ROMANIAN MATHEMATICAL MAGAZINE

SP.553 If $0 \leq a \leq b < 1$ then:

$$6\int_a^b \log\left(\frac{1+x}{1-x}\right)dx \geq (b^2-a^2)(b^2+a^2+6)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

Let be $f:[a,b]\to\mathbb{R}; f(x)=\log\left(\frac{1+x}{1-x}\right)-2x-\frac{2x^3}{3}$

$$f(x)=\log(1+x)-\log(1-x)-2x-\frac{2x^3}{3}$$

$$f'(x)=\frac{1}{1+x}+\frac{1}{1-x}-2-2x^2$$

$$f'(x)=\frac{2}{1-x^2}-2(1+x^2)=2\left(\frac{1}{1-x^2}-(1+x^2)\right)$$

$$f'(x)=2\cdot\frac{1-1+x^4}{1-x^2}=\frac{2x^4}{1-x^2}\geq 0$$

f increasing on $[a,b]\subset[0,1]\Rightarrow$

$$\min_{x\in[a,b]} f(x)=f(0)=0\Rightarrow f(x)\geq 0;(\forall)x\in[a,b]$$

$$\log\left(\frac{1+x}{1-x}\right)\geq 2x+\frac{2x^3}{3}$$

$$\int_a^b \log\left(\frac{1+x}{1-x}\right)dx\geq\int_a^b 2x\,dx+\frac{2}{3}\int_a^b x^3\,dx$$

$$=b^2-a^2+\frac{2}{3}\cdot\frac{b^4-a^4}{4}=b^2-a^2+\frac{1}{6}(b^2-a^2)(b^2+a^2)=$$

$$=\frac{1}{6}(b^2-a^2)(6+b^2+a^2)$$

$$6\int_a^b \log\left(\frac{1+x}{1-x}\right)dx\geq(b^2-a^2)(6+b^2+a^2)$$

Equality holds for $a=b$.

Solution 2 by Marin Chirciu-Romania

We prove that $\log\left(\frac{1+x}{1-x}\right)\geq 2x+\frac{2}{3}x^3, 0\leq x<1$.

Indeed: We consider the function $f(x) = \log\left(\frac{1+x}{1-x}\right) - 2x - \frac{2}{3}x^3, 0 \le x < 1$

We have $f'(x) = \frac{2x^4}{1-x^2} \ge 0 \Rightarrow f$ is increasing on $[0,1) \Rightarrow f(x) \ge f(0) = 0 \Rightarrow$

$$\Rightarrow \log\left(\frac{1+x}{1-x}\right) - 2x - \frac{2}{3}x^3 \ge 0 \Rightarrow \log\left(\frac{1+x}{1-x}\right) \ge 2x + \frac{2}{3}x^3.$$

Using $\log\left(\frac{1+x}{1-x}\right) \ge 2x + \frac{2}{3}x^3$ we obtain:

$$6\int_a^b \log\left(\frac{1+x}{1-x}\right)dx \ge 6\int_a^b \left(2x + \frac{2}{3}x^3\right)dx = 6\int_a^b \left(x^2 + \frac{x^4}{6}\right)\Bigg|_a^b =$$

$$= 6\left(b^2 - a^2 + \frac{b^4 - a^4}{6}\right) = (b^2 - a^2)(b^2 + a^2 + 6).$$

Remark: In the same way.

If $0 \le a \le b$ then:

$$6\int_a^b \log(1+x)\,dx \ge (a-b)(a^2 + ab + b^2 - 3a - 3b)$$

Marin Chirciu

Solution

We prove that $\log(1+x) \ge x - \frac{1}{2}x^2, x \ge 0$

Indeed: We consider the function $f(x) = \log(1+x) - x + \frac{1}{2}x^2, x \ge 0$.

We have $f'(x) = \frac{x^2}{1+x} \ge 0 \Rightarrow f$ is increasing on $[0,\infty) \Rightarrow f(x) \ge f(0) = 0 \Rightarrow$

$$\Rightarrow \log(1+x) - x + \frac{1}{2}x^2 \ge 0 \Rightarrow \log(1+x) \ge x - \frac{1}{2}x^2$$

Using $\log(1+x) \ge x - \frac{1}{2}x^2$ we obtain:

$$6\int_a^b \log(1+x)\,dx \ge 6\int_a^b \left(x - \frac{1}{2}x^2\right)dx = 6\left(\frac{x^2}{2} - \frac{x^3}{6}\right)\Bigg|_a^b = 6\left(\frac{b^2-a^2}{2} - \frac{b^3-a^3}{6}\right) =$$

$$= 3(b^2 - a^2) - (b^3 - a^3) = (a-b)(a^2 + ab + b^2 - 3a - 3b)$$