}, \cos z = \frac{c}{a+b}$$

Prove that:

$$\cot\frac{x}{2} + \cot\frac{y}{2} + \cot\frac{z}{2} \geq 3\sqrt{3}$$

Marin Chirciu

Solution:

Using $\cot^2\frac{x}{2} = \frac{1+\cos x}{1-\cos x} = \frac{1+\frac{a}{b+c}}{1-\frac{a}{b+c}} = \frac{b+c+a}{b+c-a} = \frac{2s}{2s-2a} = \frac{s}{s-a}$ **we obtain:**

$$LHS = \sum\sqrt{\frac{s}{s-a}} \overset{AM-GM}{\geq} 3\sqrt[3]{\prod\sqrt{\frac{s}{s-a}}} = 3\sqrt[3]{\sqrt{\frac{s^3}{r^2 s}}} = 3\sqrt[3]{\frac{s}{r}} \overset{Mitrinovic}{\geq}$$

$$\geq 3\sqrt[3]{\frac{3\sqrt{3}r}{r}} = 3\sqrt[3]{3\sqrt{3}} = 3\sqrt[3]{\left(\sqrt{3}\right)^3} = 3\sqrt{3} = RHS$$

Equality holds if and only if the triangle is equilateral.