ROMANIAN MATHEMATICAL MAGAZINE

SP.550. Prove that $(\forall)x \in (0;1)$ and $n \in \mathbb{N}^*$, we have the inequality:

$$x(1+x)(1-x^n) < \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{n+1}$$

Proposed by Gheorghe Molea – Romania

Solution 1 by proposer

$$(1-x^2)\left(x^1+x^3+x^5+\dots+x^{2n+1}\right) = x - x^{2n+3} = x(1-x^{2n+2}) < x \Rightarrow$$

$$\frac{x}{1-x^2} > x^1+x^3+x^5+\dots+x^{2n+1} > (n+1)\sqrt[n+1]{x^1\cdot x^3\cdot x^5\cdot\ldots\cdot x^{2n+1}} =$$

$$= (n+1)\sqrt[n+1]{x^{(n+1)^2}} = (n+1)x^{n+1} \Rightarrow \frac{x}{1-x^2} > (n+1)x^{n+1} \Rightarrow$$

$\frac{1}{1-x^2} > (n+1)x^n \Rightarrow x^n - x^{n+2} < \frac{1}{n+1}$, $(\forall)x \in (0;1)$ and $n \in \mathbb{N}^*$

$$x^1 - x^3 < \frac{1}{2}; x^2 - x^4 < \frac{1}{3}; x^3 - x^5 < \frac{1}{4}; \dots$$

$$x^{n-2} - x^n < \frac{1}{n-1}; x^{n-1} - x^{n+1} < \frac{1}{n};$$

$$x^n - x^{n+2} < \frac{1}{n+1}$$

By adding we obtain:

$$x + x^2 - x^{n+1} - x^{n+2} < \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{n+1}$$

$$\Rightarrow x(1+x)(1-x^n) < \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{n+1}$$

Solution 2 by Marin Chirciu-Romania

We prove through mathematical induction after $n \in \mathbb{N}^*$, the inequality from enunciation.

For $n = 1$ we write the inequality $x(1+x)(1-x) < \frac{1}{2} \Leftrightarrow 2x(1-x^2) < 1 \Leftrightarrow$

$\Leftrightarrow 2x^3 - 2x + 1 > 0.$

We consider the function $f:(0,1) \to \mathbb{R}, f(x) = 2x^3 - 2x + 1$, we have:

$f'(x) = 6x^2 - 2; f'(x) = 0 \Leftrightarrow x = \frac{1}{\sqrt{3}}$.

Preparing the variation table of the function we have:

$f'(x) < 0$ for $x \in \left(0, \frac{1}{\sqrt{3}}\right) \Rightarrow f \downarrow$ and $f'(x) > 0$ for $x \in \left(\frac{1}{\sqrt{3}}, 1\right) \Rightarrow f \uparrow$.

We have $f(0)=f(1)=0$ and $f\left(\frac{1}{\sqrt{3}}\right)=1-\frac{4}{3\sqrt{3}}>0 \Rightarrow$ point $A\left(\frac{1}{\sqrt{3}},1-\frac{4}{3\sqrt{3}}\right)$ is a point of minimum $\Rightarrow Im\, f=\left(1-\frac{4}{3\sqrt{2}},1\right)\subset(0,\infty)\Rightarrow f(x)>0, x\in(0,1)$.

Next, we assume that the inequality holds for $k\in\mathbb{N}^*$ and we show that it is true for $k+1$.

We prove: $x(1+x)\left(1-x^k\right)<\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k+1}\Rightarrow$

$$\Rightarrow x(1+x)\left(1-x^{k+1}\right)<\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k+1}+\frac{1}{k+2}.$$

It remains to prove that:

$$x(1+x)\left(1-x^{k+1}\right)<x(1+x)\left(1-x^k\right)+\frac{1}{k+2}\Leftrightarrow$$

$$\Leftrightarrow x(1+x)\left(x^k-x^{k+1}\right)<\frac{1}{k+2}\Leftrightarrow$$

$$\Leftrightarrow x^{k+1}(1-x^2)<\frac{1}{k+2}.$$

We consider the function $f:(0,1)\to\mathbb{R}, f(x)=x^{k+1}(1-x^2)-\frac{1}{k+2}$;

we have $f'(x)=x^k(k+1-(k+3)x^2)$; $f'(x)=0\Leftrightarrow\sqrt{\frac{k+1}{k+3}}$.

Preparing the variation table of the function we have:

$f'(x)>0$ for $x\in\left(0,\sqrt{\frac{k+1}{k+3}}\right)\Rightarrow f\uparrow$ and $f'(x)<0$ for $x\in\left(\sqrt{\frac{k+1}{k+3}},1\right)\Rightarrow f\downarrow$

We have $f(0)=f(1)=\frac{-1}{k+2}$ and $f\left(\sqrt{\frac{k+1}{k+3}}\right)<0\Rightarrow$ point $A\left(\sqrt{\frac{k+1}{k+3}},f\sqrt{\frac{k+1}{k+3}}\right)$ is a point of maximum with $f\sqrt{\frac{k+1}{k+3}}<0\Rightarrow f(x)<0, x\in(0,1)$

This issue is closed.