The Convergence of Summation $\sum_{n=1}^{\infty} \frac{1}{4^n}\binom{2n}{n}$

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Abstract

Through this math note, we intend to check whether the series $\sum_{n=1}^{\infty} \frac{1}{4^n}\binom{2n}{n}$ converges or diverges. The main method used to show its convergence is applying a special case of Wallis Trigonometric Integral. The steps direct us to the conclusion that the series $\sum_{n=1}^{\infty} \frac{1}{4^n}\binom{2n}{n}$ is properly divergent.

Key words and Phrases: Infinite series, double factorial, Wallis integral.

1 Preliminary

The inspiration comes when we create a rational fraction $\frac{A}{B}$ where $A = \binom{2n}{n}$ is the nth central binomial coefficient and B is the total by summing up all of $\binom{2n}{k}$ for $k = 0, 1, 2, \cdots, 2n$. Then we get the numerical form of $\frac{A}{B}$ as

$$\frac{\binom{2n}{n}}{\sum_{k=0}^{2n}\binom{2n}{k}} \quad \text{or} \quad \frac{\binom{2n}{n}}{4^n}.$$

After that, we suppose that if $b(n) = \frac{\binom{2n}{n}}{4^n}$ for all $n \in \mathbb{N}_0$, so a question arises: does the series $\sum_{n=1}^{\infty} b(n) = b(1) + b(2) + b(3) + \cdots$ converge or not?

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Definition 1.1. *Let* $(b(n))_{n\in\mathbb{N}_0} = (b(0), b(1), b(2), b(3), \cdots\cdots)$ *be an infinite sequence such that* $b(n) = \frac{1}{4^n}\binom{2n}{n}$ *for all* $n \in \mathbb{N}_0$.

Definition 1.2 (**Double Factorial Numbers**). *A double factorial number* $n!!$, *where* $n \in \mathbb{N}_0$, *is defined by an infinite sequence* $(a(n))_{n\in\mathbb{N}_0} = (a(0), a(1), a(2), a(3), \cdots\cdots)$ *where* $n!! = a(n)$, $\forall n \in \mathbb{N}_0$; $a(0) = a(1) = 1$; *and* $a(n+2) = (n+2)a(n), \forall n \in \mathbb{N}_0$. *Mathematically,*

$$n!! = \begin{cases} 1 & \text{if } n = 0. \\ \prod_{k=0}^{\frac{n-2}{2}}(n-2k) & \text{if } n \text{ is an even positive integer.} \\ \prod_{k=0}^{\frac{n-1}{2}}(n-2k) & \text{if } n \text{ is an odd positive integer.} \end{cases}$$

Lemma 1.3 (**Formula of** $(b(n))_{n\in\mathbb{N}_0}$ **in term of double factorial numbers**).

$$b(n) = \begin{cases} 1 & \text{if } n = 0. \\ \frac{(2n-1)!!}{(2n)!!} & \text{if } n \in \mathbb{N}. \end{cases}$$

Proof. It is clear that $b(n) \in \mathbb{R}^+$ for all $n \in \mathbb{N}_0$, by Definition 1.1. It is also obvious that $b(0) = 1$. Since $b(0) = 1$ and $b(1) = \frac{1}{2}\binom{2}{1} = \frac{1}{2}$ then $\frac{b(1)}{b(0)} = \frac{1}{2}$. We also have that

$$\frac{b(n+1)}{b(n)} = \frac{\frac{1}{4^{n+1}}\binom{2n+2}{n+1}}{\frac{1}{4^n}\binom{2n}{n}} = \frac{2n+1}{2n+2},\ \forall n \in \mathbb{N}.$$

In conclusion, $\frac{b(n+1)}{b(n)} = \frac{2n+1}{2n+2}$ for every $n \in \mathbb{N}_0$. Now, observe that

$$b(n+1) = \prod_{k=0}^{n}\frac{b(k+1)}{b(k)} = \prod_{k=0}^{n}\frac{2k+1}{2k+2} = \frac{(2n+1)!!}{(2n+2)!!},\ \forall n \in \mathbb{N}_0.$$

