SP.552 If $a > 0$ then find:

$$\Omega = \int_{-a}^{a} \log_a\left(\sqrt{a^2x^2+1} - ax\right)dx$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\Omega = \int_{-a}^{a} \log_a\left(\sqrt{a^2x^2+1} - ax\right)dx$$

For $y = -x \Rightarrow$

$$x = -a \Rightarrow y = a$$

$$x = a \Rightarrow y = -a$$

$$dx = -dy$$

$$\Omega = \int_{a}^{-a} \log_a\left(\sqrt{a^2y^2+1} + ay\right)(-dy)$$

$$\Omega = \int_{-a}^{a} \log_a\left(\sqrt{a^2y^2+1} + ay\right)dy = \int_{-a}^{a} \log_a\left(\sqrt{a^2x^2+1} + ax\right)dx$$

$$2\Omega = \int_{-a}^{a} \log_a\left(\sqrt{a^2x^2+1} - ax\right)dx + \int_{-a}^{a} \log_a\left(\sqrt{a^2x^2+1} + ax\right)dx$$

$$2\Omega = \int_{-a}^{a} \log_a\left(\left(\sqrt{a^2x^2+1}\right)^2 - (ax)^2\right)dx$$

$$2\Omega = \int_{-a}^{a} \log_a(a^2x^2+1-a^2x^2)\,dx$$

$$2\Omega = \int_{-a}^{a} \log_a 1\,dx \Rightarrow 2\Omega = 0 \Rightarrow \Omega = 0$$

Solution 2 by Marin Chirciu-Romania

The function $f: [-a, a] \to \mathbb{R}, f(x) = \log_a\left(\sqrt{a^2x^2+1} - ax\right)$ is odd, see:

$$f(x) + f(-x) = \log_a\left(\sqrt{a^2x^2+1} - ax\right) + \log_a\left(\sqrt{a^2x^2+1} + ax\right) =$$

$$= \log_a\left(\sqrt{a^2x^2+1} - ax\right)\left(\sqrt{a^2x^2+1} + ax\right) =$$

$$= \log_a(a^2x^2+1-a^2x^2) = \log_a 1 = 0$$

Using the fact that the integral over a symmetric interval of an odd function is zero, we deduce that:

$$\Omega = \int_{-a}^{a} \log_a\left(\sqrt{a^2x^2+1} - ax\right)dx = 0$$

Remark.

If $a > 0, a \neq 1$ then find:

$$\Omega = \int_{a}^{a} \log_a\left(\sqrt{a^2x^2+1} + ax\right)dx$$

Solution

Analogous

$$\Omega = \int_{-a}^{a} \log_a\left(\sqrt{a^2x^2+1} + ax\right)dx = 0$$