SP.551 Find:

$$\Omega = \lim_{n\to\infty} n\int_0^1 \frac{x^3(1+x^{4n-8})}{(1+x^4)^n}dx$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

Denote:

$$\Omega(n) = \int_0^1 \frac{x^3(1+x^{4n-8})}{(1+x^4)^n}dx$$

For: $y = x^4 \Rightarrow dy = 4x^3dx \Rightarrow x^3dx = \frac{1}{4}dy$

$$x = 0 \Rightarrow y = 0$$

$$x = 1 \Rightarrow y = 1$$

$$\Omega(n) = \int_0^1 \frac{(1+y^{n-2})\cdot\frac{1}{4}dy}{(1+dy)^n} =$$

$$= \frac{1}{4}\int_0^1 \frac{dy}{(1+y)^n} + \frac{1}{4}\int_0^1 \frac{y^{n-2}}{(1+y)^n}dy$$

Let's observe that:

$$\left(\frac{y}{1+y}\right)' = \frac{y'(y+1)-y(y+1)'}{(1+y)^2} = \frac{y+1-y}{(1+y)^2} = \frac{1}{(1+y)^2}$$

$$\Omega(n) = \frac{1}{4}\int_0^1 (1+y)'\,(1+y)^{-n}dy + \frac{1}{4}\int_0^1 \left(\frac{y}{1+y}\right)^{n-2}\cdot\frac{1}{(1+y)^2}dy$$

$$\Omega(n) = \frac{1}{4}\cdot\frac{(1+y)^{-n+1}}{-n+1}\Bigg|_0^1 + \frac{1}{4}\int_0^1 \left(\frac{y}{1+y}\right)^{n-2}\cdot\left(\frac{y}{1+y}\right)'dy$$

$$\Omega(n) = \frac{1}{4}\left(\frac{2^{-n+1}}{-n+1} - \frac{1}{-n+1}\right) + \frac{1}{4(n-1)}\cdot\left(\frac{y}{1+y}\right)^{n-1}\Bigg|_0^1$$

$$\Omega(n) = \frac{1}{4(1-n)2^{n-1}} + \frac{1}{4(n-1)} + \frac{1}{4(n-1)}\left(\frac{1}{2^{n-1}} - 0\right)$$

$$n\Omega(n) = \frac{n}{4(1-n)}\cdot\frac{1}{2^{n-1}} + \frac{n}{4(n-1)} + \frac{n}{4(n-1)}\cdot\frac{1}{2^{n-1}}$$

$$\Omega = \lim_{n\to\infty} n\Omega(n) = \frac{1}{4}\cdot 0 + \frac{1}{4} + \frac{1}{4}\cdot 0$$

$$\Omega = \frac{1}{4}$$

Solution 2 by Marin Chirciu-Romania

We make the substitution $x^4 = t \Rightarrow 4x^3 dx = dt$.

We obtain:

$$\int_0^1 \frac{x^3(1+x^{4n-8})}{(1+x^4)^n} dx = \frac{1}{4}\int_0^1 \frac{1+t^{n-2}}{(1+t)^n} dt = \frac{1}{4}\left(\int_0^1 \frac{1}{(1+t)^n} dt + \int_0^1 \frac{t^{n-2}}{(1+t)^n} dt\right) =$$

$$= \frac{1}{4}\left(\int_0^1 (t+1)^{-n} dt + \int_0^1 \left(\frac{t}{t+1}\right)^{n-2} \frac{1}{t^2} dt\right) =$$

$$= \frac{1}{4}\left(\frac{(t+1)^{-n+1}}{-n+1}\Bigg|_0^1 + \int_0^1 \left(\frac{t}{t+1}\right)^{n-2} \left(\frac{t}{t+1}\right)' dt\right) =$$

$$= \frac{1}{4}\left(\frac{2^{-n+1}}{-n+1} - \frac{1}{-n+1} + \frac{\left(\frac{t}{t+1}\right)^{n-2+1}}{n-2+1}\Bigg|_0^1\right) =$$

$$= \frac{1}{4}\left(\frac{2^{-n+1}}{-n+1} - \frac{1}{-n+1} + \frac{\left(\frac{1}{2}\right)^{n-1}}{n-1} - 0\right) = \frac{1}{4}\left(\frac{2^{-n+1}}{-n+1} + \frac{1}{n-1} + \frac{2^{-n+1}}{n-1}\right) =$$

$$= \frac{1}{4} \cdot \frac{1}{n-1} = \frac{1}{4(n-1)}$$

It follows $\int_0^1 \frac{x^3(1+x^{4n-8})}{(1+x^4)6n} dx = \frac{1}{4(n-1)}$.

Finally, $\Omega = \lim\limits_{n\to\infty} n\int_0^1 \frac{(1+x^{4n-8})}{(1+x^4)^n} dx = \lim\limits_{n\to\infty} n\frac{1}{4(n-1)} = \frac{1}{4}$

Remark: Let $k > 1$ fixed. Find:

$$\Omega = \lim_{n\to\infty} n\int_0^1 \frac{x^{k-1}(1+x^{kn-2k})}{(1+x^k)^n} dx$$

Marin Chiricu

Solution:

We make the substitution $x^k = t \Rightarrow kx^{k-1}dx = dt$.

We obtain

$$\int_0^1 \frac{x^{k-1}(1+x^{kn-2k})}{(1+x^k)^n} = \frac{1}{k}\int_0^1 \frac{1+t^{n-2}}{(1+t)^n} dt =$$

$$= \frac{1}{k}\left(\int_0^1 \frac{1}{(1+t)^n} dt + \int_0^1 \frac{t^{n-2}}{(1+t)^n} dt\right) = \frac{1}{k}\left(\int_0^1 (t+1)^{-n} dt + \int_0^1 \left(\frac{t}{t+1}\right)^{n-2} \frac{1}{t^2} dt\right) =$$

$$=\frac{1}{k}\left(\frac{(t+1)^{-n+1}}{-n+1}\Bigg|_0^1+\int_0^1\left(\frac{t}{t+1}\right)^{n-2}\left(\frac{t}{t+1}\right)'dt\right)=$$

$$=\frac{1}{k}\left(\frac{2^{-n+1}}{-n+1}-\frac{1}{-n+1}+\frac{\left(\frac{t}{t+1}\right)^{n-2+1}}{n-2+1}\Bigg|_0^1\right)=$$

$$=\frac{1}{k}\left(\frac{2^{-n+1}}{-n+1}-\frac{1}{-n+1}+\frac{\left(\frac{1}{2}\right)^{n-1}}{n-1}-0\right)=$$

$$=\frac{1}{k}\left(\frac{2^{-n+1}}{-n+1}+\frac{1}{n-1}+\frac{2^{-n+1}}{n-1}\right)=\frac{1}{k}\cdot\frac{1}{n-1}=\frac{1}{k(n-1)}$$

It follows $\int_0^1\frac{x^{k-1}(1+x^{kn-2k})}{(1+x^k)^n}=\frac{1}{k(n-1)}$.

Finally, $\Omega=\Omega=\lim_{n\to\infty}n\int_0^1\frac{x^{k-1}(1+x^{kn-2k})}{(1+x^k)^n}dx=\lim_{n\to\infty}\frac{1}{k(n-1)}=\frac{1}{k}$

Note: For $k=4$ **we obtain Problem SP.551 from RMM Nr. 37 – Sumer 2025**