ROMANIAN MATHEMATICAL MAGAZINE

SP.555 Find:

$$\Omega = \lim_{x\to 0}\frac{1}{x}\left(\frac{1}{x}\int_0^x \frac{dt}{t+e^t} - 1\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

Let be $f, F: (0,\infty) \to \mathbb{R}$

$$f(x) = \frac{1}{x+e^x}; F'(x) = f(x)$$

$$f'(x) = \frac{-1-e^x}{(x+e^x)^2}; f'(0) = \frac{-1-1}{(0+1)^2} = -2$$

$$\Omega = \lim_{x\to 0}\frac{1}{x}\left(\frac{1}{x}\int_0^x f(t)dt - 1\right) = \lim_{x\to 0}\frac{1}{x}\left(\frac{1}{x}(F(x)-F(0)-1)\right) =$$

$$= \lim_{x\to 0}\frac{F(x)-F(0)-x}{x^2} = \lim_{x\to 0}\frac{F'(x)-\left(F(0)\right)'-x'}{2x} =$$

$$= \lim_{x\to 0}\frac{f(x)-0-1}{2x} = \frac{1}{2}\lim_{x\to 0}\frac{f(x)-f(0)}{x} = \frac{1}{2}f'(0) = \frac{1}{2}\cdot(-2) = -1$$

Solution 2 by Marin Chirciu-Romania

$$\Omega = \lim_{x\to 0}\frac{1}{x}\left(\frac{1}{x}\int_0^x \frac{dt}{t+e^t} - 1\right)$$

$$\Omega = \lim_{x\to 0}\frac{1}{x}\left(\frac{1}{x}\int_0^x \frac{dt}{t+e^t} - 1\right) = \lim_{x\to 0}\frac{\int_0^x \frac{dt}{t+e^t} - x}{x^2} \overset{\frac{0}{0}}{\underset{l'Hospital}{=}} \lim_{x\to 0}\frac{\frac{1}{x+e^x}-1}{2x} =$$

$$= \lim_{x\to 0}\frac{1-x-e^x}{2x(x+e^x)} = \lim_{x\to 0}\frac{1}{2(x+e^x)}\lim_{x\to 0}\frac{1-x-e^x}{x} = \frac{1}{2}\lim_{x\to 0}\frac{1-x-e^x}{x} \overset{\left(\frac{0}{0}\right)}{\underset{l'Hospital}{=}}$$

$$= \frac{1}{2}\lim_{x\to 0}\frac{-1-e^x}{1} = \frac{1}{2}(-2) = -1$$

Remark: In the same way.

Find:

$$\Omega = \lim_{x\to 0}\frac{1}{x}\left(\frac{1}{x}\int_0^x \frac{dt}{\sin t+e^t} - 1\right)$$

Marin Chirciu

Solution

$$\Omega=\lim_{x\to0}\frac{1}{x}\left(\frac{1}{x}\int_0^x\frac{dt}{\sin t+e^t}-1\right)=\lim_{x\to0}\frac{\int_0^x\frac{dt}{\sin t+e^t}-x}{x^2}\underset{l'Hospital}{\overset{\left(\frac{0}{0}\right)}{=}}$$

$$=\lim_{x\to0}\frac{\frac{1}{\sin x+e^x}-1}{2x}=\lim_{x\to0}\frac{1-\sin x-e^x}{2x(\sin x+e^x)}=$$

$$=\lim_{x\to0}\frac{1}{2(\sin x+e^x)}\lim_{x\to0}\frac{1-\sin x-e^x}{x}=\frac{1}{2}\lim_{x\to0}\frac{1-\sin x-e^x}{x}\underset{l'Hospital}{\overset{\left(\frac{0}{0}\right)}{=}}$$

$$=\frac{1}{2}\lim_{x\to0}\frac{-\cos x-e^x}{1}=\frac{1}{2}(-2)=-1$$