Equivalently, $b(n) = \frac{(2n-1)!!}{(2n)!!}$ for every $n \in \mathbb{N}$. Thus, the result follows. □

2 Wallis Integral of Sine Function of an Even Positive Integer Power

Theorem 2.1. *For all even positive integers n, the following equation holds:*

$$\int_0^{\frac{\pi}{2}} \sin^n x \, dx \;=\; \frac{\pi}{2} \cdot \frac{(n-1)!!}{n!!}.$$

Proof. Using integral by part gives us

$$\begin{aligned}
\int \sin^n x \, dx \;&=\; \int \sin^{n-1} x \cdot \sin x \, dx \\
&=\; -\sin^{n-1} x \cdot \cos x \;+\; (n-1) \int \sin^{n-2} x \cdot \cos^2 x \, dx \\
&=\; -\sin^{n-1} x \cdot \cos x \;+\; (n-1) \int \sin^{n-2} x \cdot (1 - \sin^2 x) \, dx \\
&=\; -\sin^{n-1} x \cdot \cos x \;+\; (n-1) \int \sin^{n-2} x \, dx \;-\; (n-1) \int \sin^n x \, dx
\end{aligned}$$

then

$$n \int \sin^n x \, dx \;=\; -\sin^{n-1} x \cdot \cos x \;+\; (n-1) \int \sin^{n-2} x \, dx$$

$$\int \sin^n x \, dx \;=\; \frac{-\sin^{n-1} x \cdot \cos x}{n} \;+\; \frac{n-1}{n} \int \sin^{n-2} x \, dx \tag{1}$$

Suppose that $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$ for all $n \in \mathbb{N}_0$. So, we have $I_0 = \frac{\pi}{2}$.
If we set $x = 0$ and $x = \frac{\pi}{2}$ respectively as the lower and upper limit of integrals in Equation (1), then we obtain

$$I_n \;=\; \left(\frac{n-1}{n}\right) I_{n-2}, \quad \forall n \in \mathbb{N},\, n \geq 2.$$

Consequently, for every even $n \in \mathbb{N}$,

$$I_n \;=\; I_0 \cdot \prod_{k=0}^{\frac{n-2}{2}} \left(\frac{n-1-2k}{n-2k}\right) \;=\; \frac{\pi}{2} \cdot \frac{(n-1)!!}{n!!}.$$

The result is shown. □

3 Main Results

Remind that

$$\sum_{n=1}^{\infty} \frac{1}{4^n}\binom{2n}{n} = \sum_{n=1}^{\infty} b(n) = \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \tag{2}$$

By Theorem 2.1,

$$\frac{(2n-1)!!}{(2n)!!} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin^{2n} x\, dx,\ \forall n \in \mathbb{N} \tag{3}$$

Continuing the Equation (2) with (3) implies the following:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{4^n}\binom{2n}{n} = \sum_{n=1}^{\infty} \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin^{2n} x\, dx &= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \left(\sum_{n=1}^{\infty} \sin^{2n} x\, dx\right) \\
&= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{1-\sin^2 x}\, dx \\
&= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\cos^2 x}\, dx \\
&= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \left(\sec^2 x - 1\right)\, dx \\
&= \frac{2}{\pi}\Big[\tan x - x\Big]_0^{\frac{\pi}{2}} = \frac{2}{\pi}\cdot\infty = \infty.
\end{aligned}$$

From the result, $\sum_{n=1}^{\infty} \frac{1}{4^n}\binom{2n}{n}$ diverges. Since $\sum_{n=1}^{\infty} \frac{1}{4^n}\binom{2n}{n}$ goes to $+\infty$, so this series is properly divergent.

References

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